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ON DIFFERENTIABILITY PRESERVING PROPERTIES OF SEMIGROUPS ASSOCIATED WITH ONE-DIMENSIONAL SINGULAR DIFFUSIONS

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In this paper we investigate the differentiability preserving properties of the semigroup $\{T_t: t \geq 0\}$ whose infinitesimal generator is a closed extension of the one-dimensional diffusion operator $L = a(x)d^2/dx^2 + b(x)d/dx$ acting on $C^2(I)$, where I is a closed and bounded interval. Especially we treat the case in which the smoothness of the diffusion coefficient fails at the boundary. We get that $\{T_t: t \geq 0\}$ preserves the one and two-times differentiabilities but does not the three-times one of sufficiently many initial data.

1. Introduction. Given $-\infty < r_0 < r_1 < \infty$, let I be a closed interval in $(-\infty, \infty)$ with endpoints r_0 and r_1, and let $a(x)$ and $b(x)$ be continuous functions on I satisfying $a(x) \geq 0$ on I and

$$a(r_i) = 0 \leq (-1)^i b(r_i) \quad \text{for} \quad i = 0, 1. \tag{1.1}$$

We define the diffusion operator L by

$$L = a(x)\ (d^2/dx^2) + b(x)\ (d/dx). \tag{1.2}$$

Let $x(t, \omega) = \omega(t)$ for $\omega \in \Omega = C([0, \infty), I)$, $\mathscr{N}_t$ and $\mathscr{N}$ be the σ-fields generated by $\{x(s): 0 \leq s \leq t\}$ and $\{x(s): 0 \leq s\}$, respectively, and $C^n(I)$ be the space of n-times continuously differentiable, real-valued functions on I for $n = 1, 2, 3, \cdots$. A solution to the martingale problem on I for L starting at $x \in I$ is a probability measure P_x on $(\Omega, \mathscr{N})$ such that $P_x[x(0) = x] = 1$ and $\{f(x(t)) - \int_0^t Lf(x(s))\, ds;\ \mathscr{N}_t: t \geq 0\}$ is a P_x-martingale for every $f \in C^2(I)$. For each $x \in I$, the existence of such a solution follows easily; refer to [2, 10]. For the uniqueness, Yamada and Watanabe obtained a nice sufficient condition in [15]. Moreover, Dorea [1], Ethier [2], Norman [10] and others (for which we refer to the references in [1, 2]) investigated the differentiability preserving properties of Markov-semigroup $\{T_t: t \geq 0\}$ associated with the unique solutions to the martingale problem on I for L with the smooth $a(x)$ and $b(x)$. Especially Ethier proved that T_t, $t \geq 0$, maps $C^n(I)$ into itself in case $a(x) \in C^{n \vee 2}(I)$ and $b(x) \in C^n(I)$ for each $n = 1, 2, 3, \cdots$ and Norman also obtained a similar result independently.

In this paper, we study these problems for the case that $a(x) > 0$ on (r_0, r_1), $\lim_{x \to r_i} a(x)/|x - r_i| = \infty$ for $i = 0$ and 1 and so the smoothness of $a(x)$ fails at

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the boundary. For example we consider the following diffusion operator:

$$L = \{x(1-x)\}^\alpha \left\{\log \frac{1}{x(1-x)}\right\}^\beta \frac{d^2}{dx^2} + b(x)\frac{d}{dx} \quad \text{on} \quad I = [0, 1],$$

where $0 \le \alpha \le 1$, $-\infty < \beta < \infty$ ($\beta < 0$ if $\alpha = 0$; $\beta > 0$ if $\alpha = 1$) and $b(x)$ belongs to $C^2(I) \cap \{f^{(2)}(r_i) = 0$ for $i = 0$ and $1\}$ and satisfies (1.1). In this example, not only is the diffusion coefficient degenerate and undifferentiable, but Yamada-Watanabe's sufficient condition for the uniqueness does not hold for $0 \le \alpha < 1$ or for $\alpha = 1$ and $\beta > 1$ at the boundary 0 and 1. Nevertheless, as stated in the general form in Section 2, we get the uniqueness of the solution to the martingale problem and the one and two-times differentiability preserving properties of the semigroup $\{T_t: t \ge 0\}$ associated with the unique solutions for sufficiently many initial data.

For the significance and applications of these results, refer to the introduction of [1, 2, 10] and the references listed in them. Also refer to [5.8].

In Section 2, we state the main results.

In Section 3, we prepare several lemmas and, using these results, prove the main results in Section 4. In lemmas in Section 3, we chiefly engage in detailed investigations of the boundary conditions which will contain $C^2(I)$ since we take all $C^2(I)$-functions as test functions in the martingale problem. These results are useful for proving the uniqueness of the solution to the martingale problem. Moreover, using these results and applying the ideas employed in Ethier [2], we get the differentiability preserving properties of resolvent operators $(\lambda - L)^{-1}$ of the semigroup $\{T_t: t \ge 0\}$ associated with the unique solutions. Thus we obtain our main results.

Finally in Section 5, we show that the associated semigroup $\{T_t: t \ge 0\}$ does not preserve the three-times differentiability of sufficiently many initial data.

2. Notations and main results. Let $I = [r_0, r_1]$ with $-\infty < r_0 < r_1 < \infty$, G be a subinterval of I, and let $C(G)$ denote the space of real continuous functions on G. For each nonnegative integer n, we denote by $C^n(G)$ the space of n-times continuously differentiable, real-valued functions on $G(C^0(G) = C(G))$ and we let $C_0^n(I) = C^n(I) \cap \{f^{(n)}(r_i) = 0$ for $i = 0$ and $1\}$ $(C_0^0(I) = C_0(I))$, where $f^{(n)}(x)$ stands for the nth derivatives of the function f at x. We define the norm $\|\cdot\|_n$ on $C^n(I)$ and $C_0^n(I)$ by

$$\|f\|_n = \textstyle\sum_{k=0}^n \sup_{x\in I} |f^{(k)}(x)|.$$

Then, with this norm, $C^n(I)$ and $C_0^n(I)$ become Banach spaces.

Let $a(x), b(x) \in C(I^0)$ with $a(x) > 0$ on $I^0 = (r_0, r_1)$. We let the domain $D(L)$ of L defined by (1.2) for these $a(x)$ and $b(x)$ be the set of functions $f \in C(I) \cap C^2(I^0)$ satisfying $Lf(x) = a(x)f^{(2)}(x) + b(x)f^{(1)}(x) = g(x)$ on I^0 for some $g \in C(I)$ and define $Lf = g$ on I. We denote $D(L)$ also by $C(I) \cap C^2(I^0) \cap \{Lf \in C(I)\}$.

Fix $r \in I^0$. According to Feller's result, the boundary points r_0 (or r_1) are classified into the regular-boundary, the exit-boundary, the entrance-boundary and the natural-boundary. To this purpose, we introduce the following quantities:

$$u(r_i) = \int_r^{r_i} m(x)\, ds(x), \quad v(r_i) = \int_r^{r_i} s(x)\, dm(x),$$

where

$$m(x) = \int_r^x a(y)^{-1} e^{B(y)}\, dy, \quad s(x) = \int_r^x e^{-B(y)}\, dy$$

and

$$B(x) = \int_r^x b(y) a(y)^{-1}\, dy.$$

The boundary point $r_i (i = 0, 1)$ is called

regular	in case	$u(r_i) < \infty$	and	$v(r_i) < \infty$
exit	in case	$u(r_i) < \infty$	and	$v(r_i) = \infty$
entrance	in case	$u(r_i) = \infty$	and	$v(r_i) < \infty$
natural	in case	$u(r_i) = \infty$	and	$v(r_i) = \infty$

(the conditions are independent of the choice of r). Note that r_i is regular if and only if both $m(r_i)$ and $s(r_i)$ are finite. If r_i is entrance, then $m(r_i)$ is finite but $|s(r_i)| = \infty$. If $|s(r_i)| = \infty$ and $v(r_i)$ is finite, then r_i is entrance. If $|m(r_i)| = \infty$ and $u(r_i)$ is finite, then r_i is exit. Moreover if r_i is regular and $b(x) \in C(I)$, $\lim_{x \to r_i} e^{B(x)}$ has a finite limit $e^{B(r_i)}$ because

$$e^{B(x)} - 1 = \int_r^x a(y)^{-1} b(y) e^{B(y)}\, dy$$

and

$$\left| \int_r^{r_i} |a(y)^{-1} b(y) e^{B(y)}|\, dy \right| \le \|b\|_0 \left| \int_r^{r_i} a(y)^{-1} e^{B(y)}\, dy \right|$$
$$= \|b\|_0 |m(r_i)| < \infty.$$

Now we will state the uniqueness of the solution to the martingale problem on I for L in the general form as follows.

THEOREM 1. *Assume that $a(x), b(x) \in C(I)$ with $a(x) > 0$ on I^0 and* (1.1) *holds. Moreover if we assume $b(r_i) \neq 0$ $(i = 0, 1)$ in case that r_i is regular with $e^{B(r_i)} = 0$ or entrance, then for each $x \in I$ we have the uniqueness of the solution to the martingale problem on I for L starting at x. Conversely if the solution to the martingale problem on I for L starting at the boundary r_i is unique for $i = 0$ or 1 and r_i is regular with $e^{B(r_i)} = 0$ or entrance, then we have $b(r_i) \neq 0$.*

REMARK 1. For each $x \in I$, let P_x be the unique solution to the martingale problem on I for L starting at $x \in I$ and define

$$T_t f(x) = E^{P_x}[f(x(t))], \quad t \ge 0, \tag{2.1}$$

where E^{P_x} stands for the expectation by P_x. Then by results of Stroock and

Varadhan [12] $\{T_t: t \geq 0\}$ is a strongly continuous nonnegative semigroup on $C(I)$.

