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BAD RATES OF CONVERGENCE FOR THE CENTRAL LIMIT THEOREM IN HILBERT SPACE

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We show that one can smoothly renorm the Hilbert space H such that the rate of convergence in the central limit theorem becomes very bad. More precisely, let us fix a sequence $\xi_n \to 0$ and $\varepsilon > 0$. We can then construct a norm $N(\cdot)$ on the Hilbert space, and a bounded random variable X on H with the following properties:

(a) The norm $N(\cdot)$ is $(1 + \varepsilon)$ equivalent to the usual norm. It is infinitely many times differentiable, and each differential is bounded on the unit sphere.

(b) If (X_i) denotes independent copies of X, and if γ is the Gaussian measure with the same covariance as X, then the inequality

$$\mathrm{Sup}_{t>0} | P\{N(n^{-1/2} \textstyle\sum_{i=1}^n X_i) \le t\} - \gamma\{x;\, N(x) \le t\} | \ge \xi_n$$

occurs for infinitely many n.

1. Introduction. Let X be a random variable (r.v.) valued in a Banach space E. Assume X has a second moment, $EX = 0$ and that there is a Gaussian measure γ on E with the same covariance as X, i.e.,

$$E(x^*(X)y^*(X)) = \textstyle\int_E x^*(x)y^*(x)\, d\gamma(x) \quad \text{for} \quad x^*, y^* \in E^*.$$

Let (X_i) be a sequence of independent r.v. distributed as X. A way to estimate the rate of convergence in the central limit theorem is by the quantity

$$\Delta_n = \mathrm{Sup}_{t\ge 0} | P\{\| n^{-1/2} \textstyle\sum_{i=1}^n X_i\| \le t\} - \gamma\{x;\, \|x\| \le t\} |.$$

It has been shown by Kuelbs and Kurtz [2] that if the norm of E is three times differentiable with a third differential bounded on the unit sphere, if X has moments of order 7/2 and if moreover the following condition holds,

$$(*) \qquad \forall 0 \le s < t,\ \gamma\{x;\, s \le \|x\| \le t\} \le C(t - s)$$

for a constant C, then $\Delta_n = O(n^{-1/6})$. V. Paulauskas [3] reduced the moment assumption to moments of order 3.

In Hilbert space, condition (*) is always satisfied for the usual norm [2]. If moreover X has moments of order 6, then $\Delta_n = O(n^{-1/2})$ [1].

The estimate (Δ_n) depends on the precise choice of the norm. It will in particular follow from the result presented here that a small change of norm can dramatically change Δ_n. We shall prove the following:

THEOREM A. *Let $\varepsilon > 0$ and a sequence $\xi_n \to 0$. Then there is a norm $N(\cdot)$ on*

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the standard Hilbert space H and a bounded r.v. *X such that the following hold.*

(a) $\forall x \in H, \quad (1-\varepsilon)\|x\| \le N(x) \le \|x\|$

(b) $N(\cdot)$ *is infinitely many times differentiable, and each of its differentials is bounded on the unit sphere.*

(c) *For infinitely many values of n,* we have

$$\Delta_n' = \operatorname{Sup}_{t\ge 0} |P\{N(n^{-1/2} \textstyle\sum_{i=1}^n X_i) \le t\} - \gamma\{x; N(x) < t\}| \ge \xi_n.$$

In particular Kuelbs and Kurtz' result implies that the norm $N(\cdot)$ fails condition (*). So a norm can fail condition (*) even when the Banach space E is very smooth (isomorphic to H) and the norm is very smooth. Weaker examples for which condition (*) fails were previously obtained by the authors [4] (on the space ℓ^p with $p \in \mathbb{N}$, $p \ge 2$ and a norm p times differentiable) and Paulauskas (on c_0 with the sup norm).

Moreover, even if X is bounded, if E is very regular and if the norm is very smooth, no rate of convergence can be given for Δ_n without further assumptions of the type (*).

2. Methods. The construction will use finite-dimensional blocks which we shall patch together. So we start with elementary observations. Let ℓ_n^2 denote the n-dimensional Hilbert space. Let $(e_i)_{i\le n}$ be the canonical basis of ℓ_n^2. So for $x \in \ell_n^2$ we have $x = \sum_{i\le n} x_i e_i$. Let γ_n be the canonical Gaussian measure on ℓ_n^2, i.e. the measure such that the distribution of the functionals $x \to x_i$ are standard normal and independent.

