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STRONG APPROXIMATION OF EXTENDED RENEWAL PROCESSES

BY LAJOS HORVÁTH

Szeged University

We develop a strong approximation approach to extended multidimensional renewal theory. The consequences of this approximation are a Bahadur-Kiefer type representation of the renewal process in terms of partial sums, Strassen and Chung type laws of the iterated logarithm. We also give a characterization of the renewal process by four classes of deterministic curves in the sense of Révész (1982). We generalize our results to the case of non-independent and/or nonidentically distributed random vectors.

1. Introduction. Let $\mathbf{X} = (X^{(1)}, \cdots, X^{(d)})$, $\{\mathbf{X}_n, n \geq 1\}$ be a sequence of independent identically distributed random vectors (i.i.d.r.v's) in R^d, $d \geq 1$ with expectation $\boldsymbol{\mu}$, defined on some probability space $(\Omega, \mathscr{A}, P)$, and having finite νth moments:

(i) $E|X^{(i)}|^\nu < \infty$, $1 \leq i \leq d$, for some $\nu > 2$.

Let h: $R^d \to R$ be a function which satisfies the following regularity conditions:

(ii) h is homogeneous of degree one, i.e. for all $\mathbf{x} \in R^d$, $\lambda \geq 0$, $h(\lambda\mathbf{x}) = \lambda h(\mathbf{x})$,

(iii) $h(\boldsymbol{\mu}) > 0$

(iv) h has continuous partial derivatives of the second order in a neighbourhood of $\boldsymbol{\mu}$.

The partial sums of the first $[t]$ random vectors are denoted by $\mathbf{S}(t) = \sum_{i=1}^{[t]} \mathbf{x}_i$, where $[t]$ is the integer part of t. Set $\mathbf{S}(0) = 0$. We define the extended renewal process $\{N(t), t \geq 0\}$ by

$$N(t) = \min\{k\colon h(\mathbf{S}(k)) > tk^p\}, \quad 0 \leq p < 1,$$

where $N(t) = \infty$ if no such k exists. The continuity of h in $\boldsymbol{\mu}$ and Kolmogorov's law of the large numbers imply that

$$\lim_{n\to\infty} h\,((1/n)\,\mathbf{S}(n)) = h(\boldsymbol{\mu}) \quad \text{a.s.,} \tag{1.1}$$

and by the condition (ii) we get that

$$\lim_{n\to\infty} h(n^{-p}\mathbf{S}(n)) = \infty \quad \text{a.s.,} \quad 0 \leq p < 1.$$

So we obtained that the extended renewal process is well defined with probability one, i.e., $P\{N(t) = \infty\} = 0$, $t \geq 0$.

Of course, (ii) and (iii) imply in general that $\boldsymbol{\mu} \neq (0, \cdots, 0)$. If h is any norm inducing the Euclidean topology in R^d, then the conditions (ii) and (iii) are automatically satisfied and (iv) usually places a condition on the expectation

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vector $\mu = E\mathbf{X}$ only. In particular, if

$$h(\mathbf{x}) = (\sum_{i=1}^{d} x_i^2)^{1/2} \quad \text{or} \quad h(\mathbf{x}) = \sum_{i=1}^{k} |x_i|, \; \mathbf{x} = (x_1, \cdots, x_d)$$

then (iv) is satisfied if and only if the components of μ are all different from 0. If $h(\mathbf{x}) = \max_{1 \le i \le d} |x_i|$, the L_∞ norm, we find that (iv) is satisfied if and only if the components of μ are different from each other and from 0. In these cases $N(t)$ denotes the instant when the d-dimensional random walk leaves for the first time the sphere of radius t about the point $\mathbf{0}$ for the norm h.

Instead of condition (ii) we could assume that h is homogeneous of degree α, i.e. for all $\mathbf{x} \in \mathbf{R}^d$, $\lambda \ge 0$, $h(\lambda \mathbf{x}) = \lambda^\alpha h(\mathbf{x})$, but this is of course nothing more than the case when h is homogeneous of degree one.

Up to the present time, the case $p > 0$ has not been discussed when $d > 1$. Farrell (1966) proved the ordinary central limit theorem for the process $N(t)$, ($d \ge 1, p = 0$) and Bickel and Yahav (1965) generalized some elementary renewal theorems for planar walks ($d = 2, p = 0$). A functional central limit theorem for the process $N(t)$ ($d \ge 1, p = 0$) was proved by Kennedy (1971) assuming instead of (iv) that h has continuous partial derivatives of the first order *on the whole* R^d. An elementary calculation shows that Kennedy's conditions imply that $h(\mathbf{x}) = \sum_{i=1}^{d} c^{(i)} x_i$, $\mathbf{x} = (x_1, \cdots, x_d)$, where $c^{(1)}, \cdots, c^{(d)}$ are constants. Hence Kennedy's h function cannot cover any norm in R^d. When $d = 1$ and $\mu > 0$ then it follows from (iv) that $h(x) = cx$, where c is a positive constant. This case ($h(x) = x$, $0 \le p < 1$) was considered by Gut (1972, 1975) who proved functional central limit theorems for extended renewal processes of this special type.

The main result of this paper is a strong approximation of the process

$$Z(t) = (t/h(\mu))^{1/q} - N(t), \quad q = 1 - p.$$

This result will be a consequence of the best approximation of the function of partial sums

$$Y(t) = h(\mathbf{S}(t)) - th(\mu).$$

The extended renewal process $N(t)$ is based on the weighted partial sums $k^{-p}\mathbf{S}(k)$ and our approach to the approximation of Z requires the study of the asymptotic properties of the process $\hat{Y}$ defined by

$$\hat{Y}(t) = (h(\mathbf{S}(t))/t^p h(\mu))^{1/q} - t.$$

We approximate Z, Y and $\hat{Y}$ by three Gaussian processes which are constructed from the *same* Wiener process by time-transformations and prove that the rates of these approximations cannot be improved when $\nu \ge 4$. Consequences of the main theorem are functional and Chung-type laws of iterated logarithm, Bahadur-Kiefer type representation of the extended renewal process in terms of partial sums, and the characterization of the renewal process by four classes of deterministic curves in the sense of Révész (1982).