As for the differentiability preserving properties of $\{T_t: t \geq 0\}$, we treat the case where $\lim_{x\to r_i} a(x)/|x - r_i| = \infty$ holds for $i = 0$ and 1.

THEOREM 2. *Assume that* $a(x) \in C^1(I^0) \cap C(I)$, $a(x) > 0$ *on* I^0, $\lim_{x\to r_i} a^{(1)}(x)(-1)^i = \infty$ *for* $i = 0$ *and* 1, $b(x) \in C^1(I)$, *and* (1.1) *holds. Then the following conclusions are valid.*

(i) *For each* $x \in I$, *the martingale problem on* I *for* L *starting at* $x \in I$ *has a unique solution* P_x.

(ii) T_t, $t \geq 0$, *defined by* (2.1), *maps* $C^1(I)$ *into itself, the restriction of* $\{T_t: t \geq 0\}$ *to* $C^1(I)$ *is a strongly continuous semigroup in the norm* $\|\cdot\|_1$ *with* $\|T_t\|_1 \leq e^{t\xi_1}$, *and the domain of its infinitesimal generator is the restriction of* L *to* $C_0^2 \cap C^3(I^0) \cap \{Lf \in C^1(I)\}$, *where* $\xi_1 = \|b^{(1)}\|_0$.

REMARK 2. It is easily seen from the proof of Theorem 2 that, in case

$$\left| \int_r^{r_i} \frac{1}{a(x)}\, dx \right| < \infty \quad \text{for} \quad i = 0 \text{ or } 1,$$

we can replace the condition $\lim_{x\to r_i} a^{(1)}(x)(-1)^i = \infty$ by the weaker condition $\lim_{x\to r_i} a(x)/|x - r_i| = \infty$ in Theorem 2.

THEOREM 3. *Assume* $a(x) \in C^2(I^0) \cap C(I)$, $a(x) > 0$ *on* I^0,

$$\lim_{x\to r_i} a^{(1)}(x)\,(-1)^i = \infty$$

for $i = 0$ *and* 1, $\sup_{x\in I^0} a^{(2)}(x) < \infty$, $b(x) \in C^2(I)$, *and* (1.1) *holds. Then we have the following results for the semigroup* $\{T_t: t \geq 0\}$, *defined by* (2.1), *which is associated with the unique solutions to the martingale problem on* I *for* L.

(i) $\lambda - A$ *is a one-to-one map from* $\mathscr{D} = C_0^2(I) \cap C^4(I^0) \cap \{Lf \in C^2(I)\}$ *onto* $C^2(I)$ *with* $\|(\lambda - A)^{-1}\|_2 \leq (\lambda - \xi_2)^{-1}$ *if* $\lambda > \xi_2$, *where* A *stands for the infinitesimal generator of* $\{T_t: t \geq 0\}$, $\xi_2 = \max\{2\|b^{(1)}\|_0 + k, \|b^{(1)}\|_1\}$ *and* $k = \max\{0, \sup_{x\in I^0} a^{(2)}(x)\}$.

(ii) *If* $b(x) \in C_0^2(I)$, T_t, $t \geq 0$, *maps* $C_0^2(I)$ *into itself, the restriction of* $\{T_t: t \geq 0\}$ *to* $C_0^2(I)$ *is a strongly continuous semigroup in the norm* $\|\cdot\|_2$ *with* $\|T_t\|_2 \leq e^{\xi_2 t}$, *and the domain of its infinitesimal generator is the restriction of* L *to* $C_0^2(I) \cap C^4(I^0) \cap \{Lf \in C_0^2(I)\}$.

3. Some lemmas. We now prepare several lemmas for the proofs of theorems. Let notations and symbols not explained in this section be those stated in Section 2. Especially note that $a(x), b(x) \in C(I^0)$ with $a(x) > 0$ on I^0.

LEMMA 1. *Let* i *be* 0 *or* 1 *and assume that* r_i *for* L *is entrance. Then we have the following results.*

(i) *If* $\lim_{x\to r_i} b(x)(-1)^i = \infty$, $f \in C(I) \cap C^2(I^0) \cap \{Lf \in C(I^0 \cup \{r_i\})\}$ *implies* $\lim_{x\to r_i} f^{(1)}(x) = 0$.

(ii) *If* $b(x) \in C(I)$ *and* $b(r_i) \neq 0$, *we have*

$$C(I) \cap C^2(I^0) \cap \{Lf \in C(I^0 \cup \{r_i\})\}$$
$$= C(I) \cap C^1(I^0 \cup \{r_i\}) \cap C^2(I^0) \cap \{\lim_{x \to r_i} a(x) f^{(2)}(x) = 0\}.$$

PROOF. Since r_i is entrance, we have

$$|s(r_i)| = \infty \tag{3.1}$$

and

$$|m(r_i)| < \infty. \tag{3.2}$$

Further, it follows from (3.1) that

$$\liminf_{x \to r_i} b(x)(-1)^i > 0 \tag{3.3}$$

for the case $b(x) \in C(I)$ and $b(r_i) \neq 0$ as well as the case $\lim_{x \to r_i} b(x)(-1)^i = \infty$. Then, from (3.1) and (3.3), we get

$$\lim_{x \to r_i} e^{B(x)} = 0. \tag{3.4}$$

Now, for $f \in C(I) \cap C^2(I^0) \cap \{Lf \in C(I^0 \cup \{r_i\})\}$, let $g = Lf$. By solving this differential equation on I^0 (see Mandl [9], page 22, Lemma 2), we have

$$f^{(1)}(x) = e^{-B(x)}\{f^{(1)}(r) + J(x)\} \quad \text{on} \quad I^0,$$

where

$$J(x) = \int_r^x a(y)^{-1} g(y) e^{B(y)}\, dy.$$

From (3.2), $J(r_i) = \lim_{x \to r_i} J(x)$ exists and is finite. Further, it follows from $f \in C(I)$ and (3.1) that $f^{(1)}(r) = -J(r_i)$. Consequently, we have

$$f^{(1)}(x) = e^{-B(x)}\{J(x) - J(r_i)\} \quad \text{on} \quad I^0. \tag{3.5}$$

Then, applying l'Hospital's rule, it follows from (3.4) and (3.5) that

$$\lim_{x \to r_i} f^{(1)}(x) = 0 \quad \text{if} \quad \lim_{x \to r_i} b(x)(-1)^i = \infty \tag{3.6}$$

and

$$\lim_{x \to r_i} f^{(1)}(x) = g(r_i)/b(r_i) \quad \text{if} \quad b \in C(I) \quad \text{and} \quad b(r_i) \neq 0. \tag{3.7}$$

Therefore, as for assertion (i), it is obtained from (3.6). For (ii), we have from (3.7) that

$$f \in C^1(I^0 \cup \{r_i\}) \quad \text{and} \quad \lim_{x \to r_i} a(x) f^{(2)}(x) = g(r_i) - b(r_i) f^{(1)}(r_i) = 0.$$

Conversely, it is obvious that $f \in C(I) \cap C^1(I^0 \cup \{r_i\}) \cap C^2(I^0) \cap \{\lim_{x \to r_i} a(x) f^{(2)}(x) = 0\}$ implies $f \in C(I) \cap C^2(I^0) \cap \{Lf \in C(I^0 \cup \{r_i\})\}$. Hence assertion (ii) follows. □

LEMMA 2. *Let i be 0 or 1 and assume that r_i for L is regular and $b(x) \in C(I)$.*

Moreover assume $b(r_i) \neq 0$ *if* $e^{B(r_i)} = 0$. *Then we have*

$$C(I) \cap C^2(I^0) \cap \{Lf \in C(I^0 \cup \{r_i\})\} \cap \{\mu(r_i)D_s^+ f(r_i)(-1)^i = \delta(r_i)Lf(r_i)\} \\ = C(I) \cap C^1(I^0 \cup \{r_i\}) \cap C^2(I^0) \cap \{\lim_{x\to r_i} a(x)f^{(2)}(x) = 0\}, \tag{3.8}$$

where $\mu(r_i) = b(r_i)(-1)^i$, $\delta(r_i) = e^{B(r_i)}$ *and* $D_s^+ f(r_i) = \lim_{x\to r_i} f^{(1)}(x)e^{B(x)}$.

PROOF. For $f \in C(I) \cap C^2(I^0) \cap \{Lf \in C(I^0 \cup \{r_i\})\}$, we let $g = Lf$; then we have

$$f^{(1)}(x) = e^{-B(x)}\{f^{(1)}(r) + J(x)\} \quad \text{on} \quad I^0, \tag{3.9}$$

where $J(x) = \int_r^x a(y)^{-1} g(y) e^{B(y)} dy$.