OBSERVATION 1. The variables x_i^2 are equidistributed. Their expectation is one. The one dimensional central limit theorem asserts that for n large, the distribution of $\|x\|^2 = \sum x_i^2$ is approximately $N(n, \sqrt{3n})$, (since the variance of x_i^2 is 3). In particular, $\gamma_n\{\|x\| < \sqrt{n}\} > 1/3$ for n large. Moreover, the distribution of $n^{-1}\|x\|^2$ becomes very concentrated around 1.

OBSERVATION 2. Let Y_n be a r.v. valued in ℓ_n^2, such that for $i \in \{1, \cdots, n\}$ and $j \in \{-1, 1\}$ it takes the value $jn^{1/2}e_i$ with probability $1/2n$. Let $(Y_{n,i})$ be an independent sequence distributed like Y_n. Then for $q \le n$, with probability $\prod_{i=1}^q (1 - i/n) \ge (1 - q/n)^q$ the r.v. $S_{n,q} = q^{-1/2} \sum_{i=1}^q Y_{n,i}$ takes values of the type $\sum_{i\in I} a_i e_i$, where card $I = q$ and $|a_i| = n^{1/2}q^{-1/2}$. So, with the same probability, we have $\|S_{n,q}\| = n^{1/2}$. It follows that for any given q, we have

$$\lim_{n\to\infty} P\{\|S_{n,q}\| = n^{1/2}\} = 1.$$

For two integers m, n, we identify ℓ_{m+n}^2 with the spaces $\ell_m^2 \times \ell_n^2$ and $\ell_m^2 \oplus \ell_n^2$.

3. An auxiliary norm. We shall use an auxiliary norm on $\ell^2 = \ell^2(\mathbb{N})$.

PROPOSITION. *Let $\varepsilon > 0$. Then there is a norm N^0 on ℓ^2 which has the following properties*

$$\forall\, x \in \ell^2, \qquad (1-\varepsilon)\|x\| \le N^0(x) \le \|x\| \tag{3.1}$$

(3.2) N^0 is infinitely many times differentiable, and each of its derivatives is bounded on the unit sphere.

(3.3) If there exists n_0 such that $\sum_{n \neq n_0} x_n^2 \leq \varepsilon/10\ x_{n_0}^2$ then $N^0(x) = |x_{n_0}|$. Moreover, if x and y have disjoint support, $N^0(x+y) \geq N^0(x)$.

Proof. We fix a function $f: \mathbb{R} \to \mathbb{R}$ with the following properties:

(3.4) f is infinitely many times differentiable and $t \to f(t^2)$ is convex.

$$\forall t,\ |t| < \varepsilon/10,\ f(t) = 0;\ \forall t,\ |t| \geq \varepsilon \Rightarrow f(t) = t \tag{3.5}$$

$$\forall t \in \mathbb{R}^+,\ t - \varepsilon \leq f(t) \leq t. \tag{3.6}$$

The existence of such a function is elementary. We can assume $\varepsilon \leq 1/100$. For $x = (x_n) \in \ell^2$, define $A(x)$ in the following way:

First case. There is n_0 such that the following condition is satisfied:

$$P(n_0):\ x_{n_0}^2 > 10 \textstyle\sum_{n \neq n_0} x_n^2.$$

We set $A(x) = x_{n_0}^2 + f((\sum_{n \neq n_0} x_n^2))$.

Second case. The first case does not occur. We set

$$A(x) = \|x\|^2 = \textstyle\sum_n x_n^2.$$

It follows from (3.6) that

$$\forall x \in \ell^2,\ \|x\|^2 - \varepsilon \leq A(x) \leq \|x\|^2. \tag{3.7}$$

We set

$$N^0(x) = \operatorname{Inf}\{\lambda > 0;\ A(x/\lambda) \leq 1\}.$$

Let us show that N^0 is a norm. It is obvious that N^0 is homogeneous. It is enough to show that if $y(1), y(2) \in \ell^2$ with $A(y(1)), A(y(2)) \leq 1$, then $A((y(1)+y(2))/2) \leq 1$. We have

$$\begin{aligned} A((y(1)+y(2))/2) &\leq \|(y(1)+y(2))/2\|^2 \\ &= 1/2(\|y(1)\|^2 + \|y(2)\|^2) - \|(y(1)-y(2))/2\|^2. \end{aligned} \tag{3.8}$$

From (3.7) it follows that it is enough to check the case $\|y(1)\|^2 \geq 1-\varepsilon$, $\|y(2)\|^2 \geq 1-\varepsilon$, $\|(y(1)-y(2))/2\|^2 \leq \varepsilon$. We distinguish 3 cases.

