The Annals of Probability
1984, Vol. 12, No. 3, 703–713

SYNONYMITY, GENERALIZED MARTINGALES, AND SUBFILTRATIONS

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Aldous recently introduced the notion of synonymity of stochastic processes, a notion of equivalence for processes on a stochastic basis which generalizes the notion of "having the same distribution". We show that generalized martingale properties, such as the semimartingale property, are preserved under synonymity, and that synonymous semimartingales have decompositions with the same distribution law. A variation of our method yields a relatively elementary proof of the theorem of Stricker that semimartingale remains a semimartingale with respect to any subfiltration to which it is adapted.

1. Introduction. Let $\mathbf{\Omega} = (\Omega, \mathbf{F}, P, \mathbf{F}_t)_{t\in\mathbf{R}+}$ be a stochastic base, i.e. a complete probability space endowed with a filtration $(\mathbf{F}_t)_{t\in\mathbf{R}+}$—an increasing, right continuous family of σ-fields such that $\mathbf{F}_0$ contains all nullsets of P. We regard a stochastic process X on such a space $\mathbf{\Omega}$ as a family $(X_t)_{t\in\mathbf{R}+}$ together with the underlying filtration: thus we write $X = (X_t, \mathbf{F}_t)_{t\in\mathbf{R}+}$. Aldous [1] introduced the relation "X and Y are synonymous" as a notion of "sameness" for stochastic processes taken in this sense. Synonymity is similar to having the same probability law, but takes into account part of the relation between the random variables and the filtration.

DEFINITION 1.1. Let $(X_t, \mathbf{F}_t)_{t\in\mathbf{R}+}$ be a stochastic process on a stochastic base Ω and let $(Y_t, \mathbf{G}_t)_{t\in\mathbf{R}+}$ be another process on a possibly different different base. We say $X \equiv_1 Y$ ("X and Y are synonymous") iff for any $n \in N$, $t_1, \cdots, t_n, u_1, \cdots, u_n \geq 0$, and $\phi, \phi_1, \cdots, \phi_n$ bounded Borel functions $\mathbf{R}^n \to \mathbf{R}$,

$$E[\phi(E[\phi_1(X_{u_1}, \cdots, X_{u_n}) \mid \mathbf{F}_{t_1}], \cdots, E[\phi_n(X_{u_1}, \cdots, X_{u_n}) \mid \mathbf{F}_{t_n}])] = E[\phi(E[\phi_1(Y_{u_1}, \cdots, Y_{u_n}) \mid \mathbf{G}_{t_1}], \cdots, E[\phi_n(Y_{u_1}, \cdots, Y_{u_n}) \mid \mathbf{G}_{t_n}])].$$

We have elected to follow the notation of Hoover and Keisler [4] in denoting synonymity by $\equiv_1$. As is also done in that paper, we will write $X \equiv_0 Y$ to indicate that X and Y have the same probability law.

The definition of synonymity needs no assumptions about the sample path properties of the processes involved. But when X is an rcll process (one whose sample paths are right continuous with left limits), there is a process associated with X whose probability law determines the synonymity type of X.

Received February 1983; revised August 1983.

[1] The author is an NSERC Canada University Research Fellow. This research was supported by an NSERC operating grant.

AMS 1980 *subject classifications.* Primary 60G07, 60G48.

Key words and phrases. Synonymity, semimartingale, local martingale, subfiltration.

For each n let C_n be a countable set of continuous functions $\mathbf{R}^n \to [0, 1]$ sufficient to generate the Borel functions.

DEFINITION 1.2. Let X be a stochastic process. m_1X is the $\mathbf{R}^\infty$-valued martingale given by

$$m_1X_t = (E[\phi(X_{u_1}, \cdots, X_{u_n}) \mid \mathbf{F}_t]: \phi \in C_n, u_1, \cdots, u_n \in Q, n \in N). \quad \Box$$

PROPOSITION 1.3 ([4], Theorem 2.18). *Let X and Y be rcll processes. Then*

$$X \equiv_1 Y \quad \textit{iff} \quad m_1X \equiv_0 m_1Y. \quad \Box$$

This follows easily by the right continuity of the sample paths of X and Y, and the dominated convergence theorem.