Throughout this paper $\|\mathbf{x}\| = \max_{1 \le i \le d} |x_i|$, $\mathbf{x} = (x_1, \cdots, x_d)$ denotes the maximum norm in R^d. The transpose of a row-vector $\mathbf{x}$ is a column-vector denoted by $\mathbf{x}^T$. We use the abbreviations

$$\xi_n =_{\text{a.s.}} o(a(n))$$

and

$$\xi_n =_{\text{a.s.}} O(b(n))$$

where $\{\xi_n, a(n), b(n), n \geq 1\}$ are sequences of random variables, to mean that

$$\lim_{n\to\infty} \xi_n / a(n) = 0 \quad \text{a.s.}$$

and

$$P\{\limsup_{n\to\infty} |\xi_n| / |b(n)| = \infty\} = 0$$

respectively. We say that $a(n)$ is not greater than $b(n)$ almost surely ($a(n) \leq_{\text{a.s.}} b(n)$), if for almost all $\omega \in \Omega$ there is an integer $n_0 = n_0(\omega)$ such that $a(n) \leq b(n)$ for $n \geq n_0$.

2. Strong approximation of the function of partial sums. The function h has continuous derivations in μ, and setting

$$\nabla h(\mu) = (\partial h/\partial X_1, \cdots, \partial h/\partial X_d)|_{\mathbf{x}=\mu},$$

the random variable $\nabla h(\mu)\mathbf{X}^T$ has bounded νth moment under condition (i). The second moment of $\nabla h(\mu)(\mathbf{X} - \mu)^T$ will be denoted by σ^2. The following theorem is a direct generalization of the celebrated results of Komlós, Major and Tusnády (1975, 1976) and Major (1976) and coincides with them when $d = 1$ and $h(x) = x$.

THEOREM 2.1. *If the underlying probability space* $(\Omega, \mathscr{A}, P)$ *is rich enough we can define a Wiener process* $\{W(t), t \geq 0\}$ *on it such that*

$$\sup_{0\leq t\leq nT} |Y(t) - \sigma W(t)| =_{\text{a.s.}} o(n^{1/\nu})$$

for any $T > 0$, *provided that conditions* (i) *and* (ii) *and* (iv) *are satisfied.*

PROOF. The random variables $\{\nabla h(\mu)(\mathbf{X}_n - \mu)^T, n \geq 1\}$ are i.i.d. real random variables having expectation zero and finite νth moment. It follows from Komlós, Major and Tusnády (1975, 1976) and Major (1976) (cf. Theorem 2.6.3 in Csörgő and Révész, 1981) that there is a Wiener process $\{W(t), t \geq 0\}$ such that

$$\sup_{0\leq t\leq nT} \left| \sum_{i=1}^{[t]} \nabla h(\mu)(\mathbf{X}_i - \mu)^T - \sigma W(t) \right| =_{\text{a.s.}} o(n^{1/\nu}). \tag{2.1}$$

Let δ denote a positive number such that the sphere of radius δ about μ belongs to the neighborhood of μ defined in (iv). Using the Kolmogorov's law of large numbers we obtain that for almost all $\omega \in \Omega$ there is an integer $k_0 = k_0(\omega)$ such that

$$\| k^{-1}\mathbf{S}(k) - \mu \| < \delta/2, \tag{2.2}$$

if $k \geq k_0$. We divide the interval $[0, nt]$ into two parts with the random point k_0.

A two-term Taylor expansion shows that

$$\begin{aligned}
&\sup_{0\le t\le nt} | Y(t) - \sigma W(t) | \\
&\quad \le \sup_{0\le t\le k_0} | h(\mathbf{S}(t)) - th(\boldsymbol{\mu}) + \sup_{0\le t\le k_0} \sigma | W(t) | \\
&\qquad + \sup_{k_0\le t\le nT} | \nabla h(\boldsymbol{\mu}) (\mathbf{S}(t) - th(\boldsymbol{\mu}))^T - \sigma W(t) | \\
&\qquad + \sup_{k_0\le t\le nt} | \nabla^2 h(\boldsymbol{\xi}) \frac{1}{t} (\mathbf{S}(t) - th(\boldsymbol{\mu}))^T (\mathbf{S}(t) - th(\boldsymbol{\mu}))^T | \\
&\quad = a_{\mathrm{l}n} + \cdots + a_{4n},
\end{aligned}$$

where $\boldsymbol{\xi} = \boldsymbol{\xi}(t)$ is a random point satisfying

$$\| \boldsymbol{\xi}(t) - \boldsymbol{\mu} \| \le \| t^{-1}\mathbf{S}(t) - \boldsymbol{\mu} \|, \quad \text{if } k_0 \le t \le Tn. \tag{2.3}$$

The random variable k_0 is finite almost surely and therefore we get

$$a_{\mathrm{l}n} \le \max_{1\le k\le k_0} | h(\mathbf{S}(k)) | + k_0 h(\boldsymbol{\mu}) =_{\mathrm{a.s.}} O(1)$$

and

$$a_{2n} = \sigma \sup_{0\le t\le k_0} | W(t) | =_{\mathrm{a.s.}} O(1).$$

The construction of W in (2.1) implies that

$$a_{3n} =_{\mathrm{a.s.}} o(n^{1/\nu}).$$

Using (2.2) and (2.3) we obtain that

$$\| \boldsymbol{\xi}(t) - \boldsymbol{\mu} \| \le \frac{\delta}{2} + \frac{1}{n}\left(\| \boldsymbol{\mu} \| + \frac{\delta}{2}\right), \quad \text{if } k_0 \le t \le nT.$$

If $n \ge (2/\delta) \| \boldsymbol{\mu} \| + 1$, then $\boldsymbol{\xi}$ is an element of the neighbourhood of $\boldsymbol{\mu}$ defined in (iv) and therefore

$$\max_{1\le i,j\le d} \sup_{k_0\le t\le nT} \left| \left. \frac{\partial h(\mathbf{x})}{\partial x_i \partial x_j} \right|_{\mathbf{x}=\boldsymbol{\xi}} \right| =_{\mathrm{a.s.}} O(1). \tag{2.4}$$

By the condition (i) we can use the law of the iterated logarithm and an easy computation shows that

$$\max_{1\le k\le Tn} \| k^{-1/2} S(\mathbf{k}) - k^{1/2} \boldsymbol{\mu} \| =_{\mathrm{a.s.}} O((\log \log n)^{1/2})$$

and so we get

$$\sup_{1\le t\le nT} \| t^{-1/2}\mathbf{S}(t) - t^{1/2}\boldsymbol{\mu} \| =_{\mathrm{a.s.}} O((\log \log n)^{1/2}). \tag{2.5}$$

Combining (2.4) and (2.5) we obtain that

$$a_{4n} =_{\mathrm{a.s.}} O((\log \log n))$$

and the theorem is proved.

Theorem 2.1 implies, among others, that under the conditions (i), (ii) and (iv) we have

(2.6) $\limsup_{n\to\infty}(2Tn \log\log n)^{-1/2} \max_{1\le k\le nT}|h(\mathbf{S}(k)) - kh(\mu)| = \sigma$ a.s.