Case 1. $y(1)$ and $y(2)$ fail $P(n)$ for each n.
Obvious, since $A(y(i)) = \|y(i)\|^2$ for $i = 1, 2$.

Case 2. One of $y(1)$, $y(2)$ (say $y(1)$) satisfies $P(n_0)$, the other fails $P(n)$ for each n.
We have $\sum_{n \neq n_0} y_n^2(2) \geq \varepsilon$, for otherwise, since $y(2)$ fails $P(n_0)$, we would have

$\| y(2) \|^2 \le 11\varepsilon < 1 - \varepsilon$, a contradiction. So we have in fact

$$A(y(2)) = \| y(2) \|^2 = y_{n_0}^2(2) + f(\textstyle\sum_{n \neq n_0} y_n^2(2))$$

and then the result follows from the convexity of $t \to f(t^2)$.

CASE 3. Both $y(1), y(2)$ satisfy a condition of the type $P(n)$.

But it is then clear that $y(1), y(2)$ satisfy a condition $P(n)$ for the same n, and the result follows as above.

We now check conditions (3.1) to (3.3).

1st step. We first check that A is infinitely differentiable on the set U, $U = \{x \in \ell^2;\ \| x \| \ge 1/2\}$. This is done by checking that the definitions on f on the various parts of U patch smoothly. Let $x \in U$. For the convenience of notations, assume that $|x_1| \ge |x_2| \ge \cdots$.

If $x_1^2 > 10 \sum_{n>1} x_n^2$, then condition $P(1)$ is still true in a neighborhood of x. In this neighborhood, $A(x) = x_1^2 + f(\sum_{n>1} x_n^2)$ is infinitely differentiable. If $x_1^2 = 10 \sum_{n>1} x_n^2$, then $\| x \|^2 = 11 \sum_{n>1} x_n^2 \ge 1/4$, so $\sum_{n>1} x_n^2 \ge 1/44 > \varepsilon$. Let V be the set of all $y \in U$ for which $y_1^2 \ge 2 \sum_{n>1} y_n^2$ and $\sum_{n>1} y_n^2 > \varepsilon$. This is a neighborhood of U. Let $y \in V$. If y satisfies $P(n)$ for some n, then $n = 1$. It follows that

$$A(y) = y_1^2 + f(\textstyle\sum_{n>1} y_n^2) = \| y \|^2$$

since $f(t) = t$ for $t > \varepsilon$. If y fails all conditions $P(n)$ for each n, then $A(y) = \| y \|^2$. So $A(y) = \| y \|^2$ in V, and hence is infinitely differentiable.

If $x_1^2 < 10 \sum_{n>1} x_n^2$, then $A(y) = \| y \|^2$ in a neighborhood of x.

2nd step. We show that in the domain $U' = \{x;\ 1/2 < \| x \| < 2\}$ all the derivatives of A are bounded. Indeed the analysis of the second step shows that each point in U' has a neighborhood on which A coincides either with $\| \cdot \|$ or with a function f_n of the type $f_n(x) = x_n^2 + f(\sum_{i \neq n} x_i^2)$. But these functions are infinitely differentiable in U', and their derivatives are bounded.

3rd step. We check (3.1). Let $x \neq 0$. The continuity of A in U' implies $A(x/N^0(x)) = 1$. Now (3.7) implies

$$\| x/N^0(x) \|^2 - \varepsilon^2 \le 1 \le \| x/N^0(x) \|^2.$$

It follows that $N^0(x) \le \| x \| \le (1 + \varepsilon^2)^{1/2} N^0(x)$, which implies (3.1).

4th step. We check that for $x \in U$ we have $D_x A(x) \ge 1/3$. If $A(y) = \| y \|^2$ in a neighborhood of x, then $D_x A(x) = 2 \| x \|^2 \ge 1/2$ from (3.1). Otherwise $A(y) = y_n^2 + f(\sum_{i \neq n} y_i^2)$ in a neighborhood of x, then

$$D_x A(x) = 2(x_n^2 + \textstyle\sum_{i \neq n} x_n^2 f'(\sum_{i \neq n} x_i^2)) \ge 2x_n^2.$$

But since $x_n^2 \ge 10 \sum_{i \neq n} x_i^2$, we have $\| x \|^2 \le 11/10\, x_n^2$, so $D_x A(x) \ge 20/11\, \| x \|^2 \ge 1/3$ from (3.1).