In case $e^{B(r_i)} = 0$, from $b(r_i) \neq 0$, we have

$$C(I) \cap C^2(I^0) \cap \{Lf \in C(I^0 \cup \{r_i\})\} \\ \cap \{\mu(r_i)D_s^+ f(r_i)(-1)^i = \delta(r_i)Lf(r_i)\} \\ = C(I) \cap C^2(I^0) \cap \{Lf \in C(I^0 \cup \{r_i\})\} \cap \{D_s^+ f(r_i) = 0\}. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$f^{(1)}(r) + J(r_i) = \lim_{x\to r_i} f^{(1)}(x)e^{B(x)} = D_s^+ f(r_i) = 0$$

for f which belongs to the left side of (3.8) (It should be noted that r_i being regular implies that $J(r_i) = \lim_{x\to r_i} J(x)$ has a finite limit). Then, applying l'Hospital's rule to (3.9), we have

$$\lim_{x\to r_i} f^{(1)}(x) = g(r_i)b(r_i)^{-1}$$

and hence

$$f \in C^1(I^0 \cup \{r_i\})$$

and

$$\lim_{x\to r_i} a(x)f^{(2)}(x) = g(r_i) - b(r_i)f^{(1)}(r_i) = 0$$

for f which belongs to the left side of (3.8). Noting (3.10), the converse part of this case is easily seen because of $Lf \in C(I^0 \cup \{r_i\})$ and $D_s^+ f(r_i) = 0$ for f which belongs to the right side of (3.8).

In case $e^{B(r_i)} > 0$, it follows from (3.9) and $\lim_{x\to r_i} J(x)$ having a finite limit that $\lim_{x\to r_i} f^{(1)}(x)$ has a finite limit $f^{(1)'}(r_i)$. Hence we have also

$$\lim_{x\to r_i} a(x)f^{(2)}(x) = g(r_i) - b(r_i)f^{(1)}(r_i) \\ = Lf(r_i) - \mu(r_i)e^{-B(r_i)}D_s^+ f(r_i)(-1)^i = 0$$

for f which belongs to the left side of (3.8). The converse part of this case is obvious since it is easily seen that $Lf \in C(I^0 \cup \{r_i\})$ and

$$\mu(r_i)D_s^+ f(r_i)(-1)^i = \delta(r_i)Lf(r_i)$$

for f which belongs to the right side of (3.8). □

LEMMA 3. *Let $a(x), b(x) \in C(I)$ and define*

$$\bar{D} = C^1(I) \cap C^2(I^0) \cap \{\lim_{x \to r_i} a(x) f^{(2)}(x) = 0 \text{ for } i = 0 \text{ and } 1\}.$$

Then, for each positive integer n and $f \in \bar{D}$, there exists $f_n \in C^2(I)$ such that $\| f_n - f \|_0 \to 0$ and $\| Lf_n - Lf \|_0 \to 0$ as $n \to \infty$.

PROOF. For each $f \in \bar{D}$, let $Lf = g \in C(I)$ noting $\bar{D} \subset \{Lf \in C(I)\}$. Then, for $n = 1, 2, \cdots$, we can choose sequences $\{x_n\}$ and $\{y_n\} \subset I$ in such a way that $r_0 \leq y_n \leq x_n < r$, $x_n \to r_0$ as $n \to \infty$,

$$0 < x_n - y_n \leq 1/(n \,|\, f^{(2)}(x_n) \,|) \quad \text{if} \quad f^{(2)}(x_n) \neq 0,$$

$$|\, f^{(2)}(x_n) \,| \,(x_n - y_n) \leq 1/n, \quad \sup_{r_0 \leq x \leq x_n} a(x) = a(x_n),$$

and

$$a(x) \,|\, f^{(2)}(x) \,| + |\, f^{(1)}(x) - f^{(1)}(x_n) \,| \leq 1/n$$

for all $x \in [r_0, x_n]$. We can also choose sequences $\{z_n\}$ and $\{v_n\} \subset I$ for each $n = 1, 2, \cdots$ in such a way that $r < z_n \leq v_n \leq r_1$, $z_n \to r_1$ as $n \to \infty$,

$$0 < v_n - z_n \leq 1/(n \,|\, f^{(2)}(z_n) \,|) \quad \text{if} \quad f^{(2)}(z_n) \neq 0,$$

$$|\, f^{(2)}(z_n) \,| \,(v_n - z_n) \leq 1/n, \quad \sup_{z_n \leq x \leq r_1} a(x) = a(z_n)$$

and

$$a(x) \,|\, f^{(2)}(x) \,| + |\, f^{(1)}(x) - f^{(1)}(z_n) \,| \leq 1/n$$

for all $x \in [z_n, r_1]$. For each $n = 1, 2, \cdots$, let $h_n(x)$ be a continuous function on I such that $h_n(x) = f^{(2)}(x)$ for $x \in [x_n, z_n]$, $h_n(x) = 0$ for $x \in [r_0, y_n] \cup [v_n, r_1]$, $|\, h_n(x) \,| \leq |\, f^{(2)}(x_n) \,|$ for $x \in [y_n, x_n]$, and $|\, h_n(x) \,| \leq |\, f^{(2)}(z_n) \,|$ for $x \in [z_n, v_n]$. Using $h_n(x)$, we define a C^2-function $f_n(x)$ on I by

$$f_n(x) = \int_r^x \int_r^y h_n(z)\, dz\, dy + f^{(1)}(r)(x - r) + f(r).$$

Then it is easily seen that $\| f_n - f \|_0$ and $\| Lf_n - g \|_0 \to 0$ as $n \to \infty$. □

LEMMA 4. *Let i be 0 or 1 and assume that $a(x) \in C^1(I^0)$, $|\int_r^{r_i} a(x)^{-1}\, dx \,| = \infty$ and $\liminf_{x \to r_i} a(x) \,|\, x - r_i \,|^{-1} > 0$. Suppose further that $b(x)$ is of the form $b(x) = a^{(1)}(x) + \tilde{b}(x)$, where $\tilde{b}(x)$ is Lipschitz continuous on $I^0 \cup \{r_i\}$ and satisfying $\tilde{b}(r_i)(-1)^i \geq 0$. Then r_i for L is entrance.*

PROOF. Let $r_i = r_1$ (the case $r_i = r_0$ is similar). Assume $\tilde{b}(r_1) = 0$. Then there are positive constants C_1 and C_2 such that

$$a(x)^{-1}(r_1 - x) \leq C_1 \quad \text{and} \quad |\, \tilde{b}(x) \,| \,(r_1 - x)^{-1} \leq C_2 \quad \text{on} \quad [r, r_1).$$

From these and $\int_r^{r_1} 1/a = \infty$, we have that

$$v(r_1) = \int_r^{r_1} \left\{ \int_r^x a(y)^{-1} \exp\left(- \int_r^y a(z)^{-1} \tilde{b}(z)\, dz \right) dy \right\} \cdot \exp\left(\int_r^x a(y)^{-1} \tilde{b}(y)\, dy \right) dx$$

$$\leq \exp(2(r_1 - r) C_1 C_2) \int_r^{r_1} \left\{ \int_r^x C_1 (r_1 - y)^{-1}\, dy \right\} dx < \infty$$

and

$$s(r_1) = \int_r^{r_1} \frac{a(r)}{a(x)} \exp\left(- \int_r^x \frac{\tilde{b}(y)}{a(y)}\, dy \right) dx$$

$$\geq a(r) \exp(-(r_1 - r) C_1 C_2) \int_r^{r_1} \frac{dx}{a(x)} = \infty.$$

Therefore it follows easily from these that r_1 is entrance in the case $\tilde{b}(r_1) = 0$.

Next, assume $\tilde{b}(r_1) \neq 0$. Then there are positive constants δ and ε ($< r_1 - r$) such that

$$\tilde{b}(x) \leq -\delta \quad \text{for all} \quad x \in [r_1 - \varepsilon, r_1]. \tag{3.11}$$

From this, we have that

$$\int_{r_1-\varepsilon}^{r_1} \left\{ \int_{r_1-\varepsilon}^{x} \exp\left(- \int_{r_1-\varepsilon}^{y} \frac{b(z)}{a(z)}\, dz \right) dy \right\} a(x)^{-1} \exp\left(\int_{r_1-\varepsilon}^{x} \frac{b(y)}{a(y)}\, dy \right) dx$$

$$\leq \frac{1}{\delta} \int_{r_1-\varepsilon}^{r_1} \left\{ \int_{r_1-\varepsilon}^{x} \frac{-\tilde{b}(y)}{a(y)} \exp\left(- \int_{r_1-\varepsilon}^{y} \frac{\tilde{b}(z)}{a(z)}\, dz \right) dy \right\} \exp\left(\int_{r_1-\varepsilon}^{x} \frac{\tilde{b}(y)}{a(y)}\, dy \right) dx$$

$$\leq \frac{\varepsilon}{\delta} < \infty$$

and so

$$v(r_1) < \infty. \tag{3.12}$$

We also have from (3.11) and $\int_r^{r_1} 1/a = \infty$ that

$$\int_{r_1-\varepsilon}^{r_1} \exp\left(- \int_{r_1-\varepsilon}^{x} \frac{b(y)}{a(y)}\, dy \right) dx \geq \int_{r_1-\varepsilon}^{r_1} \frac{a(r_1 - \varepsilon)}{a(x)} \exp\left(\delta \int_{r_1-\varepsilon}^{x} \frac{dy}{a(y)} \right) dx$$

$$\geq a(r_1 - \varepsilon) \int_{r_1-\varepsilon}^{r_1} \frac{dx}{a(x)} = \infty$$

and so

$$s(r_1) = \infty. \tag{3.13}$$

Hence, from (3.12) and (3.13), we conclude easily that r_1 is entrance in the case $\tilde{b}(r_1) \neq 0$ also. □

REMARK 3. In the case $\tilde{b} \in C(I^0)$ and $\lim\inf_{x\to r_i}\tilde{b}(x)(-1)^i > 0$, we can drop the conditions $\lim\inf_{x\to r_i} a(x)|x - r_i|^{-1} > 0$ and the Lipschitz continuity of $\tilde{b}(x)$ on I in Lemma 4.