The corresponding strong approximation of $\hat{Y}$ is also a consequence of Theorem 2.1.

THEOREM 2.2. *On the probability space of Theorem* 2.1 *we have that*

$$\sup_{0\le t\le nT}\left| \hat{Y}(t) - \frac{\sigma}{qh(\mu)} W(t)\right| =_{\text{a.s.}} o(n^{1/\nu})$$

for any $T > 0$, *provided that conditions* (i)–(iv) *are satisfied.*

PROOF. Using (1.1) we obtain that for almost all $\omega \in \Omega$ there is an integer $k_0 = k_0(\omega)$ such that

$$\frac{1}{2} \le \frac{h(\mathbf{S}(k))}{kh(\mu)} \le \frac{3}{2}, \tag{2.7}$$

if $k \ge k_0$. The law of the iterated logarithm proved in (2.6) implies that

$$\max_{1\le k\le nT} k^{-1/2}|h(\mathbf{S}(k)) - kh(\mu)| =_{\text{a.s.}} O((\log\log n)^{1/2}). \tag{2.8}$$

As we did in the proof of Theorem 2.1, we divide $[0, nT]$ into two parts with the random point k_0 and use a two-term Taylor expansion:

$$\begin{aligned}
&\sup_{0\le t\le nT}\left| \hat{Y}(t) - \frac{\sigma}{qh(\mu)} W(t)\right| \\
&\le \sup_{0\le t\le k_0}\left| \left(\frac{h(\mathbf{S}(t))}{t^p h(\mu)}\right)^{1/q} - t \right| \\
&\quad + \sup_{0\le t\le k_0} \frac{\sigma}{qh(\mu)} |W(t)| + \sup_{k_0\le t\le nT} \frac{1}{q}\left| \left(\frac{h(\mathbf{S}(t))}{h(\mu)} - t\right) - \frac{\sigma}{h(\mu)} W(t)\right| \\
&\quad + \sup_{k_0\le t\le nT} \frac{1}{q}\left(\frac{1}{q} - 1\right) |\xi^{(1/q)-2}| \left(\frac{h(\mathbf{S}(t))}{t^p h(\mu)} - t^q\right)^2 = a_{5n} + \cdots + a_{8n},
\end{aligned}$$

where $\xi = \xi(t)$ is a random point between $(h(\mathbf{S}(t)))/(t^p h(\mu))$ and t^q. It is easy to check that

$$a_{5n} =_{\text{a.s.}} O(1) \quad \text{and} \quad a_{6n} =_{\text{a.s.}} O(1).$$

It follows from Theorem 2.1 that

$$a_{7n} =_{\text{a.s.}} o(n^{1/\nu}),$$

and elementary computation shows that

$$a_{8n} \le \frac{1}{h^2(\mu)} \frac{1}{q}\left(\frac{1}{q} - 1\right) \sup_{k_0\le t\le nT}\left| \frac{\xi}{t^q}\right|^{(1/q)-2} \cdot \left(\frac{h(\mathbf{S}(t)) - th(\mu)}{t^{1/2}}\right)^2.$$

Using (2.8) we get that

$$\sup_{1\le t\le nT} (1/t)(h(\mathbf{S}(t)) - th(\mu))^2 =_{\text{a.s.}} O(\log\log n).$$

On the other hand,

$$\min\left(\frac{h(\mathbf{S}(t))}{th(\mu)}, 1\right) \le \frac{\xi}{t^q} \le \max\left(\frac{h(\mathbf{S}(t))}{th(\mu)}, 1\right),$$

so by the help of (2.7) we obtain that

$$\sup_{k_0\le t\le Tn} \left|\frac{\xi}{t^q}\right|^{(1/q)-2} =_{\text{a.s.}} O(1).$$

Whence

$$a_{8n} =_{\text{a.s.}} O(\log\log n),$$

and the theorem is proved.

Theorem 2.2 also implies a law of the iterated logarithm:

$$\text{(2.9)}\qquad \limsup_{n\to\infty}(2Tn\log\log n)^{-1/2}\sup_{0\le t\le Tn}|\hat{Y}(t)| = \frac{\sigma}{qh(\mu)} \quad \text{a.s.}$$

The second application of Theorem 2.2 is an estimation of the modulus of continuity of $\hat{Y}$.

COROLLARY 2.3. *If the conditions* (i)–(iv) *are satisfied then*

$$\sup_{0\le t\le Tn}\sup_{0\le s\le 1}|\hat{Y}(t+s) - \hat{Y}(t)| =_{a.s.} o(n^{1/\nu}).$$

PROOF. Using Theorem 2.2 it is enough to prove that

$$\sup_{0\le t\le Tn}\sup_{0\le s\le 1}|W(t+s) - W(t)| =_{\text{a.s.}} o(n^{1/\nu}),$$

and this follows from Theorem 1.2.1 of Csörgő and Révész (1981).

3. Strong approximation of the renewal process.

The form of the main result of this paper follows the pattern of Theorems 2.1 and 2.2.

THEOREM 3.1. *On the probability space of Theorem* 2.1, *we have*

$$\sup_{0\le t\le n}\left| Z(t) - \frac{\sigma}{qh(\mu)} W\left(\left(\frac{t}{h(\mu)}\right)^{1/q}\right)\right| =_{a.s.} o(n^{1/\nu q})$$

if $2 < \nu < 4$ *and*

$$\limsup_{n\to\infty} n^{-1/4q}(\log\log n)^{-1/4}(\log n)^{-1/2}\sup_{0\le t\le n}$$

$$\cdot\left| Z(t) - \frac{\sigma}{qh(\mu)} W\left(\left(\frac{t}{h(\mu)}\right)^{1/q}\right)\right| = 2^{1/4}q^{-2}\sigma^{3/2}(h(\mu))^{-3/2-1/4q} \quad \text{a.s.},$$

if $\nu \ge 4$.

Before beginning the proof of Theorem 3.1 we prove an easy but efficient and useful lemma on the inverse of step functions. A functon ϕ, defined on $[0, \infty)$, is called a step function if there is a decomposition of $[0, \infty) = \cup_{i=1}^{\infty} [t_i, t_{i+1})$ such that $0 = t_1 < t_2 < \cdots$ and $\phi(t) = q_i^*$, $t_i \le t < t_{i+1}$, $i = 1, 2, \cdots$, where $\{q_i^*, i \ge 1\}$ is a sequence of real numbers. We assume that $q_1^* = 0$. The inverse of ϕ is

$$\psi(u) = \begin{cases} \inf\{t \ge 0: \phi(t) > u\}, \\ \infty, \quad \text{if} \quad \{t \ge 0: \phi(t) > u\} = \varnothing, \end{cases}$$

$0 \le u < \infty$. Letting $q_i = \max(q_j^*, j \le i)$ we have a direct way of looking at ψ:

$$\psi(u) = t_{i+1}, \quad \text{if} \quad q_i \le u < q_{i+1}, \ i = 1, 2, \cdots.$$

(If $q_i = q_{i+1}$ then the above definition of the inverse gives $\psi(q_i) = \psi(q_{i+1})$.)