5th step. We prove (3.2). For $x \in U'$, $t \in \mathbb{R}^+$, let $g(x, t) = A(x/t)$. The 4th step shows that the second derivative of g is nonzero. Since $g(x, N^0(x)) = 1$, the implicit function theorem (see page 67 of Henri Cartan, *Calcul Differentiel*, Hermann, Paris, 1967) shows that N^0 is differentiable on U', and we have $D_x N^0(y) = D_{\bar{x}} A(y)/D_{\bar{x}}(\bar{x})$ where $\bar{x} = x/N^0(x)$. This formula shows that N^0 is infinitely differentiable. It also shows (by induction) that the nth differential of N^0 at x is a sum of compositions of differentials of A at $\bar{x}$ divided by quantities of the type $N^0(x)^p D_{\bar{x}}(\bar{x})^q$. Hence the 3rd and 4th steps show that these differentials are bounded on U', hence on the unit sphere.

6th step. We check (3.3). Assume that $\sum_{n \neq n_0} x_n^2 \leq \varepsilon/10\, x_{n_0}^2$. Let $y = x/N^0(x)$. Then $\sum_{n \neq n_0} y_n^2 \leq \varepsilon/10\, y_{n_0}^2$. It follows that $P(n_0)$ is satisfied, and hence $A(y) = y_{n_0}^2 + f(\sum_{n \neq n_0} y_n^2)$. Since $A(y) = 1$, we have $y_{n_0}^2 \leq 1$, so $\sum_{n \neq n_0} y_n^2 \leq \varepsilon/10$. Since $f(t) = 0$ for $t \leq \varepsilon/10$, we have $1 = A(y) = y_{n_0}^2 = x_{n_0}^2 (N^0(x))^{-2}$ and the first assertion follows. The last one follows from the obvious fact that $N^0(x + y) = N^0(x - y)$. The proof is complete.

The essential part of the above construction is condition (3.3). It ensures that the unit ball of N^0 is absolutely flat in a neighborhood of the basic vectors.

4. Construction. By induction over p, we shall construct two sequences $n(p)$, $q(p)$ of integers, a sequence $a(p)$ of numbers, and a sequence Y_p of r.v. valued in $\ell^2_{n(p)}$. Let $m(p) = n(1) + \cdots + n(p)$. With natural identifications, one can consider the r.v. X_p valued in $\ell^2_{m(p)}$ given by $X_p = Y_1 + \cdots + Y_p$. Let γ_p be the Gaussian measure on $\ell^2_{m(p)}$ with the same covariance as X_p. On $\ell^2_{m(p)} = \oplus_{i=1}^p \ell^2_{n(p)}$, let N_p be the norm given by $N_p(x) = N^0(\bar{x})$, where $x = (x_1, \cdots, x_p)$, $x_i \in \ell^2_{n(p)}$ for $i = 1, \cdots, p$, and $\bar{x} = (\| x_1 \|, \| x_2 \|, \cdots, \| x_p \|, 0, \cdots)$.

The following conditions will be satisfied for all $p > 1$.

(4.1) $$\forall \omega \; \| Y_p(\omega) \| \leq 2^{-p}$$

(4.2) $$\gamma_p\{x; N_p(x) \leq a(r)\} > 2\xi_{q(r)} \quad \text{for} \quad r \leq p$$

(4.3) If $(X_p^i)_i$ is an independent sequence distributed like X_p, for each $r \leq p$ we have $P\{N_p(q(r)^{-1/2} \sum_{i \leq q(r)} X_p^i) \leq a(r)\} < \xi_{q(r)}$.

We proceed to the first step of the construction. Let $q(1)$ be large enough so that $\xi_{q(1)} < 1/6$. It follows by observations 1 and 2 that there exists $n(1)$ and an $\ell^2_{n(1)}$—valued r.v. Y_1 with $\| Y_1(\omega) \| = 1/2$ for each ω, such that we have

$$\gamma_1\{\| x \| \leq 1/2\} > 1/3$$
$$P\{\| q(1)^{-1/2} \textstyle\sum_{i \leq q(1)} X_1^i \| < 1/2\} < \xi_{q(1)}.$$

We take $a(1) < 1/2$ such that $\gamma_1\{x; \| x \| \leq a(1)\} > 1/3$, and this completes the first step of the construction, since $\| x \| = N_1(x)$ on $\ell^2_{n(1)}$.