We have assumed that X is real valued, but the construction of m_1X will go through just as well for an rcll process X taking values in any Polish (complete separable metric) space. Thus we may proceed to define $m_2X = m_1(m_1X)$ and in general

$$m_{n+1}X = m_1(m_nX).$$

Let us say

$$X \equiv_n Y \quad \text{iff} \quad m_nX \equiv_0 m_nY,$$

and

$$X \equiv Y \quad \text{iff} \quad X \equiv_n Y \quad \text{for all} \quad n.$$

The relations $\equiv_n$ and $\equiv$ can be defined for processes which are not rcll, but it is not quite as simple.

In [4], Hoover and Keisler show that the relations $\equiv_n$ grow strictly stronger as n increases, and give constructions which can be used to show that on certain very rich spaces almost all properties of stochastic processes must be preserved by the $\equiv$ relation (by "preserved" we mean that if $X \equiv Y$ and X has a property P then Y must have it also). We believe that the step from $\equiv_0$ to $\equiv_1$ is the most important of the infinitely many steps from $\equiv_0$ to $\equiv$ and that most of the more interesting and more concrete properties of stochastic processes must be preserved by synonymity. The main result of this paper verifies that this is indeed true of generalized martingale properties.

In the sequel, we will write just $(\mathbf{F}_t)$ instead of $(\mathbf{F}_t)_{t\in\mathbf{R}+}$ and we will write (m_1X_t) to indicate the natural filtration of m_1X. We will also say that a random variable or process is X- (or m_1X-) measureable to indicate that it is a measureable function of X (or m_1X).

In this paper we will show:

(1) That for an rcll process X the optional, predictable, dual optional, and dual predictable projections are m_1X-measureable;

(2) That the law of m_1X determines whether X is a semimartingale, special semimartingale, or local martingale;

(3) That for any semimartingale X one can always find a decomposition which is m_1X-measureable; when X is special this may be taken to be the canonical decomposition.

(4) We will use a variant of the characterization of semimartingales used to prove (2) and (3) in order to give a more direct proof of Stricker's theorem that any semimartingale $X = (X_t, \mathbf{F}_t)_{t\in\mathbf{R}+}$ remains a semimartingale with respect to any subfiltration $(\mathbf{G}_t)$ of $(\mathbf{F}_t)$ with respect to which X remains adapted. This proof is elementary to the extent that the only advanced results it uses are the facts that every quasimartingale is a semimartingale and that for any semimartingale

$$\sum_{i<nt} (X_{i+1/n} - X_{i/n})^2 \to [X, X]_t$$

in probability. The fundamental ideas are about the same as Stricker's, but use of Girsanov's Theorem for semimartingales is avoided.

2. Projections.

This section contains some technical results about synonymity and projections of processes which will come into our main result.

DEFINITION 2.1. The total variation process of an rcll process A is the increasing real valued process $\text{var}(A)$ given by

$$\text{var}(A)_t = \lim_{n\to\infty} \sum_{i<nt} |A_{(i+1)/n} - A_{i/n}|. \quad \square$$

LEMMA 2.2. *Let A and A' be processes of integrable variation.*

(a) *If A' is optional, then A' is the dual optional projection of A iff $E[A_0 \mid \mathbf{F}_0] = A'_0$ and for every bounded martingale M,*

$$E\left[\int_0^t M\, dA\right] = E\left[\int_0^t M\, dA'\right].$$

(b) *If A' is predictable, then A' is the dual predictable projection of A iff $E[A_0 \mid \mathbf{F}_0] = A'_0$ and for every bounded martingale M,*

$$E\left[\int_0^t M_{s-}\, dA_s\right] = E\left[\int_0^t M_{s-}\, dA'_s\right].$$

PROOF. (a) ($\Longrightarrow$) is trivial because martingales are optional. ($\Longleftarrow$) It suffices to show that

$$E\left[\int {}^o f\, dA\right] = E\left[\int {}^o f\, dA'\right]$$

whenever f is the characteristic function of a measureable rectangle $[s, t] \times F$,