LEMMA. *For any* $T \ge 0$

$$\sup_{0 \le u \le T} |\psi(u) - u| \le \sup_{0 \le t \le \psi(T)} |\phi(t) - t|.$$

PROOF. If $\psi(T) = \infty$, then $\phi(t) \le T$ on the whole $[0, \infty)$, and therefore

$$\sup_{0 \le t < \infty} |\phi(t) - t| = \infty.$$

We assume that $\psi(T) < \infty$. The inverse function ψ is also a step function. Let $\hat{q}_k$ be the first point of discontinuity of ψ after T, $T < \hat{q}_k$, and let $\hat{q}_1, \cdots, \hat{q}_{k-1}$ denote the points of discontinuity of ψ before $T (k = 1$ is possible). We have

$$\sup_{0 \le u \le T} |\psi(u) - u|$$
$$\le \max(\hat{t}_1, |\psi(\hat{q}_k -) - \hat{q}_k|, |\psi(\hat{q}_i) - \hat{q}_i|, |\psi(\hat{q}_i -) - \hat{q}_i|, 1 \le i \le k - 1),$$

where $\hat{t}_i$, $1 \le i \le k$ is defined as $\hat{q}_i = \psi(\hat{t}_i)$ and $\hat{t}_i$, $1 \le i \le k$, are points of discontinuity of ϕ. Let $\phi^*(t) = \sup\{\phi(u): 0 \le u < t\}$. We get that

$$|\psi(\hat{q}_i -) - \hat{q}_i| = |\phi^*(\hat{t}_i) - \hat{t}_i| = |\phi(\hat{t}_i) - \hat{t}_i|, \quad 1 \le i \le k,$$
$$\hat{t}_1 = \sup_{0 \le t < \hat{t}_1} |\phi(t) - t|$$

and

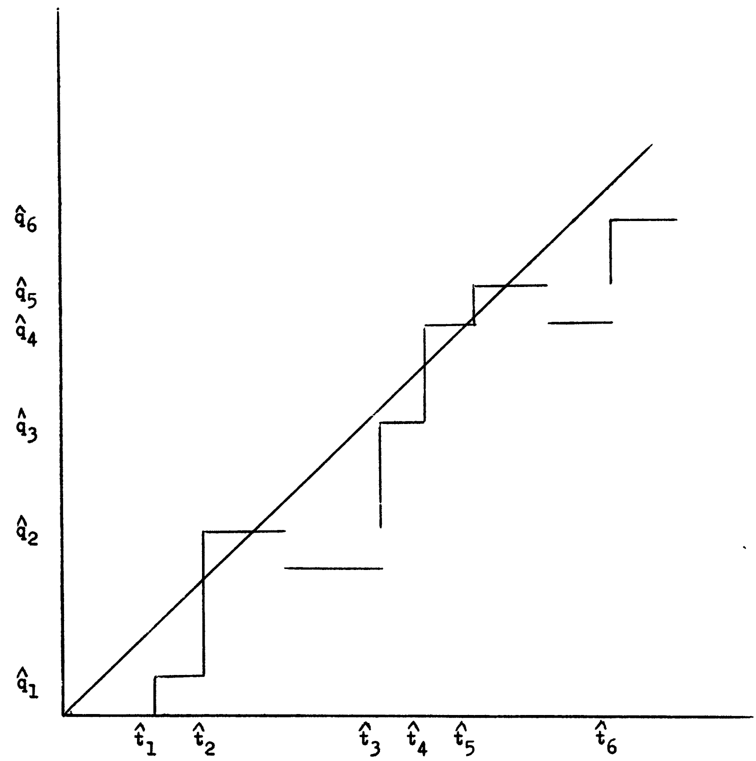
$$|\psi(\hat{q}_i) - \hat{q}_i| = |\phi^*(\hat{t}_{i+1}) - \hat{t}_{i+1}|$$
$$\le \max(|\phi(\hat{t}_{i+1}) - \hat{t}_{i+1}|, \ |\phi(\hat{t}_i) - \hat{t}_i|), \quad 1 \le i \le k - 1$$

(see figure). On the other hand, $\psi(T) = \hat{t}_k$, so we proved our lemma.

PROOF OF THEOREM 3.1. In this section we prove only the half of Theorem 3.1 in the case $\nu \ge 4$, that is, only the inequality

$$\limsup_{n\to\infty} n^{-1/4q} (\log \log n)^{-1/4} (\log n)^{-1/2} \sup_{0 \le t \le n}$$
$$\cdot \left| Z(t) - \frac{\sigma}{qh(\mu)} W\left(\left(\frac{t}{h(\mu)}\right)^{1/q}\right) \right| \le 2^{1/4} q^{-2} \sigma^{3/2} (h(\mu))^{-3/2-1/4q} \quad \text{a.s.} \tag{3.1}$$

This inequality is the most important part of Theorem 3.1 because this is enough to derive functional limit theorems, the Bahadur-Kiefer representation and other



FIGURE

strong limit theorems for the renewal process. The opposite of (3.1) will be proved only after Theorem 4.2.

We consider the time transformed renewal process $(1/n)N(n^q t^q h(\mu))$ which we can write in the form

$$\frac{1}{n} N(n^q t^q h(\mu)) = \inf\left\{s: \frac{h(\mathbf{S}(ns))}{[ns]^p} > n^q t^q h(\mu)\right\}$$

$$= \inf\left\{s: \left(\frac{h(\mathbf{S}(ns))}{[ns]^p n^q h(\mu)}\right)^{1/q} > t\right\}.$$

Introduce the process

$$\hat{Z}(t) = (t - N(t^q h(\mu))).$$

First we prove that

$$\limsup_{n\to\infty} (1/n)\ N(n^q h(\mu)) \le 1 \quad \text{a.s.} \tag{3.2}$$

For every positive ε let Λ_ε denote the following event:

$$\Lambda_\varepsilon = \{\omega: \limsup_{n\to\infty} (1/n)\ N(n^q h(\mu)) > 1 + 2\varepsilon\}.$$

