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CONVEXIFICATION IN LIMIT LAWS OF RANDOM SETS IN BANACH SPACES

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A convexification argument is offered which implies that the strong law of large numbers for random compact sets in a Banach space is valid without a convexity assumption.

A strong law of large numbers for random compact sets in a Banach space with the Minkowski addition is provided by Puri and Ralescu [6] and, among other results, by Giné, Hahn and Zinn [2] and Hess [3]. This follows the finite dimensional version introduced in [1]. The latter however allows nonconvex sets while the generalizations to the Banach space setting were proved for convex sets. A convexification argument stated in the lemma below shows that the convexity assumption can be dropped. The proof reveals information about the nature of the convergence. Since writing this note we learned that F. Hiai [4] has independently established this result, using a different method.

Some notations. Let B be a Banach space with norm $\| \;\|$. The Minkowski addition of two compact subsets K and L is $K + L = \{x + y: x \in K, y \in L\}$, and αK is the set $\{\alpha x: x \in K\}$. The closed convex hull of K is denoted coK; it is a linear operation. The Hausdorff distance $h(K, L)$ is defined as $\max(\max_{x\in K}\min_{y\in L}\| x - y \|, \max_{y\in L}\min_{x\in K}\| x - y \|)$. Convergence of sets in this paper is meant in the Hausdorff metric.

Lemma. *Let $K_1, K_2, \cdots$ be a sequence of compact sets in the Banach space B, and let K_0 be compact and convex. Suppose that*

(1) $$(1/n)(coK_1 + \cdots + coK_n) \text{ converges to } K_0 \text{ as } n \to \infty.$$

Then

(2) $$(1/n)(K_1 + \cdots + K_n) \text{ converges to } K_0 \text{ as } n \to \infty.$$

Proof. Let e be an exposed point in K_0 and let f be the bounded linear functional on B exposing e, i.e. $f(e) > f(x)$ for $x \neq e$ in K_0. For each i let $x_i \in K_i$ be a maximizer of f, i.e. $f(x_i) \geq f(x)$ for $x \in K_i$.

Claim 1. e is the limit of $y_n = (1/n)(x_1 + \cdots + x_n)$

Indeed, the point y_n is a maximizer of f in $(1/n)(coK_1 + \cdots + coK_n)$, hence by (1), and since an exposed point of a compact set is strongly exposed, it follows that every cluster point of y_n coincides with e.

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Let $0 \le \lambda \le 1$ and let $p(n) = [\lambda n]$, i.e. the largest integer smaller than or equal to λn. Let $0 \le m(n) \le n - p(n)$.

Claim 2. $(1/n)(x_{m(n)} + \cdots + x_{m(n)+p(n)})$ converge to λe.

This follows immediately from Claim 1.

We apply now the previous arguments to a finite number of exposed points $e^1, \cdots, e^k$ and let x_i^j be the corresponding maximizers. Let $\lambda_1, \cdots, \lambda_k$ be numbers in $[0, 1]$ with $\lambda_1 + \cdots + \lambda_k = 1$. Let $m_j(n) = [(\lambda_1 + \cdots + \lambda_j)n]$, this for $j = 1, \cdots, n$. Readily from the previous claim it follows:

Claim 3. If $z_{i,n} = x_i^j$ for $m_j(n) < j \le m_{j+1}(n)$ then $(1/n)(z_{1,n} + \cdots + z_{n,n})$ converge to $\lambda_1 e^1 + \cdots + \lambda_k e^k$.

To complete the proof of the lemma note that whenever

$$w_n \in (1/n)(K_1 + \cdots + K_n)$$

then by (1) every limit point of w_n is in K_0. Exposed points are such limits by Claim 1, and convex combinations of exposed points are such limits by Claim 3. Since the convex combinations of exposed points of K_0 are dense in K_0 (see e.g. Köthe [5], page 337) the lemma follows.

The application of the lemma to the strong law of large numbers is exactly as in [1]. We state it here for completeness. For terminology see either of the references [1], [2] or [6].

THEOREM. *Let $X_1(\omega), X_2(\omega) \cdots$, be a sequence of identically distributed and independent random sets with values in the space of compact sets of a Banach space, with the Hausdorff metric. If $Eh(X_1(\omega), \{0\}) < \infty$ then almost surely $(1/n)(X_1(\omega) + \cdots + X_n(\omega))$ converges to a constant convex and compact set, the expectation of coX_1.*

PROOF. The random sets $coX_i(\omega)$ satisfy (via embedding, e.g. [7]) the conditions of the strong law of large numbers in a Banach space, see [2] or [6]. (There is no need to assume separability of B since one can show that the values $X_1(\omega)$ are contained almost surely in a separable subspace.) Therefore $(1/n)(coX_1(\omega) + \cdots + coX_n(\omega))$ converges almost surely to a constant compact and convex set, the expectation of coX_1. The nonconvex case follows from the lemma.

Other limit laws which hold for convex random sets can be established for nonconvex sets with this convexification technique; for instance the ergodic theorem in Hess [3]. However, the law of iterated logarithms, and certainly the central limit theorem (see Gine, Hahn and Zinn [2]) still remain elusive.

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