$s, t \in \mathbf{R}+$, $F \in \mathbf{F}$. Let M be the martingale generated by F. Then ${}^o f = I_{[s,t]}M$, so

$$E\left[\int {}^o f \, dA\right] = E\left[\int_0^t M \, dA\right] - E\left[\int_0^s M \, dA\right]$$

$$= E\left[\int_0^t M \, dA'\right] - E\left[\int_0^s M \, dA'\right] \quad \text{(by hypothesis)}$$

$$= E\left[\int {}^o f \, dA'\right].$$

The proof of (b) is similar, but the process M_{s-} replaces M. □

PROPOSITION 2.3. (a) *Let Y be a bounded process which is X-measureable. Then the optional and predictable projections of Y with respect to $(\mathbf{F}_t)$ and $(m_1 X_t)$ are respectively the same.*

(b) *If A is an rcll $\mathbf{F}(X)$-measureable process of integrable variation, then the dual optional and dual predictable projections with respect to $(\mathbf{F}_t)$ and $(m_1 X_t)$ are respectively the same.*

PROOF. Part (a) is a simple observation which follows by examination of the proof of the existence of these projections (Dellacherie [2], Theorem V-14). (b) is almost as easy. Consider the case of dual optional projections, letting A^o denote that with respect to $(\mathbf{F}_t)$, A' that with respect to $(m_1 X_t)$. We may assume that $A_0 = 0$, since

$$A_t^o = E[A \mid \mathbf{F}_0] + (A_t - A_0)^o, \quad \text{and} \quad A_t' = E[A \mid \mathbf{F}_0] + (A_t - A_0)'.$$

By Lemma 2.2, it will suffice to show that for any bounded $(\mathbf{F}_t)$-martingale M and $t \in \mathbf{R}+$,

$$E\left[\int_0^t M_s \, dA_s'\right] = E\left[\int_0^t M_s \, dA_s\right].$$

Let M' denote the $(m_1 X_t)$-martingale $M_s' = E[M_\infty \mid m_1 X_s]$. By right continuity of M_s and the dominated convergence theorem,

$$E\left[\int_0^t M_s \, dA_s'\right] = \lim_{n\to\infty} E[\textstyle\sum_{i<nt} M_{(i+1/n)}(A_{i+1/n)} - A_{i/n})]$$

$$= \lim_{n\to\infty} E[\textstyle\sum_{i<nt} M_{(i+1/n)} E[A_{(i+1/n)} - A_{i/n} \mid F_{(i+1/n)}]]$$

$$= \lim_{n\to\infty} E[\textstyle\sum_{i<nt} E[M_{(i+1/n)} \mid m_1 X_{(i+1/n)}](A_{(i+1/n)} - A_{i/n})],$$

since $E[A_{(i+1/n)} - A_{i/n} \mid F_{(i+1/n)}]$ and $A_{(i+1/n)} - A_{i/n}$ are $m_1 X$-measureable. But $M'_{(i+1/n)} = E[M_{(i+1/n)} \mid m_1 X_{(i+1/n)}]$, so by right continuity of m and dominated convergence again, this is

$$= E\left[\int_0^t M_s' \, dA_s\right] = E\left[\int_0^t M_s' \, dA_s'\right],$$

by definition of A'. Retracing one's steps with A' in place of A shows that

$$E\left[\int_0^t M_s\,dA_s\right] = E\left[\int_0^t M_s'\,dA_s'\right],$$

which is what we had to prove. A similar argument using the left continuous modification M_{s-} of M proves the corresponding fact about dual predictable projections (cf. Rao [6] or Doleans [3]). □

We apply this fact about dual predictable projections to give in a spirit similar to 2.3 an absolute version of the theorem that a process is of locally integrable variation iff it has a generalized dual predictable projection (Meyer [5], Theorem IV-12). If A is a process of finite variation, then we let A^+ and A^- be the unique nonnegative increasing processes with $A = A^+ - A^-$, and $\mathrm{var}(A) = A^+ + A^-$.

LEMMA 2.4. *A finite variation process A is of locally integrable variation iff*

$$\lim_{N\to\infty}(\mathrm{var}(A) \wedge N)^p(t, \omega)$$

exists and is finite for almost all (t, ω). If this is the case, then

$$A^c = \lim_{N\to\infty}(A^+ \wedge N)^p - (A^- \wedge N)^p \tag{2.1}$$

has an rcll version which is the unique predictable process of finite variation such that $A - A^c$ is a local martingale null at zero.

