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ON THE LAW OF LARGE NUMBERS IN 2-UNIFORMLY SMOOTH BANACH SPACES

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In this paper we extend the Kolmogorov strong law of large numbers to random variables taking their values in a 2-uniformly smooth Banach space $(B, \| \ \|)$. In our result, the convergence of the classical series of variances is replaced by the convergence of the series having general term

$$\sup\{Ef^2(X_n)/n^2\colon \|f\|_{B'} \le 1\}.$$

Several recent papers deal with the strong law of large numbers for non-identically distributed Banach space valued random variables. Roughly speaking, the key idea in these papers is to mimic the scalar situation: the norms of the random variables fulfill the same kind of assumptions as the absolute values of the random variables in the scalar case. But in infinite dimensions, a lot of information about a random variable is lost by making only hypotheses on its norm; so under only "norm assumptions", one obtains sufficient conditions for the strong law of large numbers (the central limit theorem or the law of the iterated logarithm) which are too restrictive. On the other hand, conditions involving finite dimensional projections of a random variable with values in a general Banach space are difficult to handle.

Recently there has been substantial progress on the problem of taking into account finite dimensional assumptions on a random variable: V. Goodman, J. Kuelbs, and J. Zinn [2] found a method of investigation for Hilbert space valued random variables, which is strong enough to give a necessary and sufficient condition for the law of the iterated logarithm. M. Ledoux [4] has shown that this method extends to 2-uniformly smooth Banach spaces.

By using the same fundamental ideas as these authors, we will obtain an analogue to Kolmogorov's law of large numbers for 2-uniformly smooth Banach space valued random variables. The general term of the classical series involved in such a law of large numbers will not be "$E\|X_n\|^2/n^2$" but the smaller quantity "$\sup_{\|f\|_{B'}\le 1}(Ef^2(X_n)/n^2)$".

We will show in an example that our theorem is strong enough to conclude that the strong law of large numbers holds in situations where results as smooth as the J. Kuelbs' and J. Zinn's [3] extension of the Prohorov's law of large numbers or W. Woyczynski's [6] law of large numbers in 2-uniformly smooth Banach spaces don't apply.

1. Introduction. Consider $(X_k, k \in \mathbb{N})$ a sequence of independent random variables (r.v.) with values in a real separable Banach space $(B, \|\cdot\|)$, which is

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equipped with its Borel σ-field $\mathscr{B}$. For each integer n, we put:

$$S_n = X_1 + X_2 + \cdots + X_n.$$

One says that the strong law of large numbers holds for the sequence $(X_k, k \in \mathbb{N})$ if and only if

$$(S_n/n) \to_{n\to+\infty} 0 \quad \text{a.s.}$$

A sufficient condition for the strong law of large numbers is the following extension of the well-known Prohorov's theorem (see [5], Theorem 5.2.4):

THEOREM 1 (J. Kuelbs and J. Zinn [3]). *Let $(X_k, k \in \mathbb{N})$ be a sequence of independent r.v. with values in a real separable Banach space $(B, \|\cdot\|)$. Suppose that there exists a positive constant M, such that*

$$\forall j \in \mathbb{N} \qquad \|X_j\| \leq M\,(j/L_2 j) \qquad \text{a.s.},$$

where $L_2x = \operatorname{Log}(\operatorname{Log}(\sup(x, e^e)))$. For each integer n we define

$$\Lambda(n) = \textstyle\sum_{j=2^n+1}^{2^{n+1}} E\|X_j\|^2/2^{2(n+1)}$$

and we suppose that the following condition holds:

$$(1) \qquad \forall \varepsilon > 0 \qquad \textstyle\sum_{n=1}^{\infty} \exp(-\varepsilon/\Lambda(n)) < +\infty.$$

Then:

$$(S_n/n) \to_P 0 \iff (S_n/n) \to_{n\to+\infty} 0 \quad \text{a.s.}$$

This result is true in every Banach space. In the special case of a type 2 space, the condition (1) implies that $(S_n/n) \to_P 0$, so in that case the result can be stated simply as

$$(1) \Rightarrow (S_n/n) \to_{n\to+\infty} 0 \quad \text{a.s.}$$

In this paper we restrict our interest to a special class of type 2 spaces: the 2-uniformly smooth spaces. For the sake of completeness, we recall the definition and main properties of such a space.

