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A REPRESENTATION FOR THE INTERSECTION LOCAL TIME OF BROWNIAN MOTION IN SPACE[1]

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We present a "Tanaka-like" representation for $\alpha(x, B)$, the local time of intersection for Brownian motion in 2 and 3 dimensions, where $\alpha(x, B)$ is formally

$$\int_B \int \delta_x(\omega_t - \omega_s)\, ds\, dt.$$

Section 1. Let W_t denote three dimensional Brownian motion, and set

$$X(s, t) = W_t - W_s. \tag{1.1}$$

$X: R_+^2 \to R^3$ and induces a measure on R^3, the occupation measure μ_B of X relative to $B \subseteq R_+^2$, defined by

$$\mu_B(A) = \lambda_2(X^{-1}(A) \cap B), \tag{1.2}$$

where λ_n denotes Lebesgue measure on R^n.

In [1] we showed that $\mu_B \ll \lambda_3$, and furthermore

$$\alpha(x, B) = \frac{d\mu_{B(x)}}{d\lambda_3}, \tag{1.3}$$

for $B = [a, b] \times [c, d]$, can be chosen to be jointly continuous in x, a, b, c, d with probability one, as long as $b < c$. $\alpha(x, B)$ is called the intersection local time relative to B and plays the key role in Symanzik's approach to quantum field theory [2] and in work on polymers [3].

The main goal of this paper is to present a "Tanaka like" representation for $\alpha(x, B)$. Let us first define the occupation measure ν_K of W relative to $K \subseteq R$ by

$$\nu_K(A) = \lambda_1(W^{-1}(A) \cap K) \tag{1.4}$$

and set

$$G\nu_K(x) = \frac{1}{4\pi} \int \frac{1}{|x - y|}\, d\nu_K(y) = \frac{1}{4\pi} \int_K \frac{1}{|x - W_s|}\, ds, \tag{1.5}$$

the Newtonian potential of ν_K. We will show in Lemma 1 that $G\nu_{[a,b]}(x)$ is jointly continuous in x, a, b. Here is our representation.

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THEOREM 1. *For all x, $a < b < c < d$ with probability one,*

$$-\left(\frac{1}{2}\right)\alpha(x, [a, b] \times [c, d]) = G\nu_{[a,b]}(W_d - x) - G\nu_{[a,b]}(W_c - x) - \int_c^d \nabla G\nu_{[a,b]}(W_t - x) \cdot dW_t. \tag{1.6}$$

The integrand in the stochastic integral is defined in the course of the proof. In Section 5 we present an analogous result for planar Brownian motion. It is our hope that this combination of intersection local time, potentials and stochastic integrals will be fruitful in the study of intersections of other diffusions.

Our work on local times owes a great deal to the survey article of D. Geman and J. Horowitz [4] which in turn relies on the work of S. Berman, L. Pitt and others (see the extensive bibliography in [4]).

We use the symbol $\tilde{c}$ to denote constants, which may vary from line to line.

Section 2.

LEMMA 1. *$G\nu_{[a,b]}(x)$ is Hölder continuous of any order < 1, with probability 1.*

PROOF. We use the notation $A = [a, b]$. Let

$$R_\gamma \nu_A(x) = \int \frac{1}{|x - y|^\gamma}\, d\nu_A(y) = \int_a^b \frac{1}{|x - W_s|^\gamma}\, ds.$$