PROOF. ($\Rightarrow$) Let T be a stopping time such that $\mathrm{var}(A)_T$ is integrable. Then

$$(\mathrm{var}(A^T))^p = \lim_{N\to\infty}(\mathrm{var}(A)^T \wedge N)^p = \lim_{N\to\infty}(\mathrm{var}(A \wedge N)^p)^T$$

exists and is a finite process, and

$$\begin{aligned}(A^c)^T = (A^T)^p &= \lim_{N\to\infty}((A^+)^T \wedge N)^p - ((A^-)^T \wedge N)^p \\ &= \lim_{N\to\infty}((A^+ \wedge N)^p)^T - ((A^- \wedge N)^p)^T.\end{aligned} \tag{2.2}$$

If $\mathrm{var}(A)$ is locally integrable, we let T tend to infinity and see that A^c is finite a.s. and has an rcll version given by (2.2). Since $A^T - (A^T)^p$ is a martingale, we also find that $A - A^c$ is a local martingale when A^c is as in (2.1).

($\Leftarrow$) Let us define the predictable, increasing process $(\mathrm{var}(A))^c$ by

$$(\mathrm{var}(A))^c = \lim_{N\to\infty}(\mathrm{var}(A) \wedge N)^p \tag{2.3}$$

(rcll version). By Meyer [5], Theorem IV-12, if $(\mathrm{var}(A))^c$ is finite, then it is locally integrable. It will suffice, then, to show that whenever T is a stopping time, if $(\mathrm{var}(A))^c_T$ is integrable, then so is $\mathrm{var}(A)_T$. But

$$\begin{aligned}E[\mathrm{var}(A)_T] &= \lim_{N\to\infty}E\left[\int_0^T d(\mathrm{var}(A) \wedge N)_t\right] \\ &= \lim_{N\to\infty}E\left[\int_0^T d(\mathrm{var}(A) \wedge N)_t^p\right] = E[\mathrm{var}(A)^c_T].\end{aligned}$$

The second equality follows since $I_{[0,T]}$ is predictable, the others by monotone convergence. □

COROLLARY 2.5. *Let A be an $\mathbf{F}(X)$-measureable process. Then A is of $(\mathbf{F}_t)$-locally integrable variation iff A is of (m_1X_t)-locally integrable variation. When A is of locally integrable variation with respect to either filtration, and A^c is the process given by* (2.1), *$A - A^c$ is both an $(\mathbf{F}_t)$ and an (m_1X_t)-local martingale.*

PROOF. The process $\mathrm{var}(A)^c$ given by (2.3) is the same whether defined with respect to $(\mathbf{F}_t)$ or (m_1X_t). By Lemma 2.4, this proves the first part. A^c is also the same whether defined with respect to $(\mathbf{F}_t)$ or (m_1X_t), so the second part also follows. □

3. Synonymity and semimartingales.

DEFINITION 3.1. Let X be an rcll process.

(i) For each $n \in \mathbf{N}$, the process X^n is given by

$$X_t^n = \textstyle\sum_{i<nt} E[X_{(i+1/n)} - X_{i/n} \mid \mathbf{F}_{i/n}].$$

Thus

$$\mathrm{var}(X^n)_t = \textstyle\sum_{i<nt} | E[X_{(i+1/n)} - X_{i/n} \mid \mathbf{F}_{i/n}] |.$$

(ii) An integrable process X is a quasimartingale iff there exists a constant C such that whenever $0 \le t_0 < t_1 < \cdots < t_n$,

$$\textstyle\sum_{i\le n} | E[X_{t_{i+1}} - X_{t_i} \mid \mathbf{F}_{t_i}] | \le C. \quad \square$$

LEMMA 3.2. *A uniformly integrable process X is a quasimartingale iff $E[\mathrm{var}(X^n)_t]$ is bounded uniformly in n and t.*