DEFINITION. A Banach space $(B, \|\cdot\|)$ of dimension $n \geq 2$ is said to be 2-uniformly smooth if there exists a positive constant K such that

$$\forall (x, y) \in B^2, \quad \|x+y\|^2 + \|x-y\|^2 \leq 2\|x\|^2 + K\|y\|^2.$$

Let us write down some useful properties of 2-uniformly smooth spaces; the reader will find more details and proofs in [4].

1) The norm $\|\cdot\|$ of a 2-uniformly smooth space is uniformly Fréchet differentiable away from 0; so for each nonzero element x of B there exists a continuous linear functional $D(x)$ such that for all y in B, and all $t \in \mathbb{R}$, with $x + ty$ different from 0, one has

$$(d/dt)\ \|x + ty\| = D(x + ty)(y).$$

2) A 2-uniformly smooth space is of type 2.

3) For each nonzero $x \in B$ we define

$$F(x) = \|x\| D\left(\frac{x}{\|x\|}\right).$$

Then

$$\|F(x)\|_{B'} = \|x\|.$$

Furthermore, for every sequence $(x_1, x_2, \cdots, x_n)$ in B the following inequality holds:

$$\|\textstyle\sum_{j=1}^{n} x_j\|^2 \leq 2 \sum_{j=2}^{n} F(\sum_{k=1}^{j-1} x_k)(x_j) + C \sum_{j=1}^{n} \|x_j\|^2,$$

where C is the type 2 constant of the space B.

The law of large numbers in 2-uniformly smooth spaces has been investigated by W. A. Woyczynski. For instance, he has obtained the following result ([6] Theorem 3.1):

THEOREM 2. *Let $(M_n, n \in \mathbb{N})$ be a martingale with values in a 2-uniformly smooth space $(B, \|\cdot\|)$. If there exists $q \geq 1$ such that*

$$\sum_{n=1}^{\infty} \frac{E\|M_n - M_{n-1}\|^{2q}}{n^{q+1}} < +\infty,$$

then

$$\|M_n\| = o(n) \quad \text{a.s.}$$

This result (as Theorem 1) only involves hypotheses on the norms of the r.v. Our goal is to prove the following result which involves properties of the finite dimensional projections of the r.v.:

THEOREM 3. *Let $(X_k, k \in \mathbb{N})$ be a sequence of independent centered r.v. which take their values in a 2-uniformly smooth Banach space $(B, \|\cdot\|)$. Suppose that the following conditions hold:*

1) *There exists a positive constant k such that:*

$$\forall n \in \mathbb{N}, \quad \|X_n\| \leq k \frac{n}{\sqrt{L_2 n}} \quad \text{a.s.;}$$

2)
$$\lim_{n\to+\infty} \sum_{j=1}^{n} \frac{E\|X_j\|^2}{n^2} = 0;$$

3)
$$\sum_{j=1}^{\infty} \sup_{\|f\|_{B'}\leq 1}\left(\frac{Ef^2(X_j)}{j^2}\right) < +\infty.$$

Then the sequence $(X_k, k \in \mathbb{N})$ satisfies the strong law of large numbers.

We will first prove this result; then we will build a sequence of r.v. $(X_k, k \in \mathbb{N})$ taking values in ℓ^2, which fulfills the hypothesis of Theorem 3, but not the ones of Theorem 1 and Theorem 2.

2. Proof of Theorem 3. By applying the property 3) of 2-uniformly smooth spaces recalled above, we have, for all integers n:

$$\left(\left\| \sum_{j=1}^{n} \frac{X_j}{n} \right\| \right)^2 \leq 2 \sum_{j=2}^{n} \frac{F(\sum_{k=1}^{j-1} X_k)(X_j)}{n^2} + C \sum_{j=1}^{n} \frac{\| X_j \|^2}{n^2}.$$

So the proof of our theorem will split into 2 parts:

a) $\sum_{j=1}^{n} \frac{\| X_j \|^2}{n^2} \to_{n \to +\infty} 0$ a.s.,

b) $\sum_{j=2}^{n} \frac{F(\sum_{k=1}^{j-1} X_k)(X_j)}{n^2} \to_{n \to +\infty} 0$ a.s.