We now assume that the construction has been done up to rank p. There exists a positive number b so small that for each $r \leq p$ we have

(4.4) $$\gamma_p\{N_p(x) \leq a(r) - 2b\} > 2\xi_{q(r)} + b.$$

We can assume $b \le 2^{-p-1}$. We can now pick $q(p+1)$ so large that

$$\gamma_p\{x;\ \|x\| < \varepsilon b/2\} \ge 7\xi_{q(p+1)}.$$

It follows from observations 1 and 2 and by scaling that there exists an integer $n(p+1)$, and a r.v. Y_{p+1}, with $E(Y_{p+1}) = 0$, independent of X_p, valued in $\ell^2_{n(p+1)}$, and such that the following occurs:

(4.5) $$\forall\omega,\ \|Y_{p+1}(\omega)\| = b.$$

(4.6) If ν is the Gaussian measure on $\ell^2_{n(p+1)}$ which has the same covariance as Y_{p+1}, we have

$$\nu\{x;\ b/2 < \|x\| < b\} > 1/3;\ \nu\{x;\ \|x\| < 2b\} > 1 - b.$$

(4.7) If $(Y^i_{p+1})_i$ are independent copies of Y_{p+1}, we have

$$P\{\|q(p+1)^{-1/2} \textstyle\sum_{i\le q(p+1)} Y^i_{p+1}\| < b\} < \xi_{q(p+1)}.$$

We choose $a(p+1) < b$ such that $\nu\{x;\ b/2 < \|x\| < a(p+1)\} > 1/3$. Since Y_{p+1} and X_p are independent, we have $\gamma_{p+1} = \gamma_p \otimes \nu$.

Let $x \in \ell^2_{m(p)}$, and $y \in \ell^2_{n(p+1)}$. We can write $x = (x_1, x_2, \cdots, x_p)$ with $x_i \in \ell^2_{n(i)}$ for $1 \le i \le p$. We hence have $N_{p+1}(x, 0) = N^0(\bar{x})$, where

$$\bar{x} = (\|x_1\|, \|x_2\|, \cdots, \|x_p\|, 0, \cdots).$$

So we have $N_{p+1}(x, 0) = N_p(x)$. We have $N_{p+1}(0, y) = N^0(\bar{y})$, where $\bar{y} = (0, 0, \cdots, \|y\|, 0, \cdots)$, $\|y\|$ being the $p+1$th component. This shows that $\bar{y}$ satisfies condition $P(p+1)$ and $A(\bar{y}) = \|y\|^2$. It follows that $N_{p+1}(0, y) = \|y\|$.

For $\|y\| < 2b$ and $N_p(x) \le a(r) - 2b$, we have

$$N_{p+1}(x, y) \le N_{p+1}(x, 0) + N_{p+1}(0, y) \le N_p(x) + \|y\| \le a(r).$$

Let $r \le p$. Recall that we have

(4.4) $$\gamma_p\{x;\ N_p(x) \le a(r) - 2b\} \ge 2\xi_{q(r)} + b$$

(4.5) $$\nu\{y;\ \|y\| \le 2b\} \ge 1 - b.$$

So, we have

$$\gamma_{p+1}\{z \in \ell_{m(p+1)};\ N_{p+1}(z) \le a(r)\} \ge (2\xi_{q(r)} + b)(1 - b).$$

However if $\xi_{q(r)} \le 1/6$ and $b \le 1/2$, we have $(2\xi_{q(r)} + b)(1 - b) \ge 2\xi_{q(r)}$. This proves (4.2) for $r \le p$.

We now check a basic fact: for $\|x\| \le \varepsilon\|y\|$, we have $N_{p+1}(x, y) = \|y\|$, that is, the unit ball of N_{p+1} is flat in a neighborhood of $(0, y)$. Let $x = (x_i)_{i\le p}$ where $x_i \in \ell^2_{n(i)}$. Then $N_{p+1}(x, y) = N^0(z)$ where $Z = (\|x_1\|, \cdots, \|x_p\|, \|y\|, 0, \cdots)$. Since $\|x\|^2 = \sum_{i\le n} \|x_i\|^2 \le \varepsilon^2\|y\|^2$, we have $N^0(z) = \|y\|$ from (3.3).