This means that for each $\omega \in \Lambda_\varepsilon$ there is a random sequence of integers $n_k = n_k(\omega)$ going to infinity such that

$$N(n_k^q h(\mu)) > n_k(1 + \varepsilon). \tag{3.3}$$

It follows from (3.3) that

$$\frac{h(\mathbf{S}(n_k(1+\varepsilon)))}{[n_k(1+\varepsilon)]^p} \le n_k^q h(\mu)$$

and we obtain from this inequality by an elementary computation that

$$\frac{h(\mathbf{S}(n_k(1+\varepsilon)))}{[n_k(1+\varepsilon)]} - h(\mu) \le h(\mu)\left\{\frac{n_k^q[n_k(1+\varepsilon)]^p}{[n_k(1+\varepsilon)]} - 1\right\}.$$

It is easy to see that

$$\limsup_{k\to\infty}\left\{\frac{h(\mathbf{S}(n_k(1+\varepsilon)))}{[n_k(1+\varepsilon)]} - h(\mu)\right\} \le h(\mu)\left\{\frac{1}{(1+\varepsilon)^q} - 1\right\} < 0$$

and therefore the law of large numbers in (1.1) implies that $P(\Lambda_\varepsilon) = 0$ and we proved (3.2).

In the following step we prove a law of iterated logarithm for $\hat{Z}$:

$$\limsup_{n\to\infty}(2n\log\log n)^{-1/2}\sup_{0\le t\le n}|\hat{Z}(t)| \le \frac{\sigma}{qh(\mu)} \quad \text{a.s.} \tag{3.4}$$

Indeed, using our Lemma we have

$$\sup_{0\le t\le 1}\left|\frac{1}{n}N(n^q t^q h(\mu)) - t\right| \le \sup_{0\le t\le (1/n)N(n^q h(\mu))}\left|\left(\frac{h(\mathbf{S}(nt))}{[nt]^p n^q h(\mu)}\right)^{1/q} - t\right|.$$

Let $\varepsilon > 0$. We get from (3.2) that

$$\sup_{0\le t\le (1/n)N(n^q h(\mu))}\left|\left(\frac{h(\mathbf{S}(nt))}{[nt]^p n^q h(\mu)}\right)^{1/q} - t\right| \le_{\text{a.s.}} \sup_{0\le t\le 1+\varepsilon}\left|\left(\frac{h(\mathbf{S}(nt))}{[nt]^p n^q h(\mu)}\right)^{1/q} - t\right|,$$

and by the help of (2.9) it is easy to see that

$$(2n\log\log n)^{-1/2}\sup_{0\le t\le 1+\varepsilon}\left|\left(\frac{h(\mathbf{S}(nt))}{[nt]^p h(\mu)}\right)^{1/q} - nt\right| \le_{\text{a.s.}} \frac{(1+\varepsilon)\sigma}{qh(\mu)},$$

and (3.4) is proved.

Now consider the following decomposition of $\hat{Z}(t)$:

$$\hat{Z}(t) = \hat{Y}(N(t^q h(\mu))) + \left(t - \left(\frac{h(\mathbf{S}(N(t^q h(\mu))))}{h(\mu)(N(t^q h(\mu)))^p}\right)^{1/q}\right). \tag{3.5}$$

We approximate the first term in (3.5) and show that the second term is almost surely less than the rate of the approximation. Clearly,

$$\Delta_n^1 = \sup_{0\le t\le n} | \hat{Y}(N(t^q h(\mu))) - \frac{\sigma}{qh(\mu)} W(t) |$$

$$\le \sup_{0\le t\le n} | \hat{Y}(N(t^q h(\mu))) - \frac{\sigma}{qh(\mu)} W(N(t^q h(\mu))) |$$

$$+ \sup_{0\le t\le n} \frac{\sigma}{qh(\mu)} | W(t) - W(N(t^q h(\mu))) | = a_{9n} + a_{10n}.$$

Using (3.2) and Theorem 2.2 we get

$$a_{9n} =_{\text{a.s.}} o(n^{1/\nu}). \tag{3.6}$$

By (3.2) and (3.4) we have

$$\begin{aligned} &\sup_{0\le t\le n} | W(N(t^q h(\mu))) - W(t) | \\ &\qquad \le_{\text{a.s.}} \sup_{0\le t\le n(1+\varepsilon)} \sup_{0\le s\le g(n)} | W(t+s) - W(t) |, \end{aligned} \tag{3.7}$$

where ε is an arbitrary positive number and

$$g(n) = (1+\varepsilon)(\sigma/(qh(\mu))(2n \log\log n)^{1/2}.$$

Theorem 1.2.1 of Csörgő and Révész (1981) shows that

$$\begin{aligned} &\lim\sup_{n\to\infty} (g(n)\log n)^{-1/2} \sup_{0\le t\le n(1+\varepsilon)} \sup_{0\le s\le g(n)} \\ &\qquad \cdot | W(t+s) - W(t) | = 1 \quad \text{a.s.} \end{aligned} \tag{3.8}$$

We get from (3.6), (3.7) and (3.8) that

$$\Delta_n^1 =_{\text{a.s.}} o(n^{1/\nu}), \quad \text{if} \quad 2 < \nu < 4$$

and

$$\lim\sup_{n\to\infty} n^{-1/4} (\log\log n)^{-1/4} (\log n)^{-1/2} \Delta_n^1 \le 2^{1/4} \left(\frac{\sigma}{qh(\mu)}\right)^{3/2} \quad \text{a.s.},$$

if $\nu \ge 4$.

In order to prove (3.1) and Theorem 3.1 in the case $2 < \nu < 4$ we have to estimate

$$\Delta_n^2 = \sup_{0\le t\le n} \left| t - \left(\frac{h(\mathbf{S}(N(t^q h(\mu))))}{h(\mu)(N(t^q h(\mu)))^p}\right)^{1/q} \right|.$$

We show that Δ_n^2 is not greater then the largest jump of $\hat{Y}(t)$ on $[0, N(n^q h(\mu))]$. Let $\{\nu_i,\ i \ge 0\}$, $\nu_0 = 0$ denote the points of discontinuity of $N(t)$, increasing in order, and $\nu_k \le n^q h(\mu) < \nu_{k+1}$. If $t_{i-1} = (\nu_{i-1}/h(\mu))^{1/q} \le t < (\nu_i/h(\mu))^{1/q} = t_i$, then by the definition of $N(t)$ we have that

$$\frac{h(\mathbf{S}(N(t^q h(\mu))))}{(N(t^q h(\mu)))^p} = \nu_i,$$

and therefore

$$\text{(3.9)} \qquad \sup_{0\le t<t_k}\left|\left(\frac{h(\mathbf{S}(N(t^q h(\mu))))}{h(\mu)(N(t^q h(\mu)))^p}\right)^{1/q} - t\right| = \max_{1\le i\le k}(t_i - t_{i-1}).$$