PROOF. The forward direction is obvious. To prove the converse, let $0 \le t_0 < \cdots < t_k$ and $\varepsilon > 0$ be given. By uniform integrability and right continuity of X, choose n and $i_0 < \cdots < i_k$ such that for $j \le k$, $E[| X_{t_j} - X_{i_j/n} |] < \varepsilon/k$. Then

$$\begin{aligned} &E[\textstyle\sum | E[X_{t_{i+1}} - X_{t_i} \mid \mathbf{F}_{t_i}] |] \\ &\qquad \le E[\textstyle\sum | E[X_{i_{j+1}/n} - X_{i_j/n} \mid \mathbf{F}_{i_j/n}] |] + 2\varepsilon \\ &\qquad \le E[\textstyle\sum_j E[\sum_{i_j \le i < i_{j+1}} | E[X_{(i+1)/n} - X_{i/n} \mid \mathbf{F}_{i/n}] | \mid \mathbf{F}_{i_j/n}]] + 2\varepsilon \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Jensen's inequality)} \\ &\qquad = E[\textstyle\sum_{i<i_k} | E[X_{(i+1)/n} - X_{i/n} \mid \mathbf{F}_{i/n}] |] + 2\varepsilon \le C + 2\varepsilon \end{aligned}$$

for the constant C that bounds $\mathrm{var}(X^n)_t$. □

The following characterization of bounded semimartingales will make it easy to show that the semimartingale property of X is determined by the law of m_1X.

THEOREM 3.3. *Let X be a bounded rcll process. X is a semimartingale iff for every t there is a strictly positive random variable N such that, for all $n \in \mathbf{N}$,*

$$E[N \operatorname{var}(X^n)_t] \le 1.$$

PROOF. ($\Rightarrow$) Observe first of all that since X is a bounded semimartingale, it is a local quasimartingale. Let T_m, $m \in \mathbf{N}$, be a sequence of stopping times such that $T_m \to \infty$ and X^{T_m} is a quasimartingale. Choose an increasing sequence of constants K_m, $m \in \mathbf{N}$, such that for each m,

$$E[\operatorname{var}((X^{T_m})^n)] + 1 \le K_m.$$

Define $N = N_t$ by

$$N = (K_m + 2C)^{-1}2^{-m} \quad \text{on} \quad T_{m-1} < t \le T_m \quad (\text{with } T_0 = 0),$$

where C bounds $|X|$, and let T_m^n denote the stopping time

$$T_m^n = \inf\{i/n\colon i/n \ge T_m\}. \tag{3.1}$$

Then, for each m,

$$\begin{aligned} E[\operatorname{var}((X^{T_m})^n)_t - \operatorname{var}((X^n)^{T_m})_t] &\le E[\textstyle\sum_{T_m^n=(i+1)/n} E[\,|X_{T_m} - X_{T_m^n}|\,|\,\mathbf{F}_{i/n}]] \\ &\le E[\,|X_{T_m} - X_{T_m^n}|\,] \to 0 \quad \text{in} \quad L^1, \end{aligned} \tag{3.2}$$

by right continuity of X, since $|X_{T_m} - X_{T_m^n}|$ is bounded by $2C$. Now,

$$\begin{aligned} E[N_t \operatorname{var}(X^n)_t] &\le \textstyle\sum_m E[I_{(T_{m-1},T_m]}(t) N_t \operatorname{var}(X^n)_{T_m}] \\ &\le \textstyle\sum_m E[I_{(T_{m-1},T_m]}(t) N_t (\operatorname{var}((X^{T_m})^n)_\infty + 2C)] \\ &\le \textstyle\sum_m (K_m + 2C)^{-1}2^{-m}(K_m + 2C) \le 1. \end{aligned}$$

($\Leftarrow$) Given t, choose N satisfying the hypothesis and let $N_s = E[N \mid \mathbf{F}_s]$. Define stopping times T_m, $m \in \mathbf{N}$ by

$$T_m = \min(t, \inf\{s \mid 1/N_s \ge m\}). \tag{3.3}$$

Since N is positive, N_s is never zero, so $T_m \to \infty$. We claim that X^{T_m} is a quasimartingale. Since X is bounded and $T_m \le t$, by Lemma 3.2 it suffices to show that, for each m, $E[\operatorname{var}((X^{T_m})^n)_t]$ is bounded uniformly in n. By (3.2), to show this it suffices to show that $E[\operatorname{var}(X^n)_{T_m}]$ is bounded uniformly in n. But since $N_s > 1/m$ for $s < T_m$,

$$\begin{aligned} E[\operatorname{var}((X^n)_{T_m})_t] &\le m \textstyle\sum_{i<nT_m} E[N_{i/n} \,|\, E[X_{(i+1/n)} - X_{i/n} \mid \mathbf{F}_{i/n}]\,|\,] \\ &\le m \textstyle\sum_{i<nt} E[N_{i/n} \,|\, E[X_{(i+1/n)} - X_{i/n} \mid \mathbf{F}_{i/n}]\,|\,] \\ &= mE[N \operatorname{var}(X^n)_t] \le m. \quad \square \end{aligned}$$