We have

$$\begin{aligned}
\sup_x \mathbb{E}(R_\gamma \nu_A(x)) &= \sup_X \mathbb{E} \int_a^b \frac{1}{|x - W_s|^\gamma}\, ds \\
&= \sup_x \int_a^b \int \frac{1}{|u - x|^\gamma} \exp(-|u|^2/2s)\, du \frac{ds}{(2\pi s)^{3/2}} \\
&= \sup_x \int \frac{1}{|u - x|^\gamma} \left(\int_{b^{-1}}^{a^{-1}} \exp(-|u|^2 t/2) \frac{dt}{\sqrt{t}}\right) \frac{du}{(2\pi)^{3/2}} \qquad (2.1)\\
&\le \sup_x \int \frac{\exp(-|u|^2/4b)}{|u - x|^\gamma} \left(\int_0^\infty \exp(-|u|^2 t/4) \frac{dt}{\sqrt{t}}\right) \frac{du}{(2\pi)^{3/2}} \\
&\le \sup_x \int \frac{\exp(-|u|^2/4b)}{|u - x|^\gamma} \cdot \frac{1}{|u|} \frac{du}{(2\pi)^{3/2}}\, \delta < \infty
\end{aligned}$$

for any $\gamma < 2$.

Using the Markov property in the form of Khasminskii's lemma [5, page 461], we have

$$\sup_x \mathbb{E}(R_\gamma \nu_A(x))^k \le k!\delta^K. \tag{2.2}$$

The inequality [5, page 468]

$$||z - x|^{-\beta} - |z - y|^{-\beta}| \le c|x - y|^\alpha (|z - x|^{-(\alpha+\beta)} + |z - y|^{-(\alpha+\beta)})$$

valid for any $\alpha < 1$ now shows that

$$\mathbb{E}(R_\beta \nu_A(x) - R_\beta \nu_A(y))^k \le \tilde{c}\,|x - y|^{\alpha k} \tag{2.3}$$

for any $\alpha < 1$, $\alpha + \beta < 2$.

The multidimensional form of Kolmogorov's lemma [10] applied to (2.3) shows that a.s.

$$|R_\beta \nu_A(x) - R_\beta \nu_A(y)| \le \tilde{c}\,|x - y|^{\alpha} \tag{2.4}$$

for all rational x, y in $B_n = \{z \mid |z| \le n\}$ and any $\alpha < 1$, $\alpha + \beta < 2$.

Hence $R_\beta \nu_A(\cdot)$ for any $\beta < 2$ is bounded on B_n when restricted to the rationals, hence by Fatou's lemma it is bounded for all $x \in B_n$. This means that the family of functions of s, $a \le s \le b$,

$$\frac{1}{|x - W_s|^\beta}$$

indexed by $x \in B_n$ is uniformly integrable. This shows that (2.4) holds for all x, $y \in B_n$.

REMARK. It is clear from the proof that $G\nu_{[a,b]}(x)$ is jointly continuous in x, a, b.

Section 3. In this section we present a proof of Theorem 1 based on the existence of α established in [1]. In Section 4 we show how to establish Theorem 1 independently of [1], and indeed recover the existence and continuity of α as a corollary.

Let g be a C^∞ function on R^3 with $g(p) = 1$ for $|p| \le 1$ and $g(p) = 0$ for $|p| \ge 2$. The Fourier transform f of g is in S. Set $f_n(x) = n^3 f(nx)$ so that $\int f_n(x)\, d^3x = 1$.

We use the abbreviations

$$\begin{aligned} A &= [a, b] \\ B &= [a, b] \times [c, d]. \end{aligned} \tag{3.1}$$

By Lemma 1, $G\nu_A(x)$ is continuous and bounded in x, (in fact $G\nu_A(x) \to 0$ as $|x| \to \infty$ since ν_A has compact support). Therefore

$$f_n * G\nu_A(x) \to G\nu_A(x). \tag{3.2}$$

$f_n * G\nu_A(x)$ is a bounded C^∞ function of x, which depends on the path only up to time $b < c$ so that we can apply Ito's formula

$$\begin{aligned} &f_n * G\nu_A(W_d - x) - f_n * G\nu_A(W_c - x) \\ &\quad = \int_c^d \nabla f_n * G\nu_A(W_t - x) \cdot dW_t + \frac{1}{2}\int_c^d \Delta f_n * G\nu_A(W_t - x)\, dt \\ &\quad = \int_c^d \nabla f_n * G\nu_A(W_t - x) \cdot dW_t - \frac{1}{2}\int_c^d f_n * \nu_A(W_t - x)\, dt \end{aligned} \tag{3.3}$$

since $-\Delta(f_n * G\nu_A) = f_n * \nu_A$.