We have to consider the interval $t_k \le t \le n$ which may be only one point. We obtain by the definition of $N(t)$ that

$$\begin{aligned}
&\sup_{t_k\le t\le n}|(h(\mathbf{S}(N(t^q h(\mu)))))^{1/q}(h(\mu)(N(t^q h(\mu)))^p)^{-1/q} - t|\\
\text{(3.10)} \qquad &= \sup_{t_k\le t\le n}|(h(\mathbf{S}(N(n^q h(\mu)))))^{1/q}(h(\mu)(N(n^q h(\mu)))^p)^{-1/q} - t|\\
&= \max(t_k - t_{k-1}, t_{k+1} - t_k).
\end{aligned}$$

On the other hand, we have $\nu_{k+1.} = h(\mathbf{S}(N(n^q h(\mu))))(N(n^q h(\mu)))^{-p}$. Using (3.9) and (3.10) we obtain that

$$\Delta_n^2 \le \max_{1\le i\le k-1}(t_i - t_{i-1}),$$

and therefore Δ_n^2 is not greater than the largest jump of $\hat{Y}(t)$ on $[0, N(n^q h(\mu))]$. Theorem 2.2, Corollary 2.1 and (2.2) then imply that

$$\text{(3.11)} \qquad \Delta_n^2 =_{\text{a.s.}} o(n^{1/\nu}).$$

4. Applications. First application of the main theorem is the determination of the set of the limit points of the process

$$\xi_n(t) = \frac{q}{\sigma}(h(\mu))^{1+(1/2q)}(2n^{1/q}\log\log n)^{-1/2}\left(N(nt) - \left(\frac{nt}{h(\mu)}\right)^{1/q}\right).$$

Let $\mathscr{S}$ be Strassen's set of absolutely continuous functions (with respect to Lebesgue measure) such that

$$\text{(4.1)} \qquad f(0) = 0 \text{ and } \int_0^1 (f'(t))^2\,dt \le 1.$$

THEOREM 4.1. *If the conditions* (i)–(iv) *are satisfied then the sequence* $\{\xi_n(t), 0 \le t \le 1, n \ge 1\}$ *is a.s. relatively compact with respect to the supremum norm and the set of its limit points is* $\mathscr{S}^* = \{f(t^{1/q}): f \in \mathscr{S}\}$. *Consequently*

$$\limsup_{n\to\infty}(n^{1/q}\log\log n)^{-1/2}\sup_{0\le t\le n}\left|N(t) - \left(\frac{t}{h(\mu)}\right)^{1/q}\right| = 2^{1/2}\frac{\sigma}{q}(h(\mu))^{-(1+1/2q)}.$$

PROOF. Using Theorem 3.1 and (3.1) it is enough to determine the limit points of the process

$$\xi_n^*(t) = (h(\mu))^{1/2q}(2n^{1/q}\log\log n)^{-1/2}W\left(\left(\frac{nt}{h(\mu)}\right)^{1/q}\right).$$

The limit points of $\xi_n^*(t)$ is the same as the limit points of

$$(2n \log \log n)^{-1/2} W(nt^{1/q})$$

which is $\mathscr{S}^*$ by Strassen's functional law of iterated logarithm (Theorem 1.3.2 in Csörgő and Révész, 1981).

Bahadur (1966) was the first to investigate the distance between the empirical distribution function and its inverse, the empirical quantile function. Kiefer (1970) determined the exact rate of this distance. Shorack (1982) gave a new proof of Kiefer's theorem based on the strong approximation of the quantile process, or, generally, on the "Hungarian construction". The following theorem shows that a Bahadur-Kiefer type representation is also true for the renewal process in terms of partial sums. Set

$$\Delta_n = \sup_{0 \le t \le n} \left| N(t) + \frac{h\left(\mathbf{S}\left(\left(\frac{t}{h(\mu)}\right)^{1/q}\right)\right)}{qh(\mu)} - \left(\frac{t}{h(\mu)}\right)^{1/q}\left(1 + \frac{1}{q}\right) \right|.$$

THEOREM 4.2. *If conditions* (i)–(iv) *are satisfied and* $2 < \nu < 4$ *then*

$$\Delta_n =_{\text{a.s.}} o(n^{1/\nu q}),$$

and if $\nu \ge 4$, *then*

$$\limsup_{n\to\infty} n^{-1/4q}(\log \log n)^{-1/4}(\log n)^{-1/2}\Delta_n$$

$$= 2^{1/4}\left(\frac{\sigma}{qh(\mu)}\right)^{3/2}(h(\mu))^{-1/4q}q^{-1/2} \quad \text{a.s.}$$

PROOF. If $2 < \nu < 4$ then the statement follows immediately from Theorems 2.1 and 3.1 because the function of the partial sums and the renewal process are approximated by the *same* Wiener process.

Let $\nu \ge 4$ and

$$\tilde{\Delta}_n = \sup_{0 \le t \le n} \left| N(t^q h(\mu)) + \left(\frac{h(\mathbf{S}(t))}{t^p h(\mu)}\right)^{1/q} - 2t \right|.$$

First we prove that

$$\limsup_{n\to\infty}(n \log \log n)^{-1/4}(\log n)^{-1/2}\tilde{\Delta}_n = 2^{1/4}\left(\frac{\sigma}{qh(\mu)}\right)^{3/2} \quad \text{a.s.} \tag{4.2}$$

Using Theorem 2.2 and (3.1) we get that

$$\limsup_{n\to\infty}(n \log \log n)^{-1/4}(\log n)^{-1/2}\tilde{\Delta}_n \le 2^{1/4}\left(\frac{\sigma}{qh(\mu)}\right)^{3/2} \quad \text{a.s.,}$$

because we approximated $\hat{Z}$ and $\hat{Y}$ by the *same* Wiener process. Theorem 4.2

implies that the sequence

$$\left\{(2n \log\log n)^{-1/2} \frac{qh(\mu)}{\sigma} (N(n^q t^q h(\mu)) - nt),\quad 0 \le t \le 1,\quad n \le 1\right\}$$

is a.s. relatively compact with respect to the supremum norm and the set of its limit points is $\mathscr{S}$. Consider the function $h_\delta(t)$ that equals t and $1 - \delta$ according as $0 \le t \le 1 - \delta$ and $1 - \delta \le t \le 1$, where $0 < \delta < 1$. For each δ, $h_\delta \in \mathscr{S}$ so it follows from the relative compactness of

$$\frac{qh(\mu)}{\sigma} (2n \log\log n)^{-1/2} (N(n^q t^q h(\mu)) - nt)$$

that there is a sequence of integer-valued random variables $n_k = n_k(\omega)$ such that

$$\lim_{k\to\infty} \sup_{0\le t\le 1} \left| \frac{qh(\mu)}{\sigma} (2n_k \log\log n_k)^{-1/2} (N(n_k^q t^q h(\mu)) - n_k t) - h_\delta(t) \right| = 0 \quad \text{a.s.} \tag{4.3}$$