To prove a) it is clearly sufficient to show:

$$\sum_{j=1}^{n} \frac{\| X_j \|^2 / j}{n} \to_{n \to +\infty} 0 \quad \text{a.s.}$$

It follows easily from hypothesis 2) that

$$\lim_{n \to +\infty} \sum_{j=1}^{n} \frac{E \| X_j \|^2 / j}{n} = 0,$$

and so:

$$\sum_{j=1}^{n} \frac{\| X_j \|^2 / j}{n} \to_P 0.$$

Now put for each integer j:

$$Y_j = \frac{\| X_j \|^2}{j}.$$

By hypothesis 1) there exists $K > 0$ such that:

$$\forall j \in \mathbb{N}, \quad \| Y_j \| \leq K\,(j / L_2 j) \quad \text{a.s.}$$

Furthermore:

$$\Lambda(n) = \sum_{j=2^n+1}^{2^{n+1}} \frac{E \| Y_j \|^2}{2^{2(n+1)}} \leq K' \sum_{j=2^n+1}^{2^{n+1}} \frac{E \| X_j \|^2}{2^{2(n+1)}(\ln n)}.$$

So, for each $\varepsilon > 0$:

$$\sum_{n=1}^{\infty} \exp(-\varepsilon / \Lambda(n)) < +\infty.$$

Property a) is therefore an easy consequence of Theorem 1 applied to the r.v. $(Y_j, j \in \mathbb{N})$.

To prove b) we consider the martingale:

$$Z_n = \sum_{j=2}^{n} F\left(\sum_{k=1}^{j-1} X_k\right)(X_j),$$

the stochastic basis being $\mathscr{F}_n = \sigma(X_1, \cdots, X_n)$. We will show that it fulfills the hypothesis of the following convergence theorem:

LEMMA (Chow [1]). *Let $(X_i, \mathscr{F}_i, i \geq 1)$ be a martingale difference sequence. Let a_i be $\mathscr{F}_{i-1}$ measurable for each $i \geq 1$ and $0 < a_i \uparrow +\infty$ a.s. Then:*

$$\sum_{i=1}^{\infty} \frac{E[|X_i|^p \mid \mathscr{F}_{i-1}]}{a_i^p} < +\infty \quad \text{a.s.}$$

for some $0 < p \leq 2$ implies:

$$(S_n/a_n) \to_{n\to+\infty} 0 \quad \text{a.s.}$$

The space B being 2-uniformly smooth, one has:

$$E\left(\sum_{j=2}^{\infty} \frac{E[F^2(\sum_{k=1}^{j-1} X_k)(X_j) \mid \mathscr{F}_{j-1}]}{j^4}\right)$$

$$= \sum_{j=2}^{\infty} \frac{E[F^2(\sum_{k=1}^{j-1} X_k)(X_j)]}{j^4}$$

$$\leq \sum_{j=2}^{\infty} \frac{E\| S_{j-1}\|^2}{j^2} \sup_{\|f\|_{B'}\leq 1} \frac{Ef^2(X_j)}{j^2}$$

$$\leq C\left(\sup_l \sum_{k=1}^{l} \frac{E\|X_k\|^2}{l^2}\right) \sum_{j=2}^{\infty} \sup_{\|f\|_{B'}\leq 1} \frac{Ef^2(X_j)}{j^2}.$$

So it follows by applying hypothesis 3):

$$\sum_{j=2}^{\infty} \frac{E[F^2(\sum_{k=1}^{j-1} X_k)(X_j) \mid \mathscr{F}_{j-1}]}{j^4} < +\infty \quad \text{a.s.}$$

By Chow's Lemma one has:

$$(Z_n/n^2) \to_{n\to+\infty} 0 \quad \text{a.s.},$$

and this ends the proof of Theorem 3.

3. An example of application. Consider an independent triangular array of independent real values r.v.: $\{\lambda_j^n, j = 1, 2, \cdots, [n^{1/6}]\}$, the notation [ ] denoting the integer part of a number.

For each fixed n, the λ_j^n have the same distribution; they are Cauchy r.v. truncated at the level $n^{5/6}/L_3 n$, where the function L_3 is defined in a way analogous to L_2.