Because of (1.3) we have

$$\int_c^d f_n * \nu_A(W_t - x)\, dt = \int_c^b \int f_n(W_t - x - y)\, d\nu_A(y)\, dt$$

$$(3.4) \qquad = \int_c^d \int_a^b f_n(W_t - W_s - x) ds\, dt$$

$$= \int f_n(y - x) \alpha(y, B)\, dy = f_n * \alpha(\cdot, B)(x).$$

By (3.3) and (3.4)

$$(3.5) \qquad \begin{aligned} &-f_n * \alpha(\cdot, B)(x)/2 \\ &= f_n * G\nu_A(W_d - x) - f_n * G\nu_A(W_c - x) - \int_x^d \nabla f_n * G\nu_A(W_t - x) \cdot dW_t. \end{aligned}$$

By the results of [1] $\alpha(x, B)$ is a continuous compactly supported function of x so that letting $n \to \infty$ in (3.5) we have

$$(3.6) \qquad \begin{aligned} -\alpha(x, B)/2 &= G\nu_A(W_d - x) - G\nu_A(W_c - x) \\ &\quad - \lim_{n\to\infty} \int_c^d \nabla f_n * G\nu_A(W_t - x) \cdot dW_t. \end{aligned}$$

To obtain Theorem 1 we will define the stochastic integral in (1.6) as the limit in (3.6). In order to do this we need to show that this limit is indeed a stochastic integral. However, it follows from Lemma 1 and (2.2) that $G\nu_A(x)$ is a bounded continuous $L^2(dP)$ valued function of x, (dP is the measure for Brownian motion) so that (3.2) is true in $L^2(dP)$. Similarly $f_n * \alpha(\cdot, B)(x) \to \alpha(x, B)$ in $L^2(\Omega)$ by the proof of Lemma 1 of [1]. Hence

$$\int_c^d \nabla f_n * G\nu_A(W_t - x) \cdot dW_t$$

converges in $L^2(dP)$. By a basic property of stochastic integrals, [7, page 25], this limit is the stochastic integral of the $L^2(dP \times dt)$ limit of

$$\nabla f_n * G\nu_A(W_t - x)$$

which we have denoted in Theorem 1 by $\nabla G\nu_A(W_t - x)$. (While we do not know that $G\nu_A a(x)$ is differentiable, our limit will be a distributional derivative.)

Section 4. In this section we directly establish Theorem 1 and obtain the existence and joint continuity of $\alpha(x, B)$ as a corollary. This is analogous to the now standard procedure in 1-dimension: first establish Tanaka's formula, and then use it to study Brownian local time (see e.g. [6], [7], [8]).

Let

$$(4.1) \qquad L_n = L_n(t, x, A, \omega) = \nabla f_n * G\nu_A(W_t - x).$$

Since as distributions

$$\widehat{\nabla f_n * G\nu_A}(p) = p/|p|^2\hat{f}_n(p)\hat{\nu}_A(p) \in L'(R^3), \tag{4.2}$$

$$\begin{aligned} L_n &= \int \exp(ip \cdot (W_t - x))p/|p|^2\hat{f}_n(p)\hat{\nu}_A(p)\, d^3p \\ &= \int\int_a^b \exp(ip \cdot (W_t - W_s - x))p/|p|^2\hat{f}_n(p)\, ds\, dp. \end{aligned} \tag{4.3}$$