By the help of the decomposition in (3.5), and by (3.11), Theorem 2.2, we obtain that

$$\begin{aligned}\limsup_{n\to\infty} (n \log\log n)^{-1/4} (\log n)^{-1/2} \tilde{\Delta}_n &= \limsup_{n\to\infty} (n \log\log n)^{-1/4} (\log n)^{-1/2} \frac{\sigma}{h(\mu) q} \\ &\quad \sup_{0\le t\le 1} | W(N(n^q t^q h(\mu))) - W(nt) | \quad \text{a.s.}\end{aligned} \tag{4.4}$$

Using a modification of part (iii) of Theorem 1.2.1 of Csörgő and Révész (1981) we get that

$$\begin{aligned}\lim_{n\to\infty} (n \log\log n)^{-1/4} (\log n)^{-1/2} \sup_{0\le t\le 1} &\left| W\left(nt + h_\delta(t) \frac{\sigma}{qh(\mu)} (2n \log\log n)^{1/2}\right) - W(nt) \right| \\ &= (1-\delta)^{1/2} 2^{1/4} \left(\frac{\sigma}{qh(\mu)}\right)^{1/2} \quad \text{a.s.}\end{aligned} \tag{4.5}$$

Since $\delta > 0$ is arbitrary, (4.3), (4.4) and (4.5) imply that

$$\limsup_{n\to\infty} (n \log\log n)^{-1/4} (\log n)^{-1/2} \tilde{\Delta}_n \ge 2^{1/4} \left(\frac{\sigma}{qh(\mu)}\right)^{3/2} \quad \text{a.s.,}$$

and we proved (4.2). On the other hand, (4.2) implies that

$$\limsup_{n\to\infty} n^{-1/(4q)}(\log\log n)^{1/4}(\log n)^{-1/2}\sup_{0\le t\le n}$$

$$\left| N(t) + \left(\left(\frac{h(\mu)}{t}\right)^{p/q}(h(\mu))^{-1}h\left(\mathbf{S}\left(\left(\frac{t}{h(\mu)}\right)^{1/q}\right)\right)\right)^{1/q} - 2\left(\frac{t}{h(\mu)}\right)^{1/q}\right|$$

$$= 2^{1/4}\left(\frac{\sigma}{qh(\mu)}\right)^{3/2}(h(\mu))^{-(1/4)q}q^{1/2} \quad \text{a.s.}$$

Using Theorem 2.1 and 2.2 we then get that

$$\sup_{0\le t\le n}\left|\left(\frac{h(\mathbf{S}(t))}{t^p h(\mu)}\right)^{1/q} - \frac{1}{q}\frac{h(\mathbf{S}(t))}{h(\mu)} - t\left(1-\frac{1}{q}\right)\right| =_{\text{a.s.}} o(n^{1/\nu}),$$

and therefore we proved our theorem.

The obtained Bahadur-Kiefer representation of the renewal process immediately implies that

$$\limsup_{n\to\infty} n^{-1/4q}(\log\log n)^{-1/4}(\log n)^{-1/2}\sup_{0\le t\le n}$$

$$\left| Z(t) - \frac{\sigma}{qh(\mu)} W\left(\left(\frac{t}{h(\mu)}\right)^{1/q}\right)\right| \ge 2^{1/4}\left(\frac{\sigma}{qh(\mu)}\right)^{3/2}(h(\mu))^{-1/4q}q^{-1/2} \quad \text{a.s.},$$

where W is the Wiener process defined in Theorem 2.1 and we therefore also finished the remaining part of the proof of Theorem 3.1.

Theorem 3.1 implies not only the law of functional iterated logarithm as above but a Chung-type law of iterated logarithm as well.

THEOREM 4.3. *If the conditions* (i)–(iv) *are satisfied then*

$$\liminf_{n\to\infty}(n^{-q}\log\log n)^{1/2}\sup_{0\le t\le n}\left| N(t) - \left(\frac{t}{h(\mu)}\right)^{1/q}\right|$$

$$= 8^{-1/2}\,\pi\,\frac{\sigma}{q}\,(h(\mu))^{-\left(1+\frac{1}{2q}\right)} \quad \text{a.s.}$$

PROOF. This theorem immediately follows from Theorem 3.1 and the Chung (1948) law of iterated logarithm for the Wiener process.

These two laws of the iterated logarithm give an almost sure characterization of the path behaviour of the renewal process but a more detailed one can be given using the concept of upper-upper, upper-lower, lower-upper and lower-lower classes introduced by Lévy (1948) and Révész (1982). They say that a function ϕ belongs to the upper-upper class of the process X if the inequality $X(n) \le_{\text{a.s.}} \phi(n)$ holds and ϕ belongs to the upper-lower class if $X(n) > \phi(n)$ holds infinitely often almost surely. The definition of lower-upper and lower-lower classes are analogous. We investigate these classes for the following processes:

$$\eta_1(n) = N(n) - \left(\frac{n}{h(\mu)}\right)^{1/q},$$

$$\eta_2(n) = \left| N(n) - \left(\frac{n}{h(\mu)}\right)^{1/q} \right|,$$

$$\eta_3(n) = \sup_{0\le t\le n}\left(N(t) - \left(\frac{t}{h(\mu)}\right)^{1/q}\right)$$

and

$$\eta_4(n) = \sup_{0\le t\le n}\left| N(t) - \left(\frac{t}{h(\mu)}\right)^{1/q} \right|.$$

The characterizations in question are given in the form of integral tests concerning the convergence of divergence of integrals

$$I_1(\phi) = \int_1^\infty t^{-1}\phi(t)\exp\left(-\frac{1}{2}\,\phi^2(t)\right)dt,$$

$$I_2(\phi) = \int_1^\infty t^{-1}\phi(t)\,dt$$

and

$$I_3(\phi) = \int_1^\infty t^{-1}(\phi(t))^{-2}\exp\left(-\frac{\pi^2}{8}\,(\phi(t))^{-2}\right)dt.$$