LEMMA 5. *Let i be 0 or 1 and assume that* $|\int_r^{r_i} dx/a(x)| < \infty$ *and* $\lim_{x\to r_i} a(x)|x - r_i|^{-1} = \infty$. *Suppose further that* $a(x) \in C^1(I^0)$ *and* $b(x)$ *is of the form* $b(x) = a^{(1)}(x) + \tilde{b}(x)$ *for a continuous function* $\tilde{b}(x)$ *on* $I^0 \cup \{r_i\}$. *Then we have*

$$\begin{aligned} C(I) \cap C^2(I^0) \cap \{Lf \in C(I^0 \cup \{r_i\})\} \cap \{D_s^+ f(r_i) = 0\} \\ = C(I) \cap C^2(I^0) \cap C^1(I^0 \cup \{r_i\}) \cap \{f^{(1)}(r_i) = 0\} \\ \cap \{Lf \in C(I^0 \cup \{r_i\})\}. \end{aligned} \tag{3.14}$$

PROOF. Let f belong to the left side of (3.14) and $g = Lf$. Then we have

$$f^{(1)}(x) = \frac{a(r)}{a(x)} \exp\left(-\int_r^x \frac{\tilde{b}(y)}{a(y)}\, dy\right) \{f^{(1)}(r) + J(x)\} \quad \text{on} \quad I^0, \tag{3.15}$$

where

$$J(x) = \int_r^x a(r)^{-1} g(y) \exp\left(\int_r^y a(z)^{-1}\tilde{b}(z)\, dz\right) dy.$$

Moreover we have

$$\lim_{x\to r_i}\{f^{(1)}(r) + J(x)\} = \lim_{x\to r_i} f^{(1)}(x) e^{B(x)} = D_s^+ f(r_i) = 0 \tag{3.16}$$

and $J(x) \in C^1(I^0 \cup \{r_i\})$ because of $|\int_r^{r_i} 1/a| < \infty$. Then, applying the mean value theorem to $J(x)$, we get $J(x) + f^{(1)}(r) = O(|x - r_i|)$. It therefore follows from (3.15) and $|\int_r^{r_i} 1/a| < \infty$ that

$$f^{(1)}(x) = (1/a(x))O(|x - r_i|).$$

Consequently, from $\lim_{x\to r_i} a(x)/|x - r_i| = \infty$, we conclude $\lim_{x\to r_i} f^{(1)}(x) = 0$. As for the converse part of (3.14), it follows easily from

$$D_s^+ f(r_i) = \lim_{x\to r_i} \frac{f^{(1)}(x)a(x)}{a(r)} \exp\left(\int_r^x \frac{\tilde{b}(y)}{a(y)}\, dy\right) = 0$$

for f which belongs to the right side of (3.14). □

LEMMA 6. *Assume that*

$$\lim\inf_{x\to r_i} \frac{a(x)}{|x - r_i|} > 0, \tag{3.17}$$

(3.18) $$|b(x) - b(y)| \leq C|x - y|$$

for all $x, y \in I$ and some positive constant C, and

(3.19) $$b(r_i)(-1)^i \geq 0.$$

Then we have

(i) *r_i is exit if $|\int_r^{r_i} a(x)^{-1}\,dx| = \infty$ and $b(r_i) = 0$,*

(ii) *r_i is entrance if $b(r_i) \neq 0$ and $|s(r_i)| = \infty$ (therefore $|\int_r^{r_i} a(x)^{-1}\,dx| = \infty$),*

(iii) *r_i is regular otherwise* (i.e. $|\int_r^{r_i} a(x)^{-1}\,dx| < \infty$ *or both* $b(r_i) \neq 0$ *and* $|s(r_i)| < \infty$).

Moreover define $D = D_0 \cap D_1$, where

(a) *$D_i = D(L) \cap \{Lf(r_i) = 0\}$ in case* (i),

(b) *$D_i = D(L)$ in case* (ii)

and

(c) *$D_i = D(L) \cap \{\mu(r_i)D_s^+ f(r_i)(-1)^i = \delta(r_i)Lf(r_i)\}$ in case* (iii) *($\mu(r_i)$ and $\delta(r_i)$ are those defined in Lemma* 2).

Then we have the following results:

(iv) *$D = \bar{D} = \bar{D}_0 \cap \bar{D}_1$, where*

$$\bar{D}_i = \tilde{D} \cap C(I) \cap \{\lim_{x\to r_i} b(x)f^{(1)}(x) = 0\}$$

$$\text{if} \quad \left| \int_r^{r_i} \frac{dx}{a(x)} \right| = \infty \quad \text{and} \quad b(r_i) = 0,$$

$$\bar{D}_i = \tilde{D} \cap C^1(I^0 \cup \{r_i\}) \quad \text{otherwise}$$

and

$$\tilde{D} = C^2(I^0) \cap \{\lim_{x\to r_i} a(x)f^{(2)}(x) = 0\}.$$

PROOF. (i) It follows from (3.17), (3.18) and $b(r_i) = 0$ that

(3.20) $$\left| \int_r^{r_i} \left| \frac{b(x)}{a(x)} \right| dx \right| < \infty$$

and hence, from (3.17) again,

(3.21) $$u(r_i) < \infty.$$

We also have from (3.20) and $|\int_r^{r_i} 1/a| = \infty$ that

(3.22) $$|m(r_i)| = \infty.$$

Consequently (3.21) and (3.22) imply that r_i is exit.

(ii) From (3.18), (3.19) and $b(r_i) \neq 0$, we have $\lim\inf_{x\to r_i} b(x)(-1)^i > 0$ easily and so, by similar calculations to those done in the proof of Lemma 4, $v(r_i) < \infty$. Hence we conclude easily from this and $|s(r_i)| = \infty$ that r_i is entrance.

(iii) If $|\int_r^{r_i} 1/a| < \infty$, then we have $u(r_i)$, $v(r_i) < \infty$ by simple calculations and hence r_i is regular. If $b(r_i) \neq 0$, then, from (3.18) and (3.19), the same calculations as done in (ii) yield $v(r_i) < \infty$. Hence, from this, we conclude easily that r_i is regular if $b(r_i) \neq 0$ and $|s(r_i)| < \infty$. Thus (iii) follows.

(iv) From the results of (i), (ii) and (iii), we have $b(r_i) \neq 0$ in case r_i is entrance or in case r_i is regular and $e^{B(r_i)} = 0$. Therefore, from (ii) of Lemma 1 and Lemma 2, we get $D_i = \bar{D}_i$ in case (b) or (c). Then the only thing left to prove is $D_i = \bar{D}_i$ in the case (a). For $f \in D_i$, let $g = Lf$. Then solving this equation yields

$$f^{(1)}(x) = e^{-B(x)}\left\{f^{(1)}(r) + \int_r^x \frac{g(y)}{a(y)} e^{B(y)}\, dy\right\} \quad \text{on} \quad I^0$$

and so, from (3.18), (3.20) and $b(r_i) = 0$, it follows that

$$|b(x)f^{(1)}(x)| \leq C_1|x - r_i| + C_2\|g\|_0|x - r_i|\left|\int_r^x \frac{dy}{a(y)}\right| \quad \text{on} \quad I^0$$

for some positive constants C_1 and C_2. Since, by (3.17), the second term of the right side vanishes at r_i, we therefore have

$$\lim_{x\to r_i} b(x)f^{(1)}(x) = 0$$

and consequently, because of $Lf(r_i) = 0$,

$$\lim_{x\to r_i} a(x)f^{(2)}(x) = Lf(r_i) - \lim_{x\to r_i} b(x)f^{(1)}(x) = 0.$$

From these we conclude $f \in \bar{D}_i$, that is, $D_i \subset \bar{D}_i$. The converse part $D_i \supset \bar{D}_i$ is obvious because of $Lf(r_i) = \lim_{x\to r_i}\{a(x)f^{(2)}(x) + b(x)f^{(1)}(x)\} = 0$ for $f \in \bar{D}_i$ and this completes the proof of Lemma 6. □

4. Proofs of main results.

PROOF OF THEOREM 1. *The first half.* Since $a(x) > 0$ on I^0, it suffices to show the uniqueness of the solution starting at the boundary. We prove the uniqueness for the following two cases since the other cases can be reduced to these cases.

CASE 1. The boundary points r_0 and r_1 are exit or natural. We will show that the path starting at the boundary point r_i never enters into I^0 in this case. Then the uniqueness follows immediately.