COROLLARY 3.4. (i) *A process X is an $(\mathbf{F}_t)$-semimartingale iff it is an $(m_1 X_t)$-semimartingale. When either statement holds, X has a decomposition $X = A + M$*

where A is a process of finite variation and M is both an $(\mathbf{F}_t)$- and an (m_1X_t)-local martingale.

(ii) *X is an $(\mathbf{F}_t)$-special semimartingale iff it is an (m_1X_t)-special semimartingale, and when either statement holds, the canonical decomposition of X (that is, the unique one with A predictable and $A_0 = 0$) with respect to each filtration is the same.*

(iii) *X is an $(\mathbf{F}_t)$-local martingale iff it is an (m_1X_t)-local martingale.*

PROOF. We will begin by arguing that (i) holds when X is bounded and reduce the other cases to this case. Since $\mathrm{var}(X^m)$ is the same with respect to $(\mathbf{F}_t)$ and (m_1X_t), it follows immediately by Theorem 3.3 that if X is a bounded (m_1X_t)-semimartingale, it is also an $(\mathbf{F}_t)$-semimartingale. Furthermore, since $\mathrm{var}(X^m)$ is m_1X-measureable, for any positive random variable N, $E[N \mid m_1X]$ is positive and

$$E[N\,\mathrm{var}(X^m)_t] = E[E[N \mid m_1X]\mathrm{var}(X^m)_t],$$

so the converse also holds. Now suppose X is bounded (m_1X_t)-semimartingale. Let N be as in Theorem 3.3, let N_s be the (m_1X_t)-martingale generated by N, and let T_m, $m \in \mathbf{N}$, be as in (3.1). For each m, X^{T_m} is a quasimartingale (with respect to either $(\mathbf{F}_t)$ or (m_1X_t)). By Rao [7], its canonical compensator A^{T_m} with respect to $(\mathbf{F}_t)$ is given by

$$A_t^{T_m} = \lim_{n\to\infty}(X^{T_m})_t^n$$

in $\sigma(L^1, L^\infty)$ (the topology induced on L^1 by its duality with L^∞ = the weak topology on L^1). By (3.2)

$$\lim_{n\to\infty} \mid (X^{T_m})_t^n - (X^n)_t^{T_m} \mid = 0 \quad \text{in} \quad L^1.$$

Hence

$$A_t^{T_m} = \lim_{n\to\infty}(X^n)_t^{T_m} \quad \text{in} \quad \sigma(L^1, L^\infty).$$

The same argument shows that the canonical (m_1X_t)-compensator of X^{T_m} is this same limit, since X^n is the same with respect to both $(\mathbf{F}_t)$ and (m_1X_t). Thus X^{T_m} has the same canonical decomposition with respect to $(\mathbf{F}_t)$ and (m_1X_t). The same is true of X because the canonical compensator A of X is simply the unique process A such that for each m, A^{T_m} is the canonical compensator of X^{T_m}.

Having completed this long preliminary, we can prove the three statements of the theorem quite quickly. To prove (i), let V be the process

$$V_t = \textstyle\sum_{\{s\le t:\, |\Delta X_s|>1\}} \Delta X_s \quad (\text{including } \Delta X_0 = X_0).$$

Then $X - V$ has bounded jumps, and V is of finite variation. If U_m, $m \in N$, is the sequence of stopping times

$$U_m = \inf\{t: (X - V)_t \ge m\},$$

then for each m, $(X - V)^{U_m}$ is bounded and X-measureable. The foregoing proof works for any X-measureable process, hence, for each m, $(X - V)^{U_m}$ is an $(\mathbf{F}_t)$-semimartingale iff it is an (m_1X_t)-semimartingale, hence the same is true of X.