We first show that L_n is a Cauchy sequence in $L^2(dP \times dt)$. Using the fact that

$$g_{m,n}(p) = \hat{f}_n(p) - \hat{f}_m(p) = 0 \text{ if } |p| \le m \le n$$

we have

$$\begin{aligned} &\int_c^d \mathbb{E}\,|L_n - L_m|^2 dt \\ &\quad = \int_c^d dt \int_a^b\int_a^b dr\, ds \int\int dp\, dq \exp(-(p-q)\cdot x)\frac{p\cdot q}{|p|^2|q|^2} \\ &\qquad \cdot g_{m,n}(p)q_{m,n}(q)\,\mathbb{E}(\exp(ip\cdot(W_t - W_s) - iq\cdot(W_t - W_r))) \\ &\quad \le 2\int_c^d dt \int_a^b ds \int_a^s dr \int_{|p|,|q|\ge m}\int \exp(-(s-r)|q|^2/2) \\ &\qquad \cdot \exp(-(t-s)|p-q|^2/2)\frac{1}{|p|\,|q|}\, dp\, dq \\ &\quad \le \int_{|p|,|q|\ge m}\int |q|^{-3}|p|^{-1}\exp(-\varepsilon|p-q|^2/2)\, dp\, dq, \\ &\qquad (t - s \ge c - b \doteq \varepsilon) \\ &\quad \le \tilde{c}\int_{|q|\ge m}\int |q|^{-3}|p-q|^{-1}\exp(-\varepsilon|p|^2/2)\, dp\, dq \\ &\quad \le \tilde{c}\int_{|q|\ge m} |q|^{-4}\, dq \to 0 \quad \text{as} \quad n \ge m \to \infty \end{aligned} \tag{4.4}$$

where the last inequality is obtained by arguing separately on the region of p space where $|p| \ge |q|/2$ and where $|p| \le |q|/2$.

Denote this limit by L. We will find a version of $\int_c^d L \cdot dW_t$, jointly continuous in x, a, b, c, d.

We will show that for a subsequence n

$$\begin{aligned} &\mathbb{E}\left(\sup_n\left(\int_c^d L_n(t, x, a, b)\cdot dW_t - \int_{c'}^{d'} L_n(t, x', a', b')\, dW_t\right)^{2k}\right) \\ &\qquad \le \tilde{c}_k|(x, a, b, c, d) - (x', a', b', c', d')|^{k/5}. \end{aligned} \tag{4.5}$$

The multidimensional version of Kolmogorov's lemma [10] then shows that a.s.

$$\left| \int_c^d L_n(t, x, a, b) \cdot dW_t - \int_{c'}^{d'} L_n(t, x', a', b') \cdot dW_t \right| \leq \tilde{c}(\omega) \,|\, (x, a, b, c, d) - (x', a', b', c', d') \,|^{1/12} \tag{4.6}$$

uniformly in n for all rational arguments (locally). This allows us to find versions of $\int_c^d L_n \cdot dW_t$ satisfying (4.6) for *all* arguments locally. With these versions (3.3) is now true a.s. for all (x, a, b, c, d) simultaneously.

Going to a subsequence in (4.4) we see that $\int_c^d L_n \cdot dW_t$ converges in $L^2(dP)$ and a.s. simultaneously for all rational (x, a, b, c, d). Using (4.6), we find that this limit, $\int_c^d L \cdot dW_t$ also satisfies the bound in (4.6)—first for all rational arguments, then with a suitable version—for all (x, a, b, c, d).

We now prove (4.5). Considerations such as those in (4.4) will show that it suffices to prove (4.5) without $\sup_n$, but for $\tilde{c}_k$ independent in n. We bound our expectation by three terms:

$$\begin{aligned} &E\left(\int_c^d L_n(t, x, a, b) \cdot dW_t - \int_c^d L_n(t, x', a, b) \cdot dW_t\right)^{2k} \\ &+ \mathbb{E}\left(\int_c^d L_n(t, x', a, b) \cdot dW_t - \int_c^d L_n(t, x', a', b') \cdot dW_t\right)^{2k} \\ &+ \mathbb{E}\left(\int_c^d L_n(t, x', a', b') \cdot dW_t - \int_{c'}^{d'} L_n(t, x', a', b') \cdot dW_t\right)^{2k}. \end{aligned} \tag{4.7}$$