For $x \in I$, let P_x be a solution to the martingale problem on I for L starting at x and E_x denote the expectation by P_x. Let

$$u_\varepsilon(x) = \int_{r_0+\varepsilon}^x e^{-B(y)}\, dy \int_{r_0+\varepsilon}^y a(z)^{-1}e^{B(z)}\, dz + 1 \in C^2(I^0)$$

for sufficiently small $\varepsilon > 0$. It is obvious that $u_\varepsilon > 0$, $Lu_\varepsilon \leq u_\varepsilon$ on I^0 and $\lim_{\varepsilon\downarrow 0} u_\varepsilon(x) = \infty$ for each $x \in I^0$ since r_0 is exit or natural. For each $n \geq 1$ and any $\alpha \in (r_0 + \varepsilon, r_1)$, define the stopping times $\tau_n(\varepsilon, \alpha)$ and $\sigma_n(\varepsilon)$ as follows:

$$\tau_n(\varepsilon, \alpha) = \inf\{t \geq \sigma_{n-1}(\varepsilon)\colon x(t) = r_0 + \varepsilon/2 \text{ or } \alpha\}$$

and

$$\sigma_n(\varepsilon) = \inf\{t \geq \tau_n(\varepsilon, \alpha): x(t) = r_0 + \varepsilon\},$$

where $\tau_0(\varepsilon, \alpha) = \sigma_0(\varepsilon) = 0$. Further let $\tau(y)$ be the first hitting time to $\{y\}$ for $y \in I^0$. Then applying (b) of Theorem 2.1 in Stroock and Varadhan [13] and the optional sampling theorem to $f_\varepsilon(t, x) = e^{-t}u_\varepsilon(x)$, we have

$$\begin{aligned} &E_{r_0+\varepsilon}[\exp(-\tau_1(\varepsilon, \alpha))u_\varepsilon(x(\tau_1(\varepsilon, \alpha))) - u_\varepsilon(x(0))] \\ &= \lim_{t\uparrow\infty} E_{r_0+\varepsilon}[\exp(-\tau_1(\varepsilon, \alpha)\wedge t)u_\varepsilon(x(\tau_1(\varepsilon, \alpha)\wedge t)) - u_\varepsilon(x(0))] \qquad (4.1) \\ &= \lim_{t\uparrow\infty} E_{r_0+\varepsilon}\left[\int_0^{\tau_1(\varepsilon,\alpha)\wedge t} e^{-u}(Lu_\varepsilon - u_\varepsilon)(x(u))\, du\right] \leq 0. \end{aligned}$$

On the other hand, from $0 \leq \sigma_{i-1}(\varepsilon) \leq \tau_i(\varepsilon, \alpha) \leq \sigma_i(\varepsilon) \leq \infty$ and $u_\varepsilon(r_0 + \varepsilon/2) > u_\varepsilon(r_0 + \varepsilon) = 1$, we have

$$\begin{aligned} &u_\varepsilon(\alpha)E_{r_0+\varepsilon}[e^{-\tau(\alpha)}] \\ &= E_{r_0+\varepsilon}[e^{-\tau(\alpha)}u_\varepsilon(x(\tau(\alpha)))] \\ &= \lim_{n\to\infty} E_{r_0+\varepsilon}[e^{-\tau_n(\varepsilon,\alpha)\wedge\tau(\alpha)}u_\varepsilon(x(\tau_n(\varepsilon, \alpha) \wedge \tau(\alpha)))] \\ &= \lim_{n\to\infty} E_{r_0+\varepsilon}[\textstyle\sum_{i=1}^n\{\{e^{-\tau_i(\varepsilon,\alpha)\wedge\tau(\alpha)}u_\varepsilon(x(\tau_i(\varepsilon, \alpha) \wedge \tau(\alpha))) \\ &\qquad - e^{-\sigma_{i-1}(\varepsilon)\wedge\tau(\alpha)}u_\varepsilon(x(\sigma_{i-1}(\varepsilon) \wedge \tau(\alpha)))\} \\ &\qquad + \{e^{-\sigma_{i-1}(\varepsilon)\wedge\tau(\alpha)}u_\varepsilon(x(\sigma_{i-1}(\varepsilon) \wedge \tau(\alpha))) \\ &\qquad - e^{-\tau_{i-1}(\varepsilon,\alpha)\wedge\tau(\alpha)}u_\varepsilon(x(\tau_{i-1}(\varepsilon,\alpha) \wedge \tau(\alpha)))\}\} + u_\varepsilon(x(0))] \\ &= \lim_{n\to\infty} \textstyle\sum_{i=1}^n \{E_{r_0+\varepsilon}[\{\sigma_{i-1}(\varepsilon) < \tau(\alpha)\}: e^{-\tau_i(\varepsilon,\alpha)}u_\varepsilon(x(\tau_i(\varepsilon, \alpha))) \\ &\qquad - e^{-\sigma_{i-1}(\varepsilon)}u_\varepsilon(r_0 + \varepsilon)] + E_{r_0+\varepsilon}[\{\sigma_{i-1}(\varepsilon) \leq \tau(\alpha)\}: \\ &\qquad e^{-\sigma_{i-1}(\varepsilon)}u_\varepsilon(r_0 + \varepsilon) - e^{-\tau_{i-1}(\varepsilon,\alpha)}u_\varepsilon(r_0 + \varepsilon/2)]\} + 1 \\ &\leq \limsup_{n\to\infty} \textstyle\sum_{i=1}^n E_{r_0+\varepsilon}[\{\sigma_{i-1}(\varepsilon) < \tau(\alpha)\}: e^{-\tau_i(\varepsilon,\alpha)}u_\varepsilon(x(\tau_i(\varepsilon, \alpha))) \\ &\qquad - e^{-\sigma_{i-1}(\varepsilon)}u_\varepsilon(r_0 + \varepsilon)] + 1 \\ &= \limsup_{n\to\infty} \textstyle\sum_{i=1}^n \lim_{t\uparrow\infty} E_{r_0+\varepsilon}[\{\sigma_{i-1}(\varepsilon) < \tau(\alpha) \wedge t\}: e^{-\sigma_{i-1}(\varepsilon)} \\ &\qquad \cdot E_{r_0+\varepsilon}[e^{-\tau_1(\varepsilon,\alpha)(\theta_{i-1,t}\omega)}u_\varepsilon(x(\tau_1(\varepsilon, \alpha)(\theta_{i-1,t}\omega))) \\ &\qquad - u_\varepsilon(r_0 + \varepsilon) \mid \mathscr{N}_{\sigma_{i-1}(\varepsilon)\wedge t}]] + 1 \end{aligned}$$

for $r_0 + \varepsilon < \alpha$, where $\mathscr{N}_{\sigma_{i-1}(\varepsilon)\wedge t}$ is the σ-field associated with $\sigma_{i-1}(\varepsilon)\wedge t$, $\theta_{i-1,t}$ the shift operator such that $x(s, \theta_{i-1,t}\omega) = x(s + \sigma_{i-1}(\varepsilon)\wedge t, \omega)$ and $a\wedge b = \min\{a, b\}$. Note that $P_{r_0+\varepsilon}[\theta_{i-1,t}^{-1}[\cdot] \mid \mathscr{N}_{\sigma_{i-1}(\varepsilon)\wedge t}](\omega)$ is a solution to the martingale problem on I for L starting at $x(\sigma_{i-1}(\varepsilon)\wedge t, \omega)P_{r_0+\varepsilon}$ – a.s. and $P_{r_0+\varepsilon}$ is unique on $\mathscr{N}_{\tau_1(\varepsilon,\alpha)\wedge s}$ for all $s \geq 0$, where $\mathscr{N}_{\tau_1(\varepsilon,\alpha)\wedge s}$ is the σ-field associated with $\tau_1(\varepsilon, \alpha)\wedge s$. Consequently it follows that

$$
\begin{aligned}
&u_\varepsilon(\alpha)E_{r_0+\varepsilon}[e^{-\tau(\alpha)}]\\
&\quad\le \limsup_{n\to\infty} \textstyle\sum_{i=1}^{n} \lim_{t\uparrow\infty} E_{r_0+\varepsilon}[\{\sigma_{i-1}(\varepsilon) < \tau(\alpha)\wedge t\}\colon e^{-\sigma_{i-1}(\varepsilon)}]\\
&\qquad\cdot E_{r_0+\varepsilon}[e^{-\tau_1(\varepsilon,\alpha)}u_\varepsilon(x(\tau_1(\varepsilon,\alpha))) - u_\varepsilon(x(0))] + 1\\
&\quad\le \limsup_{n\to\infty} E_{r_0+\varepsilon}[e^{-\tau_1(\varepsilon,\alpha)}u_\varepsilon(x(\tau_1(\varepsilon,\alpha))) - u_\varepsilon(x(0))]\\
&\qquad\cdot\{\textstyle\sum_{i=1}^{n} E_{r_0+\varepsilon}[\{\sigma_{i-1}(\varepsilon) < \tau(\alpha)\}\colon e^{-\sigma_{i-1}(\varepsilon)}]\} + 1.
\end{aligned}
\tag{4.2}
$$

Then combining (4.1) and (4.2), we get

$$E_{r_0+\varepsilon}[e^{-\tau(\alpha)}] \le 1/u_\varepsilon(\alpha) \quad \text{for any } \varepsilon > 0 \quad \text{and} \quad \alpha \in (r_0+\varepsilon, r_1).$$

So we have

$$\lim_{\varepsilon\downarrow 0} E_{r_0+\varepsilon}[e^{-\tau(\alpha)}] = 0.$$

It therefore follows from this that, for any $t \ge 0$ and $\alpha \in I^0$,

$$\lim_{x\downarrow r_0} P_x[\tau(\alpha) \le t] = 0 \quad \text{uniformly for any solution } P_x, \tag{4.3}$$

that is, given $t \ge 0$, $\alpha \in I^0$ and $\varepsilon > 0$, we can choose some $\delta > 0$ such that $P_x[\tau(\alpha) \le t] < \varepsilon$ for any $x \in (r_0, r_0+\delta)$ and any solution P_x starting at x.