We now check (4.2) for $r = p + 1$. We have

$$\gamma_{p+1}\{(x, y);\ N_{p+1}(x, y) \le a(p+1)\}$$
$$\ge \int_{b/2\le\|y\|\le a(p+1)} \gamma_p\{x;\ N_{p+1}(x, y) \le a(p+1)\}\, d\nu(y).$$

For $b/2 \leq \|y\| \leq a(p+1)$, we have

$$\gamma_p\{x;\ N_{p+1}(x, y) \leq a(p+1)\} \geq \gamma_p\{x;\ \|x\| \leq \varepsilon b/2\} \geq 7\xi_{q(p+1)}.$$

So (4.6) implies (4.2) for $r = p + 1$. It follows from (3.4) and the definition of N_{p+1} that $N_{p+1}(x, y) \geq \sup(N_p(x), \|y\|)$. Hence (4.3) follows by induction hypothesis for $r \leq p$ and from (4.7) for $r = p + 1$. The construction is completed.

PROOF OF THEOREM A. We identify H to the Hilbertian sum $\oplus_{p=1}^{\infty} \ell^2_{n(p)}$. Under this identification, we can write $x = (x_p)$ with $x_p \in \ell^2_{n(p)}$. Let $N(x) = N^0(x^0)$, where $x^0 = (\|x_p\|)_p$. Then (a) and (b) are consequences of (3.1) and (3.2).

Since $\|Y_p(\omega)\| \leq 2^{-p}$ for each p, it follows that $X_p = \sum_{i\leq p} Y_i$ converges a.e. to a random variable X with $\|X(\omega)\| \leq 2$. Since the variables Y_p are independent, the Gaussian measure on H with the same covariance as X identifies with the limit γ of the γ_p. This limit is the product measure when each $\ell_{n(p)}$ is provided with the Gaussian measure ν_p having the same covariance as Y_p.

For each r and $p \geq r$, we have $\gamma_p\{x \in H;\ N_p(x) \leq a(r)\} > 2\xi_{q(r)}$. For each x, we have $N(x) = \lim_p N_p(x)$. For $\eta > 0$,

$$\{x;\ N(x) < a(r) + \eta\} \subset \limsup\{x;\ N_p(x) < a(r) + \eta\}.$$

So, $\gamma\{x;\ N(x) < a(r) + \eta\} \geq 2\xi_{q(r)}$ and $\gamma\{x;\ N(x) \leq a(r)\} \geq 2\xi_{q(r)}$. Moreover, using (3.3) and (4.7) we have

$$P\{N(q(r)^{-1/2} \textstyle\sum_{i\leq q(r)} X^i) \leq a(r)\} \leq P\{N_r(q(r)^{-1/2} \textstyle\sum_{i\leq q(r)} Y^i_r) \leq a(r)\} \leq \xi_{q(r)}.$$

This proves (c) and finishes the proof.

The basic idea of the above proof is quite simple. To make it clearer, we will explain why the norm N fails condition (*) without using the theorem of Kuelbs and Kurtz. The measure ν_{p+1} on $\ell^2_{n(p+1)}$ which was chosen at the step $p + 1$ is in fact extremely concentrated around the sphere S of radius $a(p + 1)$ and this degree of concentration can be chosen independently of $a(p + 1)$. The sphere of $\ell^2_{m(p+1)}$ of radius $a(p + 1)$ contains the points $(0, y)$ with $\|y\| = a(p + 1)$. But it is also flat at these points in the direction of $\ell^2_{m(p)}$, and in fact it contains $B(0, a(p + 1)\varepsilon/10) \times S$ where $B(0, a(p + 1)\varepsilon/10)$ is the ball in $\ell^2_{m(p)}$. It then follows that there is a very narrow annulus A_{p+1},

$$A_{p+1} = \{x;\ a(p+1) - \delta_p \leq N_{p+1}(x) \leq a(p+1) + \delta_p\},$$

which measure is of the order of $\gamma_p(B(0, a(p + 1)\varepsilon/10))$, but the width of A_{p+1} can be arbitrarily small. On the further steps of the construction, the measures γ_q, $q \geq p$ are close enough to γ_{p+1} so that

$$\gamma_q\{x \in \ell^2_{n(q)};\quad a(p+1) - \delta_p \leq N_q(x) \leq a(p+1) + \delta_p\}$$

is very close to $\gamma_{p+1}(A_{p+1})$. This is achieved by taking the sequence $a(p)$ decreasing fast enough.

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