Using the bounds [6, page 110]

$$\mathbb{E}\left(\left(\int_S f_t \, dW_t\right)^{2k}\right) \leq c\mathbb{E}\left(\int_S f_t^2 \, dt\right)^k$$

and

$$|\exp(iu \cdot x) - \exp(iu \cdot x')| \leq \tilde{c}_\varepsilon |u|^\varepsilon |x - y|^\varepsilon, \quad \text{for} \quad \varepsilon < 1.$$

We can bound the first term in (4.7) by

$$|x - x'|^{2k\varepsilon} \int_{[c,d]^k \times [a,b]^{2k} \times R^{4k}} \textstyle\prod_{j=1}^k (|p_\ell| \; |q_\ell|)^\varepsilon F \, dp \, dq \, dr \, ds \, dt \tag{4.8}$$

where

$$F = \prod_{j=1}^k \frac{p_j \cdot q_j}{(|p_j| \; |q_j|)^2} \cdot \mathbb{E}(\exp(i \textstyle\sum_{\ell=1}^k p_\ell (W_{t_\ell} - W_{s_\ell}) + i \sum_{\ell=1}^k q_\ell \cdot (W_{t_\ell} - W_{r_\ell}))). \tag{4.9}$$

For the second and third term in (4.7) we have the bound

$$\int_{S^k}\int_{R^{4k}} F\,dr\,ds\,dt\,dp\,dq \le |S|^{k\varepsilon}\int_{R^{4k}}\left(\int_{S^k} F^{1/(1-\varepsilon)}\,dr\,ds\,dt\right)^{1-\varepsilon} dp\,dq \tag{4.10}$$

with S respectively $[c, d] \times ([a, b]\Delta(a', b'])^2$ and $([c, d]\Delta(c', d']) \times [a, b]^2$.

We will show how to obtain uniform bounds on

$$\int\int F\,dr\,ds\,dt\,dp\,dq, \tag{4.11}$$

after which the reader can easily verify that the integrals in (4.8) and (4.10) are bounded for ε small; 1/10 will do. This will prove (4.5).

We have

$$\begin{aligned}\int\int F\,dr\,ds\,dt\,dq &\le \tilde{c}\,\textstyle\sum_\pi \displaystyle\int\int \textstyle\prod_{j=1}^{2K} |u_j|^{-1}(1 + |\sum_{i=1}^{j} u_i|^2)^{-1} \prod_{\ell=1}^{K} \\ &\qquad \cdot (1 + |\textstyle\sum_{i=1}^{2\ell} u_{\pi(i)}|^2)^{-1}\,du_1\cdots du_{2K}\end{aligned}$$

where the last inequality uses the fact that the Brownian motion has independent increments,

$$\int_0^h \exp(-sp^2)\,ds \le \tilde{c}(1 + p^2)^{-1},$$

and all $t_i \ge c > b \ge$ all s_j. The sum is over all permutations π of $\{1, \cdots, 2K\}$.

Using Hölder's inequality this is bounded by

$$\begin{aligned}&\le \tilde{c}\left(\int \textstyle\prod_{j=1}^{2K} (1 + |\sum_{i=1}^{j} u_i|^2)^{-5/3}\,du_i\right)^{3/5} \\ &\qquad \cdot \left(\int \textstyle\prod_{j=1}^{K} [|u_{2j-1}|\;|u_{2j}|\;(1 + |\sum_{i=1}^{2j} u_i|^2)]^{-5/2}\,du\right)^{2/5} \\ &\le \tilde{c}\left(\int\cdots\int \textstyle\prod_{j=1}^{K} (|u_{2j-1}|\;|u_{2j}|)^{-5/2}\right. \\ &\qquad \left.\cdot\,(1 + |\textstyle\sum_{i=1}^{2j} u_i|^2)^{-5/2}\,du_1\cdots du_{2n}\right)^{2/5}.\end{aligned} \tag{4.12}$$