Let P_{r_0} be a solution starting at r_0, $\mathscr{N}_{\tau(y)\wedge t}$ the σ-field associated with $\tau(y)\wedge t$ and $\theta_{\tau(y)\wedge t}$ the shift operator such that $x(s, \theta_{\tau(y)\wedge t}\omega) = x(s+\tau(y)\wedge t, \omega)$. Then we have

$$
\begin{aligned}
P_{r_0}[\tau(\alpha) \le t] &\le P_{r_0}[\{\tau(y) \le t\} \cap \{\tau(\alpha)(\theta_{\tau(y)\wedge t}\omega) \le t\}]\\
&= E_{r_0}[\{\tau(y) \le t\}\colon P_{r_0}[\theta^{-1}_{\tau(y)\wedge t}[\tau(\alpha) \le t] \mid \mathscr{N}_{\tau(y)\wedge t}]]
\end{aligned}
\tag{4.4}
$$

for $r_0 < y < \alpha$. So noting that $P_{r_0}[\theta^{-1}_{\tau(y)\wedge t}[\cdot] \mid \mathscr{N}_{\tau(y)\wedge t}](\omega)$ is a solution to the martingale problem on I for L starting at $x(\tau(y)\wedge t, \omega)$ P_{r_0} − a.s. and letting $y \downarrow r_0$, we get from (4.3) and (4.4) that $P_{r_0}[\tau(\alpha) \le t] = 0$ for any $t \ge 0$. It therefore follows that $P_{r_0}[\tau(\alpha) < \infty] = 0$ for any $\alpha \in I^0$, that is, $P_{r_0}[x(t) = r_0$ for all $t \ge 0] = 1$. Applying the same way to a solution P_{r_1} starting at r_1, $P_{r_1}[x(t) = r_1$ for all $t \ge 0] = 1$ also follows.

CASE 2. The boundary points r_0 and r_1 are regular or entrance. Let $D = C(I) \cap C^2(I^0) \cap \{Lf \in C(I)\} \cap \{\mu(r_i)D_s^+ f(r_i)(-1)^i = \delta(r_i)Lf(r_i)$ if r_i is regular$\}$, $\mu(r_i) = b(r_i)(-1)^i$ and $\delta(r_i) = e^{B(r_i)}$. Then, by Feller's result, the restriction $L|_D$ of L to D generates a strongly continuous contraction semigroup $\{T_t\colon t \ge 0\}$ on $C(I)$ (for details, refer to Mandl [9], Chapter II). On the other hand, it follows from assumptions (ii) of Lemma 1 and Lemma 2 that $D = C^2(I^0) \cap C^1(I) \cap \{\lim_{x\to r_i} a(x)f^{(2)}(x) = 0$ for $i = 0$ and $1\}$. Hence it follows from this result and Lemma 3 that $L|_D$ is the closure of the restriction of L to $C^2(I)$, that is, $C^2(I)$ is a core for $L|_D$. Consequently, by standard arguments, we get the uniqueness of the solution to the martingale problem on I for L (especially, starting at r_i).

The latter half. Let $r_i = r_1$ and suppose that the conclusion does not hold. That is, let r_1 be regular or entrance, $b(r_1) = 0$, and $e^{B(r_1)} = 0$ if r_1 is regular. As for r_0, we can assume without loss of generality that r_0 for L is the natural boundary point because of the local property of the solution to the martingale problem (see Theorem 6.6.1 in [14]).

Let δ_1 be a probability measure on $(\Omega, \mathscr{N})$ such that $\delta_1[x(t) = r_1 \text{ for all } t \geq 0] = 1$. Then obviously δ_1 is a solution starting at r_1.

On the other hand, let $D = C(I) \cap C^2(I^0) \cap \{Lf \in C(I)\} \cap \{D_s^+ f(r_1) = 0 \text{ if } r_1 \text{ is regular}\}$. Then D contains $C^2(I)$ because of $e^{B(r_1)} = 0$ in case r_1 is regular and, by Feller's result, the restriction $L|_D$ of L to D generates a strongly continuous contraction semigroup $\{T_t: t \geq 0\}$ on $C(I)$. Consequently, in the same way as that of Theorem 4.1 in Stroock and Varadhan [12], we can construct a solution Q_{r_1} to the martingale problem on I for L starting at r_1 such that $T_t f(r_1) = \bar{E}_{r_1}[f(x(t))]$ for $f \in C(I)$. Here $\bar{E}_{r_1}$ stands for the expectation by Q_{r_1}.

Now we will show $Q_{r_1} \neq \delta_1$. By Theorems 61.2 and 61.3 in Itô [7], there exists a function $u(x)$ on I such that $u \in C((r_0, r_1]) \cap C^2(I^0) \cap \{Lf \in C((r_0, r_1])\}$, $D_s^+ u(r_1) = 0$, $u(r_1) > 0$ and $(1-L)u = 0$ on $(r_0, r_1]$ ($u(x)$ is also positive and decreasing on I^0). Let $h(x)$ be a $C^2(R^1)$-function which is equal to 0 and 1 in some neighborhoods U_0 and $U_1 (U_0 \cap U_1 = \phi)$ of r_0 and r_1, respectively, and define $v = hu$. Then it is easily seen that $v \in C(I) \cap C^2(I^0) \cap \{Lf \in C(I)\}$, $v = u$ on $U_1 \cap I$ and $D_s^+ v(r_1) = 0$. Hence we have $v \in D$, $v(r_1) > 0$ and $g(r_1) = 0$, where $g = (1-L)v \in C(I)$. Consequently it follows that

$$\bar{E}_{r_1}\left[\int_0^\infty e^{-t} g(x(t))\, dt\right] = \int_0^\infty e^{-t} T_t g(r_1)\, dt = (1 - L)^{-1} g(r_1)$$
$$= v(r_1) > 0.$$

This implies $Q_{r_1}[x(t) = r_1 \text{ for all } t \geq 0] < 1$ and so we have $Q_{r_1} \neq \delta_1$.

Thus the martingale problem on I for L starting at r_1 has two solutions at least and it contradicts the uniqueness. Similarly, we get the same result for $r_i = r_0$. Hence the theorem is proved. □

REMARK 4. In Theorem 1, we add the assumption that there exists some positive constant K such that $b(x)(-1)^i \leq K|x - r_i|$ for all $x \in I$ and $i = 0, 1$, from which follows $b(r_i) = 0$. Then an application of Gronwall's inequality to $E_{r_i}[|x(t) - r_i|]$ implies the uniqueness of the solution P_{r_i} to the martingale problem on I for L starting at r_i. Consequently it follows from this fact and the latter half of Theorem 1 that r_i is neither regular with $e^{B(r_i)} = 0$ nor entrance. Moreover we get the result that, in case $|\int_r^{r_i} 1/a| = \infty$, r_i is neither regular nor entrance. Here we have used the fact that, if r_i is regular, $|\int_r^{r_i} 1/a| = \infty$ implies $e^{B(r_i)} = 0$ since $\lim_{x \to r_i} e^{B(x)}$ has a finite limit $e^{B(r_i)}$ and

$$|m(r_i)| = \left| \int_r^{r_i} a(x)^{-1} e^{B(x)}\, dx \right| < \infty.$$

This result is also found in the proof of Lemma 2 in Ethier [2].

REMARK 5. Using Proposition 3 in Dorea [1] and applying Theorem 4.1 in Stroock and Varadhan [12], we have that the infinitesimal generator of $\{T_t: t \geq 0\}$, defined by (2.1), is equal to the restriction of L to D, which is defined by

$$D = C(I) \cap C^2(I^0) \cap \{Lf \in C(I)\}$$
$$\cap \{Lf(r_i) = 0 \text{ if } r_i \text{ is exit; } \mu(r_i)D_s^+f(r_i)(-1)^i = \delta(r_i)Lf(r_i)$$
$$\text{if } r_i \text{ is regular for } i = 0 \text{ and } 1\},$$
$$\mu(r_i) = b(r_i)(-1)^i, \quad \delta(r_i) = e^{B(r_i)}.$$

REMARK 6. In Theorem 1, if r_i is regular or entrance for $i = 0$ and 1, $C^2(I)$ is a core for the infinitesimal generator A of the semigroup $\{T_t: t \geq 0\}$ associated with unique solutions to the martingale problem. If $b(x)$ is Lipschitz continuous on I, we can show that $C^2(I)$ is a core for A also in the case of $r_i (i = 0, 1)$ being exit. But in the other cases, we do not know whether $C^2(I)$ is a core for A or not.