If $(X - V)^{U_m} = B^{U_m} + L^{U_m}$ is a simultaneous $(\mathbf{F}_t)$ and $(m_1 X_t)$-decomposition of $(X - V)^{U_m}$, then $X = (B + V) + L$ is a simultaneous decomposition of X.

(ii) X is a special semimartingale iff V is locally integrable. By Corollary 2.5, this property is the same with respect to both $(\mathbf{F}_t)$ and $(m_1 X_t)$, and the unique predictable process V^p such that $V - V^p$ is a local martingale is the same for each filtration. The proof for the bounded case shows that the process B in (i) is the canonical compensator of $X - V$, hence if $A = B + V^p - X_0$ and $M = X_0 + L + (V - V^p)$, then $X = A + M$ is the canonical decomposition of X, good for both filtrations.

(iii) This follows from (ii), because a local martingale is just a special semimartingale whose canonical compensator is the zero process. □

COROLLARY 3.5. *Suppose $X \equiv_1 Y$. Then if X is a semimartingale, so is Y, and there are decompositions $X = A + M$ and $Y = B + L$ such that $(A, M) \equiv_0 (B, L)$. If X is a special semimartingale, then so is Y, and if (A, M) is the canonical decomposition of X, then (B, L) can be taken to be the canonical decomposition of Y. If X is a local martingale, so is Y.*

PROOF. The proof of Corollary 3.4(i) gives a measureable function h such that when X is a semimartingale, $h(m_1 X) = (A, M)$ is a decomposition of X. To prove the first statement in the Corollary, just let $(B, L) = h(m_1 Y)$. Likewise, the proof of 3.4(ii) gives a measureable function g such that, when X and Y are special semimartingales, $g(m_1 X) = (A, M)$ and $g(m_1 Y) = (B, L)$ are the canonical decompositions of X and Y. If $X \equiv_1 Y$, then $m_1 X \equiv_0 m_1 Y$, and this proves the second statement. If X is a local martingale, then, by the foregoing, Y is a special semimartingale whose canonical compensator is the zero process, hence a local martingale. □

4. Proof of a theorem of Stricker.

By making an easy modification of our characterization of bounded semimartingales (Theorem 3.3) we get a proof of the main theorem of Stricker [8] (Corollary 4.2 below).

THEOREM 4.1. *Let X be a bounded process such that, for some subsequence n_m of $\mathbf{N}$, there is a constant C such that for all m*

$$\sum (X_{(i+1/n_m)} - X_{i/n_m})^2 \leq C.$$

Then X is a semimartingale iff for each t there is a positive, bounded random variable N such that for some constant K

$$E[\textstyle\sum_{i<nt} | E[N(X_{(i+1/n)} - X_{i/n}) \mid \mathbf{F}_{i/n}] |] \leq K.$$

PROOF. Given t, choose N, and, as before, let $N_s = E[N \mid \mathbf{F}_s]$. Observe that

$$E[N(X_{(i+1/n)} - X_{i/n}) \mid \mathbf{F}_{i/n}].$$
$$= E[N_{i/n}(X_{(i+1/n)} - X_{i/n}) \mid \mathbf{F}_{i/n}] + E[(N_{(i+1/n)} - N_{i/n})(X_{(i+1/n)} - X_{i/n}) \mid \mathbf{F}_{i/n}].$$

Hence

$$
\begin{aligned}
E[\textstyle\sum_{i<nt} N \mid E[(X_{(i+1/n)} - X_{i/n}) \mid \mathbf{F}_{i/n}] \mid] \\
&- E[\textstyle\sum_{i<nt} \mid E[(N_{(i+1/n)} - N_{i/n})(X_{(i+1/n)} - X_{i/n}) \mid \mathbf{F}_{i/n}] \mid] \\
&\leq E[\textstyle\sum_{i<nt} \mid E[N(X_{(i+1/n)} - X_{i/n}) \mid \mathbf{F}_{i/n}] \mid] \\
&\leq E[\textstyle\sum_{i<nt} N \mid E[(X_{(i+1/n)} - X_{i/n}) \mid \mathbf{F}_{i/n}] \mid] \\
&\quad + E[\textstyle\sum_{i<nt} \mid E[(N_{(i+1/n)} - N_{i/n})(X_{(i+1/n)} - X_{i/n}) \mid \mathbf{F}_{i/n}] \mid \;].
\end{aligned}
$$