THEOREM 4.4. *We assume that the conditions* (i)–(iv) *are satisfied and* ϕ *is a nondecreasing function. Then*

$$P\left\{\eta_i(n) \ge \sigma q^{-1}(h(\mu))^{-(1+2/q)}n^{1/2q}\phi\left(\left(\frac{n}{h(\mu)}\right)^{1/q}\right)\ i.o.\right\} = \begin{cases}1, & if\ I_1 = \infty\\ 0, & if\ I_1 < \infty,\end{cases}$$

$i = 1, 2, 3, 4,$

$$P\left\{\eta_3(n) \le \sigma q^{-1}(h(\mu))^{-(1+2/q)}\phi\left(\left(\frac{n}{h(\mu)}\right)^{1/q}\right)\ i.o.\right\} = \begin{cases}1, & if\ I_2 = \infty\\ 0, & if\ I_2 < \infty\end{cases}$$

and

$$P\left\{\eta_4(n) \le \sigma q^{-1}(h(\mu))^{-(1+2/q)}n^{1/2q}\phi\left(\left(\frac{n}{h(\mu)}\right)^{1/q}\right)\ i.o.\right\} = \begin{cases}1, & if\ I_3 = \infty\\ 0, & if\ I_3 < \infty.\end{cases}$$

PROOF. The rates in Theorem 3.1 go to infinity faster than $n^{1/2q}\log n$; therefore, as it was pointed out by Jain, Jogdeo and Stout (1975), the upper-upper and the upper-lower classes of $\eta_i(n)$, $1 \le i \le 4$ and lower-lower classes of $\eta_4(n)$ are identical with the corresponding classes of the corresponding functionals of the approximating Gaussian processes. Using again Theorem 3.1 and the fact that

$$I_2(\phi(t) \pm t^{-\lambda}) < \infty \Leftrightarrow I_2(\phi) < \infty, \quad \lambda > 0,$$

we get that

$$P\left\{\eta_3(n) \le \sigma q^{-1}(h(\mu))^{-(1+2/q)} n^{1/2q}\phi\left(\left(\frac{n}{h(\mu)}\right)^{1/q}\right) \text{ i.o.}\right\}$$

$$= P\left\{\sup_{0\le t\le (n/h(\mu))^{1/q}} W(t) \le \left(\frac{n}{h(\mu)}\right)^{1/2q}\phi\left(\left(\frac{n}{h(\mu)}\right)^{1/q}\right) \text{ i.o.}\right\}.$$

So we obtained that it is enough to determine the classes in question for the indicated functionals of the Gaussian process which is a time-transformed Wiener process. The characterizations of these classes for the time-transformed Wiener process $W((t/h(\mu))^{1/q})$ can easily be deduced from the well-known theorems of Kolmogorov, Petrovski, Erdös and Feller (see, for example, Itô and McKean (1965, page 163), Hirsch (1954), Chung (1948), Itô and McKean (1965, page 547)).

5. Nonindependent and/or nonidentically distributed random vectors. Instead of (i), the condition of independence and identical distribution of the random vectors $\{\mathbf{X}_n, n \ge 1\}$ we assume that

(v) $E\mathbf{X}_n = \mu$ for each n,

(vi) $\max_{0\le t\le n} \| \mathbf{S}(t) - t\mu \| =_{\text{a.s.}} O((n \log\log n)^{1/2})$

and

(vii) $E(\nabla h(\mu)(\mathbf{X}_n - \mu)^T)^2 = \sigma^2$ for each n.

In this case $\{\nabla h(\mu)(\mathbf{X}_n - \mu)^T, n \ge 1\}$ is a sequence of random variables which are not independent, nor identically distributed, but under further conditions on the dependence and the distribution we can define a Wiener process such that

$$\sup_{0\le t\le n} \left| \sum_{i=1}^{[t]} \nabla h(\mu)(\mathbf{X}_i - \mu)^T - \sigma W(t) \right| =_{\text{a.s.}} o(r(n)) \tag{5.1}$$

and

$$\limsup_{n\to\infty} r(n)(n \log\log n)^{-1/2} < \infty,$$

where $r(n)$ is a nondecreasing, regularly varying sequence. The conditions which are sufficient for (5.1) were studied in many papers, we refer only to Philipp and Stout (1975), an excellent early survey of the problem, and to Berkes and Philipp (1979).

Reading through the previous sections we can see that we used in fact only (v)–(vii) when the random vectors were i.i.d., the strong approximation in (2.1) and the law of the iterated logarithm which followed from (2.1) as well. So we conclude that (5.1) always implies the appropriate strong approximation of the renewal process.

THEOREM 5.1. *If the conditions* (ii)–(vii) *are satisfied then* (5.1) *implies that*

$$\sup_{0\le t\le n}\left| Z(t)-\frac{\sigma}{qh(\mu)}W\left(\left(\frac{t}{h(\mu)}\right)^{1/q}\right)\right| =_{a.s.} o\left(r\left(\left(\frac{n}{h(\mu)}\right)^{1/q}\right)\right),$$

if

$$\lim_{n\to\infty}\frac{1}{r(n)}(n\log\log n)^{1/4}(\log n)^{1/2}=0$$

and

$$\limsup_{n\to\infty} n^{-1/4q}(\log\log n)^{-1/4}(\log n)^{-1/2}\sup_{0\le t\le n}\left| Z(t)-\frac{\sigma}{qh(\mu)}W\left(\left(\frac{t}{h(\mu)}\right)^{1/q}\right)\right|$$

$$=2^{1/4}q^{-2}\left(\frac{\sigma}{h(\mu)}\right)^{3/2}(h(\mu))^{-1/(4q)}\quad a.s.,$$

if

$$\limsup_{n\to\infty} n^{-1/4}(\log\log n)^{-1/4}(\log n)^{-1/2}r(n)<\infty.$$

The results of Section 4, all being consequences of the approximation of the renewal process, are true for the case $r(n) = n^{1/2}(\log n)^{-2}$. If we assume that $\{\mathbf{X}, \mathbf{X}_n, n \ge 1\}$ are i.i.d.r.v's and $\mathbf{X}$ have only finite second moments, then (5.1) is true with $r(n) = (n \log\log n)^{1/2}$ by Strassen's approximation theorem (see, for example Csörgő and Révész, 1981, Theorem 0.2.). We obtain now from Theorem 5.1 that conditions (ii)–(iv) and $E|X^{(i)}|^2 < \infty$, $1 \le i \le d$, imply Theorem 4.1.

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BOLYAI INSTITUTE
SZEGED UNIVERSITY
ARADI VÉRTANÚK TERE 1.
H-6720 SZEGED
HUNGARY