REMARK 7. In the exit boundary case, the fact that the path starting at the boundary point r_i never enters into I^0 has already been shown in Gihman and Skorohod [4] pages 163–165. On the other hand, the technique used in Case 1 in the proof of Theorem 1 is useful for the multidimensional case and the related topics will be stated elsewhere.

REMARK 8. Let $D = C(I) \cap C^2(I^0) \cap \{Lf \in C(I)\} \cap \{Lf(r_i) = 0$ if r_i is exit; $\mu(r_i)D_s^+f(r_i)(-1)^i = \delta(r_i)Lf(r_i)$ if r_i is regular for $i = 0$ and $1\}$ for some nonnegative constants $\mu(r_i)$ and $\delta(r_i)$. That is, D is Feller's boundary condition without killings and jumps at the boundary. By some calculations, we have the fact that, if the conditions of the first assertion of Theorem 1 hold, $\mu(r_i)$ and $\delta(r_i)$ $(i = 0, 1)$ such that $D \subset C^2(I)$ and $\mu(r_i) + \delta(r_i) = 1$ exist and are unique in case r_0 and r_1 are regular. Then applying Theorem 12.2.4 in Stroock and Varadhan [14], we get the first assertion of Theorem 1 also from this fact and further properties of one-dimensional diffusion process. But we omit its proof.

PROOF OF THEOREM 2. (i) From [2] (or [11]) follows the existence of the solution to the martingale problem on I for L starting at any $x \in I$. Moreover from (i), (ii) and (iii) of Lemma 6, we see that the assumptions of the first assertion of Theorem 1 hold under the assumptions of Theorem 2. Therefore the uniqueness follows immediately.

(ii) It follows from (i), (ii) and (iii) of Lemma 6 and Feller's result that $L|_D$, which is the restriction of L to D defined by (a)–(c) of Lemma 6, generates a strongly continuous contraction semigroup $\{S_t: t \geq 0\}$ on $C(I)$. Moreover, from (iv) of Lemma 6, we have easily that $C^2(I) \subset D$. Then, in the same way as that of Theorem 4.1 in Stroock and Varadhan [12], we have solutions $Q_x (x \in I)$ to the martingale problem on I for L associated with $\{S_t: t \geq 0\}$. Hence it follows from the uniqueness of the solution that $\{S_t: t \geq 0\}$ is equal to $\{T_t: t \geq 0\}$, that is, $L|_D$ is equal to the infinitesimal generator of $\{T_t: t \geq 0\}$. Therefore, in order to prove assertion (ii), it suffices to show that (a) $C_0^2(I) \cap C^3(I^0) \cap \{Lf \in C^1(I)\}$ is contained in D and a dense subset of $C^1(I)$ (with respect to $\|\cdot\|_1$), and (b) the equation $(\lambda - L)u = f$ has a (unique) solution u in $C_0^2(I) \cap C^3(I^0) \cap \{Lf \in C^1(I)\}$ for all $f \in C^1(I)$ if $\lambda > \|b^{(1)}\|_0$ and this u satisfies $\|u\|_1 \leq \|f\|_1(\lambda - \|b^{(1)}\|_0)^{-1}$. By simple calculations, (a) follows from (iv) of Lemma 6 and the fact that $C^3(I) \cap \{f^{(2)}$ has a compact support in $I^0\}$ is contained in $C_0^2(I) \cap C^3(I^0) \cap \{Lf \in C^1(I)\}$ and dense in $C^1(I)$.

Now we will show (b). First we note that it follows from Feller's result that

$$\|u\|_0 \leq \lambda^{-1}\|f\|_0 \quad \text{if} \quad u \in D \quad \text{and} \quad (\lambda - L)u = f \in C(I). \tag{4.5}$$

Define the differential operator H by

$$Hf = af^{(2)} + (a^{(1)} + b)f^{(1)} \quad \text{on } I^0,$$
$$Hf(r_i) = \lim_{x\to r_i} Hf(x) \quad \text{for} \quad i = 0 \quad \text{and} \quad 1$$

with the domain

$$D(H) = C(I) \cap C^2(I^0) \cap \{Hf \in C(I)\}.$$

In case $|\int_r^{r_i} a(x)^{-1}\,dx| = \infty$ for $i = 0$ or 1, r_i for H is entrance by $\lim_{x\to r_i} a^{(1)}(x) \cdot (-1)^i = \infty$ and Lemma 4 and, in case $|\int_r^{r_i} a(x)^{-1}\,dx| < \infty$ for $i = 0$ or 1, r_i for H is regular by the simple calculation. Hence it follows from

$$\lim_{x\to r_i} a^{(1)}(x)(-1)^i = \infty,$$

(i) of Lemma 1 and Lemma 5 that

$$\bar{D}(H) \subset C^1(I) \cap \{f^{(1)}(r_i) = 0 \text{ for } i = 0 \text{ and } 1\}, \tag{4.6}$$

where

$$\bar{D}(H) = C(I) \cap C^2(I^0) \cap \{Hf \in C(I)\}$$
$$\cap \left\{\lim_{x\to r_i} f^{(1)}(x)\exp\left(\int_r^x \frac{a^{(1)}(y) + b(y)}{a(y)}\,dy\right) = 0 \quad \text{if } r_i (i = 0, 1) \text{ is regular}\right\}$$

(more precisely, we have $\bar{D}(H) = C^1(I) \cap C^2(I^0) \cap \{f^{(1)}(r_i) = 0 \text{ for } i = 0 \text{ and } 1\} \cap \{Hf \in C(I)\}$). Moreover the restriction $H|_{\bar{D}(H)}$ of H to $\bar{D}(H)$ generates a strongly continuous contraction semigroup on $C(I)$ by Feller's result. Then, if we define the bounded operator V on $C(I)$ by $Vf = b^{(1)}f$ for all $f \in C(I)$, by Theorem 13.2.1 of Hille and Phillips [6], the operator $\bar{H} = H + V$ defined on $\bar{D}(H)$ generates a strongly continuous semigroup on $C(I)$. Consequently, for each $f \in C^1(I)$ and $\lambda > \|b^{(1)}\|_0$, the equation $(\lambda - \bar{H})v = f^{(1)}$ has a unique solution v in $\bar{D}(H)$ and, moreover, v satisfies

$$\|v\|_0 \leq \|f^{(1)}\|_0(\lambda - \|b^{(1)}\|_0)^{-1}. \tag{4.7}$$

We now define $u_\theta(x) = \theta + \int_r^x v(y)\,dy$ for this v and some $\theta \in R^1$. Then, since we have

$$[(\lambda - L)u_\theta]^{(1)} = (\lambda - \bar{H})u_\theta^{(1)} = (\lambda - \bar{H})v = f^{(1)} \quad \text{on} \quad I^0,$$

there is some $\theta_0 \in R^1$ such that

$$(4.8) \qquad (\lambda - L)u_{\theta_0} = f.$$

Further, from (4.6), we have

$$(4.9) \qquad u_{\theta_0} \in C_0^2(I) \cap C^3(I^0)$$

and so, from (a) and (4.5),

$$(4.10) \qquad \| u_{\theta_0} \|_0 \leq \lambda^{-1} \| f \|_0.$$

Hence (b) follows from (4.7), (4.8), (4.9) and (4.10). Thus the proof is complete. □

REMARK 9. The result of (4.9) depends on the differentiability of f. Indeed let $I = [0, 1]$, $a(x) = x(1 - x)\log 1/x(1 - x)$ and $b(x) \equiv 0$. Then the boundary $r_0 = 0$ and $r_1 = 1$ are exit and $u(\in C(I))$ and $v(\in C(I))$ satisfy $(1 - L)v = u$ and $Lv(r_i) = 0$ for $i = 0, 1$, where $u = x(1 - x)\{2(1 - 3x + 3x^2)(\log x(1 - x))^{-1} + 2x(1 - x) - 2(1 - 2x)^2(\log x(1 - x))^{-2}\}$ and $v = x(1 - x)(\log x(1 - x))^{-1}$. Hence $v \in D(A)$ and $v = \int_0^\infty e^{-t}T_t u\, dt = (1 - A)^{-1}u$. But v is not differentiable at 0 and 1. Next let $G_1(x, y)$ be Green function with respect to $dm(x)$, then $v(x) = \int_I G_1(x, y)u(y)dm(y)$ for above $u(x)$ and $v(x)$. If $\partial G_1(x, y)/\partial x$ exists for all $(x, y) \in I \times I$ and there is a measurable function $g(y)$ such that $|\partial G_1/\partial x| \leq |g|$ and $\int_I |g|\, dm(y) < \infty$, then v belongs to $C^1(I)$ and this is a contradictory result. Consequently $G_1(x, y)$ cannot be a nice function. Thus, from these arguments, we see that it will be very difficult that we obtain the results of Theorem 2 (and Theorem 3) from eigen-differential expansions for Green functions and transition densities.

PROOF OF THEOREM 3. (i) From the fact mentioned in the proof of (ii) of Theorem 2, it suffices to prove that (a) $\mathscr{D} = C_0^2(I) \cap C^4(I^0) \cap \{Lf \in C^2(I)\}$ is contained in D which is defined by (a)–(c) of Lemma 6, and (b) the equation $(\lambda - L)u = f$ has a (unique) solution u in $\mathscr{D}$ for all $f \in C^2(I)$ if $\lambda > \xi_2$ and u satisfies $\| u \|_2 \leq \| f \|_2(\lambda - \xi_2)^{-1}$. By simple calculations (a) follows easily from (iv) of Lemma 6.