We use the inequality

$$\frac{1}{1 + |\sum_{i=1}^{2j} u_i|^2} \le \tilde{c}\,\frac{1 + |u_{2j}|^2}{1 + |\sum_{i=1}^{2j-1} u_i|^2}$$

raised to the ¾ power to bound the integral in (4.12) by

$$(4.13)\quad \tilde{c}\int\cdots\int \prod_{j=1}^{K} |u_{2j-1}|^{-5/2}\Big(1+\Big|\sum_{i=1}^{2j-1} u_i\Big|^2\Big)^{-3/4}|u_{2j}|^{-1} \cdot \Big(1+\Big|\sum_{i=1}^{2j} u_i\Big|^2\Big)^{-7/4}\,du_1\cdots du_{2K}$$

which is finite by induction on j if we can bound

$$(4.14)\quad \sup_a \int\int |u|^{-5/2}(1+|u+a|^2)^{-3/4}|v|^{-1} \cdot (1+|u+v+a|^2)^{-7/4}\,du\,dv.$$

But

$$\sup_b \int |v|^{-1}(1+|v+b|^2)^{-7/4}\,dv<\infty \text{ trivially and}$$

$$\begin{aligned}\sup_a &\int |u|^{-5/2}(1+|u+a|^2)^{-3/4}\,du\\ &\le \tilde{c}+\sup_a\int (1+|u|)^{-5/2}(1+|u+a|^2)^{-3/4}\,du\\ &\le \tilde{c}+\sup_a\left(\int (1+|u|)^{-15/4}\,du\right)^{2/3}\left(\int(1+|u+a|^2)^{-9/4}\,du\right)^{1/3}<\infty.\end{aligned}$$

Return to (3.3) and (3.4) and integrate with respect to a continuous function of compact support $h(x)$ to obtain

$$\begin{aligned}(4.9)\quad -\frac{1}{2}&\int_c^d\int_a^b f_n * h(W_t-W_s)\,ds\,dt\\ &= f_n * h * G\nu_A(W_d) - f_n * h * G\nu_A(W_c) - \int h(x)\left(\int_c^d L_n\cdot dW_t\right)dx.\end{aligned}$$

Equicontinuity, (4.6), guarantees that

$$\int_c^d L_n\cdot dW_t \to \int_c^d L\cdot dW_t(\text{locally})$$

uniformly along a path dependent subsequence. The $n\to\infty$ limit yields

$$\begin{aligned}-\frac{1}{2}&\int_c^d\int_a^b h(W_t-W_s)\,ds\,dt\\ &=\int h(x)\left(G\nu_A(W_d-x)-G\nu_A(W_c-x)-\int_c^d L\cdot dW_t\right)dx.\end{aligned}$$

This establishes formula (1.5), and at the same time establishes the existence of $\alpha(x,B)$. Formula (1.5) then yields the joint continuity of $\alpha(x,B)$ for $a<b<c<d$.

Section 5. In this section we briefly describe how to proceed in 2 dimensions. The problem here is that the logarithmic potential is not bounded at ∞. Therefore in place of the $G(x) = 1/(4\pi|x|)$ of 3-dimensions, we use $K(x)$ the Fourier transform of $1/(1 + |p|^2)$. Note that $K(x) > 0$, and falls off exponentially at ∞, with a logarithmic singularity at $x = 0$. As in Lemma 1 we can show $K\nu_A(x)$ is continuous. Since $(-\Delta + 1) * K = \delta$ we obtain

THEOREM 2. *For all x, $a < b < c < d$ with probability one*

$$\begin{aligned} -\left(\frac{1}{2}\right) \alpha(x, [a, b] \times [c, d]) &= K\nu_{[a,b]}(W_d - x) - K\nu_{[a,b]}(W_c - x) \\ &\quad - \int_c^d \nabla K\nu_{[a,b]}(W_t - x)\, dW_t \\ &\quad - \frac{1}{2} \int_c^d K\nu_{[a,b]}(W_t - x)\, dt. \end{aligned}$$

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