Thus, by Theorem 3.3, to prove this Theorem, it suffices to bound the last term above. But

$$
\begin{aligned}
E[\textstyle\sum_{i<nt} \mid E[(N_{(i+1/n)} - N_{i/n})(X_{(i+1/n)} - X_{i/n}) \mid \mathbf{F}_{i/n}] \mid] \\
&\leq \textstyle\sum_{i<nt} E[\mid (N_{(i+1/n)} - N_{i/n})(X_{(i+1/n)} - X_{i/n}) \mid] \\
&\leq (\textstyle\sum_{i<nt} E[(X_{(i+1/n)} - X_{i/n})^2])^{1/2}(\textstyle\sum_{i<nt} E[(N_{(i+1/n)} - N_{i/n})^2])^{1/2} \\
&\leq C^{1/2} \| N \|_2 < \infty,
\end{aligned}
$$

the first two inequalities following by Jensen and Cauchy-Schwartz, respectively, and the last inequality holding when $n = n_m$ is an element of the subsequence in the hypothesis. But because X and N are bounded and X is right continuous, the bound for arbitrarily large n implies it for all n (the proof is similar to that of Lemma 3.2). □

COROLLARY 4.2 (Stricker's Theorem). *If X is an $(\mathbf{F}_t)$-semimartingale, and $(\mathbf{G}_t)$ is a subfiltration of $(\mathbf{F}_t)$ such that X is $(\mathbf{G}_t)$-adapted, then X is a $(\mathbf{G}_t)$-semimartingale.*

PROOF. Suppose that X satisfies the hypothesis of Theorem 4.1. Observe that, by Jensen's inequality, for any n, N and t,

$$
\begin{aligned}
E[\textstyle\sum_{i<nt} \mid E[N(X_{(i+1/n)} - X_{i/n}) \mid \mathbf{G}_{i/n}] \mid] \\
&\leq E[\textstyle\sum_{i<nt} E[\mid E[N(X_{(i+1/n)} - X_{i/n}) \mid \mathbf{F}_{i/n}] \mid \mid \mathbf{G}_{i/n}]] \\
&= E[\textstyle\sum_{i<nt} \mid E[N(X_{(i+1/n)} - X_{i/n}) \mid \mathbf{F}_{i/n}] \mid].
\end{aligned}
$$

Thus, by Theorem 4.1, if X is an $(\mathbf{F}_t)$-semimartingale, then it is also a $(\mathbf{G}_t)$-semimartingale. The remainder of the proof, then, consists merely in reducing the general case to this one. First, by replacing X with $X - V$, where V is as in Theorem 3.3, we may take X_0 and the jumps of X to be uniformly bounded by 1. If we let

$$T_n = \inf\{t: \mid X_t \mid + [X, X]_t > n\},$$

then both X^{T_n} and $[X^{T_n}, X^{T_n}]$ are bounded by $n + 1$. Since the T_n's are all $\mathbf{F}(X)$-measureable, and X is a semimartingale iff each X^{T_n} is, we may assume that X and $[X, X]$ are bounded. Now by Meyer [5], Theorem VI-4,

$$[X, X]_\infty = \lim_{n\to\infty} \textstyle\sum (X_{(i+1/n)} - X_{i/n})^2 \quad \text{in probability.}$$

Choose a subsequence n_m of $\mathbf{N}$ such that for each m,

$$P(|\sum (X_{(i+1/n_m)} - X_{i/n_m})^2 - [X, X]_\infty| > 2^{-m}) < 2^{-m}, \tag{4.1}$$

and let

$$U_k = \inf\{t\colon \text{for some } m, \textstyle\sum_{i<nt} (X_{(i+1/n_m)} - X_{i/n_m})^2 > k\}.$$

Then, by (4.1), $U_k \to \infty$ a.s., and

$$\sum (X^{U_k}_{(i+1/n_m)} - X^{U_k}_{i/n_m})^2 \le k + 2\,\|X\|^2_\infty.$$

Thus X^{U_k} satisfies the hypothesis of Theorem 4.1, and hence the Corollary holds for each X^{U_k}. But, X is a semimartingale iff for each k, X^{U_k} is a semimartingale, so the result follows for X. □

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