As for (b), we will show that $U = u_{\theta_0}$ obtained in the proof of (ii) in Theorem 2 satisfies the assertions of (b) for all $f \in C^2(I)$: in the proof of Theorem 2, we had, for each $f \in C^2(I)$,

$$(4.11) \qquad U = u_{\theta_0} \in C_0^2(I) \cap C^3(I^0) \cap \{Lu \in C^1(I)\} \subset D,$$

$$(4.12) \qquad (\lambda - L)U = f,$$

$$(4.13) \qquad \| U \|_0 \leq \| f \|_0 \lambda^{-1} \quad \text{and} \quad \| U^{(1)} \|_0 \leq \| f^{(1)} \|_0 (\lambda - \| b^{(1)} \|_0)^{-1}$$

(see (4.7), (4.8), (4.9) and (4.10)). First, differentiating both sides of (4.12) yields

$$(4.14) \qquad \lambda U^{(1)} - aU^{(3)} - (a^{(1)} + b)U^{(2)} - b^{(1)}U^{(1)} = f^{(1)} \quad \text{on} \quad I^0.$$

Solving (4.14), we get

$$U^{(2)}(x) = \frac{a(r)}{a(x)} \exp\left(-\int_r^x \frac{b(y)}{a(y)}\,dy\right)$$

$$\cdot \left\{U^{(2)}(r) + \int_r^x \frac{(\lambda U^{(1)}(y) - f^{(1)}(y) - b^{(1)}(y)U^{(1)}(y))}{a(r)} \exp\left(\int_r^y \frac{b(z)}{a(z)}\,dz\right) dy\right\}$$

and so $U^{(2)} \in C^2(I^0)$, that is,

$$U \in C^4(I^0). \tag{4.15}$$

Further differentiating both sides of (4.14) yields

$$\lambda U^{(2)} - MU^{(2)} - (k + 2b^{(1)})U^{(2)} - b^{(2)}U^{(1)} = f^{(2)} \quad \text{on} \quad I^0,$$

where $M = a(d^2/dx^2) + (2a^{(1)} + b)(d/dx) + a^{(2)} - k$. It is easy to check that M is a dispersive (s) operator on I^0 for all $f \in C^2(I^0)$. Hence, noting $U^{(2)}(r_i) = 0$ for $i = 0$ and 1, it follows that

$$\|U^{(2)}\|_0 \le \{(k + 2\|b^{(1)}\|_0)\|U^{(2)}\|_0 + \|b^{(2)}\|_0\|U^{(1)}\|_0 + \|f^{(2)}\|_0\}\lambda^{-1} \quad \text{for} \quad \lambda > k + 2\|b^{(1)}\|_0. \tag{4.16}$$

Combining (4.13) and (4.16), we have

$$\|U\|_2 \le \|f\|_2 \cdot \max\left\{\frac{1}{\lambda - k - 2\|b^{(1)}\|_0}, \frac{\|b^{(2)}\|_0}{(\lambda - k - 2\|b^{(1)}\|_0)(\lambda - \|b^{(1)}\|_0)} + \frac{1}{\lambda - \|b^{(1)}\|_0}\right\}$$

$$\text{for} \quad \lambda > k + 2\|b^{(1)}\|_0.$$

Find the minimum C such that $\max\{(\lambda - k - 2\|b^{(1)}\|_0)^{-1}, (\lambda - \|b^{(1)}\|_0)^{-1} + (\lambda - k - 2\|b^{(1)}\|_0)^{-1}(\lambda - \|b^{(1)}\|_0)^{-1}\|b^{(2)}\|_0\} \le (\lambda - C)^{-1}$ for all $\lambda > C$. Then, by simple calculations, it is equal to $\xi_2 = \max\{\|b^{(1)}\|_1, 2\|b^{(1)}\|_0 + k\}$. We have therefore

$$\|U\|_2 \le \|f\|_2(\lambda - \xi_2)^{-1} \quad \text{for all} \quad \lambda > \xi_2. \tag{4.17}$$

Thus the assertions of (b) follow from (4.12), (4.15) and (4.17) and this completes the proof of (i).

(ii) From (4.12) and the results of (i), we have only to prove that $\mathscr{D}_0 = C_0^2(I) \cap C^4(I^0) \cap \{Lf \in C_0^2(I)\}$ is dense in $C_0^2(I)$ (with respect to $\|\cdot\|_2$). But this assertion follows from the fact that $C^4(I) \cap \{f^{(2)}$ has a compact support in $I^0\}$ is dense in $C_0^2(I)$ and contained in $\mathscr{D}_0$ in the case $b(x) \in C_0^2(I)$ and the proof of Theorem 3 is complete. □

5. A remark on the three-times differentiability preserving property. In this section, we consider the three-times differentiability preserving property of the semigroup $\{T_t: t \ge 0\}$ associated with the unique solutions to the martingale problem on I for L for sufficiently many initial data.

For simplicity, we consider the following example. Let $I = [0, 1]$, $a(x) = \{x(1-x)\}^{1/2}$, $b(x) \in C^3(I)$, A with the domain $D(A)$ be the infinitesimal generator of $\{T_t\colon t \geq 0\}$, and let $u = (\lambda - A)^{-1}f$ for $f \in C^3(I)$ and $\lambda > 0$. Then it follows from Theorems 2 and 3 that $u \in C_0^2(I) \cap C^4(I^0)$, $(\lambda - L)u = f$, $u^{(1)} \in \bar{D}(H)$, $(\lambda - H)u^{(1)} - b^{(1)}u^{(1)} = f^{(1)}$ and that the boundaries 0 and 1 for H are regular, where $H = a(d^2/dx^2) + (a^{(1)} + b)\, d/dx$ and $\bar{D}(H) = C(I) \cap C^2(I^0) \cap \{Hf \in C(I)\} \cap \{\lim_{x\to r_i} f^{(1)}(x)\exp\{\int_{1/2}^{x}(a^{(1)}(y) + b(y))a(y)^{-1}\,dy\} = 0$ for $r_i = 0$ and $1\}$. Further from (3.15) and (3.16), we have

$$u^{(2)}(x) = \frac{1}{\{x(1-x)\}^{1/2}} \exp\left(-\int_{1/2}^{x} \frac{b(y)}{\{y(1-y)\}^{1/2}}\,dy\right) \tag{5.1}$$
$$\cdot \int_{r_i}^{x} (\lambda u^{(1)}(y) - f^{(1)}(y) - b^{(1)}(y)u^{(1)}(y))\exp\left(\int_{1/2}^{y} \frac{b(z)}{\{z(1-z)\}^{1/2}}\,dz\right) dy$$

for $r_i = 0$ and 1. Consequently it follows easily from (5.1) that, if $\lambda u^{(1)}(r_i) = f^{(1)}(r_i) + b^{(1)}(r_i)u^{(1)}(r_i)$, then $u \in C_0^3(I) \cap C_0^2(I)$ and, if $\lambda u^{(1)}(r_i) \neq f^{(1)}(r_i) + b^{(1)}(r_i)u^{(1)}(r_i)$, then u is not three-times differentiable at $r_i(= 0$ or $1)$. Let E be a closed subspace of $C^3(I)$ with respect to $\|\cdot\|_3$ such that $T_tE \subset E$ for all $t \geq 0$ and T_t is strongly continuous on E with respect to $\|\cdot\|_3$. Then it is obvious by above results that E is contained in $C_0^2(I) \cap C_0^3(I)$. But it is easily seen that E is not a dense subset of $C^1(I)$ with respect to $\|\cdot\|_1$, because, if we assume that E is dense in $C^1(I)$ with respect to $\|\cdot\|_1$, then $F = \{f^{(1)}\colon f \in E\}$ is dense in $C(I)$ with respect to $\|\cdot\|_0$. Moreover it follows from above results that, if $(\lambda - H - b^{(1)})u = f$ on I for $f \in F$ and $u \in \bar{D}(H)$, we have $\lambda u(r_i) = f(r_i) + b^{(1)}(r_i)u(r_i)$, that is, $Hu(r_i) = 0$ for $i = 0$ and 1. Hence it follows from the boundedness of $(\lambda - H - b^{(1)})^{-1}$ from $C(I)$ onto $\bar{D}(H)$ that

$$Hu(r_i) = 0 \quad \text{for all} \quad u \in \bar{D}(H) \quad \text{and} \quad i = 0 \text{ and } 1. \tag{5.2}$$

On the other hand, by an argument similar to that given in the proof of the latter half of Theorem 1, we have a $v_i \in \bar{D}(H)$ such that $v_i(r_i) > 0$ and $(\lambda - H)v_i(r_i) = 0$ for each $i = 0$ and 1. Then $Hv_i(r_i) = \lambda v_i(r_i) \neq 0$ and this contradicts (5.2). Thus we see that E is not a sufficiently large set in $C^1(I)$.

Now let $\Gamma = \{$the set of linear function on $I\}$ and $b(x) \in \Gamma$. Then it follows easily from the martingale property that $T_t\,\Gamma \subset \Gamma$ for all $t \geq 0$ and T_t is strongly continuous on Γ with respect to $\|\cdot\|_3$. That is, Γ is a nonempty trivial example of E. But we do not know whether there exists an E that is larger than Γ.

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