Consider ℓ^2 equipped with its canonical basis $(e_n, n \in \mathbb{N})$. It is a Hilbert space and so it is of course a 2-uniformly smooth space. Define the following sequence

$(X_n, n \in \mathbb{N})$ of ℓ^2-valued r.v.:

$$X_n = \sum_{j=1}^{[n^{1/6}]} \lambda_j^n e_j.$$

We will first check that this sequence of r.v. fulfills the hypotheses of Theorem 3:

1) It is clear that there exists $k > 0$ such that: $\forall\, n \in \mathbb{N}$, $\| X_n \| \le k(n/\sqrt{L_2 n})$ a.s.

2) $E \| X_n \|^2 = \sum_{j=1}^{[n^{1/6}]} E(\lambda_j^n)^2 \le K(n/L_3 n)$, so: $\lim_{n\to+\infty} \sum_{j=1}^{n} E \| X_j \|^2/n^2 = 0$.

3) $\sup_{\|f\|_{\ell^2} \le 1} Ef^2(X_n) =$

$$\sup_{\left\{\substack{a_1, \ldots, a_{[n^{1/6}]}: \\ \sum a_i^2 \le 1}\right\}} 2 \sum_{k=1}^{[n^{1/6}]} \frac{a_k^2}{\pi} \int_0^{n^{5/6}/L_3 n} \frac{x^2}{1+x^2}\, dx \le \frac{2}{\pi} \frac{n^{5/6}}{L_3 n}.$$

It follows:

$$\sum_{j=1}^{\infty} \sup_{\|f\|_{\ell^2} \le 1} \frac{Ef^2(X_j)}{j^2} < +\infty.$$

By applying Theorem 3, we conclude that the sequence (X_n) satisfies the strong law of large numbers.

We will see now that the hypothesis:

$$\text{(2)} \qquad \forall\, \varepsilon > 0 \ \sum_{n=1}^{\infty} \exp(-\varepsilon/\Lambda(n)) < +\infty$$

of Theorem 1 is not fulfilled. To see this, notice,

$$E \| X_n \|^2 \ge K' \ (n/L_3 n),$$

so

$$\Lambda(n) \ge (K''/L_2 n),$$

and condition (2) clearly fails to hold. So Theorem 3 allows one to reach situations in which Theorem 1 is helpless. What about Theorem 2?

Let $q \ge 1$; then:

$$E \| X_n \|^{2q} = E\Big(\sum_{j=1}^{[n^{1/6}]} (\lambda_j^n)^2\Big)^q \ge \Big(\sum_{j=1}^{[n^{1/6}]} E(\lambda_j^n)^2\Big)^q \ge (K')^q \ (n/L_3 n)^q.$$

It follows that for each integer n:

$$\frac{E \| X_n \|^{2q}}{n^{q+1}} \ge (K')^q \frac{1}{n(L_3 n)^q}:$$

so a $q \ge 1$ doesn't exist for which the hypothesis of Theorem 2 is fulfilled.

Theorem 3 also allows us to conclude that the strong law of large numbers holds in some situations in which Theorem 2 does not apply.

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REFERENCES

[1] CHOW, Y. S. (1965). Local convergence of martingales and the law of large numbers. *Ann. Math. Statist.* **36** 552–558.

[2] GOODMAN, V., KUELBS, J., and ZINN, J. (1981). Some results on the law of the iterated logarithm in Banach space with applications to weighted empirical processes. *Ann. Probab.* **9** 713–752.

[3] KUELBS, J. and ZINN, J. (1979). Some stability results for vector valued random variables. *Ann. Probab.* **7** 75–84.

[4] LEDOUX, M. (1982). Sur les théoremes limites dans certains espaces de Banach lisses. Probability in Banach spaces 4—Oberwolfach 1982. *Lecture Notes in Math* **990** 150–169.

[5] STOUT, W. F. (1974). *Almost Sure Convergence.* Academic, New York.

[6] WOYCZYNSKI, W. A. (1981). Asymptotic behavior of martingales in Banach spaces II, Martingale theory in harmonic analysis and Banach spaces, Cleveland 1981. *Lecture Notes in Math* **939** 216–225.

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