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DEPENDENCE BY REVERSE REGULAR RULE

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A family of concepts of dependence by reverse regular (DRR) rule is introduced. Interrelationships among DRR random variables are investigated. Relationships with other concepts of negative dependence are discussed.

1. Introduction. In Lee (1985), a family of concepts of positive dependence by total positivity was considered. This generalized work of Shaked (1977) was done only in the bivariate case. Because of recent work by Karlin and Rinott (1980) and of Block, Savits and Shaked (1982) on negative dependence analogs of total positivity, it is now possible to consider families of negatively dependent distributions analogous to those of Lee (1985).

Since negative dependence concepts are not simply mirror images of positive dependence properties (except in the bivariate case), results are not immediate. For example, Theorem 5.1 of Karlin (1968, page 123) for TP_2 densities, which was heavily used in Lee (1985), does not have an immediate analog for a simple reverse TP_2 type of concept. Consequently we use a more specialized version of a reverse TP_2 type of concept due to Karlin and Rinott (1980) called strongly multivariate reverse regular rule of order 2 (S-MRR_2).

In Section 2, the concept of dependence by reverse regular rule is given and some properties are derived. The main result is Proposition 2.7 which shows that if a distribution is dependent by reverse regular rule of a certain order, then it is dependent by reverse regular rule for all higher orders. Various other properties are given.

Section 3 considers other concepts of negative dependence and how they are related to the families studied here. In particular we consider concepts due to Brindley and Thompson (1972) and Ebrahimi and Ghosh (1981).

2. Definition and properties of dependence by reverse regular rule. A function K is said to be *reverse regular of order 2* (RR_2) on $S_1 \times S_2$ if $K(x, y) \geq 0$ and if $K(x, y)K(x', y') \leq K(x, y')K(x', y)$ whenever $x \leq x'$, $y \leq y'$, for $x, x' \in S_1$ and $y, y' \in S_2$ (see Karlin, 1968, page 12).

There are several recent papers in which reverse regular functions are discussed. Ebrahimi and Ghosh (1981) considered the condition that the joint density of a random vector and all its marginals are RR_2 in pairs. Block, Savits and Shaked (1981) defined a probability *measure* μ to be RR_2 *in pairs* if the set

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function $\hat{\mu}(I_1, \cdots, I_n) = \hat{\mu}(I_1 \times I_2 \times \cdots \times I_n)$ is RR_2 in the pairs I_i, I_j for all $1 \le i < j \le n$, with the remaining variables held fixed. According to Karlin and Rinott (1980) a nonnegative function satisfying the property

$$f(\mathbf{x} \wedge \mathbf{y}) f(\mathbf{x} \vee \mathbf{y}) \le f(\mathbf{x}) f(\mathbf{y}) \quad \text{for all} \quad \mathbf{x}, \mathbf{y}$$

is said to be *multivariate reverse regular of order 2* (MRR_2). Since the composition formulas do not in general preserve the MRR_2 property, these authors introduced the following stronger notion.

DEFINITION 2.1. (Karlin and Rinott, 1980). An MRR_2 function $f(\mathbf{x})$ is said to be *strongly*-MRR_2 (S-MRR_2) if for any set of PF_2 functions $\{\phi_\nu\}$, each resulting marginal

$$g(x_{\nu_1}, \cdots, x_{\nu_k}) = \int \cdots \int f(x_1, \cdots, x_n) \phi_1(x_{j_1}) \cdots \phi_{n-k}(x_{j_{n-k}})\, dx_{j_1} \cdots dx_{j_{n-k}}$$

is MRR_2 in the variables $x_{\nu_1}, \cdots, x_{\nu_k}$, where $\{\nu_1, \cdots, \nu_k\}$ and $\{j_1, \cdots, j_{n-k}\}$ are complementary sets of indices $\{1, \cdots, n\}$ with $n \ge 3$.

NOTE. Let $T_1, \cdots, T_n$ be random variables with density f (with respect to a product measure of σ-finite measures), then by Remark (vi) of Block et al. (1982), μ is RR_2 in pairs if and only if

$$\int \cdots \int [\textstyle\prod_{k \ne i,j} \chi_{I_k}(t_k)] f(t_1, \cdots, t_n) [\prod_{k \ne i,j} dt_k]$$

is RR_2 in the unintegrated variables t_i and t_j for all choice of intervals $I_k (k \ne i, j)$ in R', where χ_I denotes the indicator function of I.

We give here a counterexample which shows that the RR_2 in pairs definition of Block et al. (1982) does not imply the S-MRR_2 definition of Karlin and Rinott (1980).

EXAMPLE. Let X, Y, Z be distributed such that $P(X = 1, Y = 1, Z = 0) = P(X = 1, Y = 0, Z = 0) = P(X = 0, Y = 1, Z = 0) = P(X = 0, Y = 0, Z = 1) = 1/4$. Let $I \subset R$ be any interval. We will check in the following cases that X, Y, Z is RR_2 in pairs according to Block et al. (1982). We use Remark vi of Section 2 of that paper.

Case 1, $\{0, 1\} \subset I$:

$$P(X = x, Y = y, Z \in I) = P(X = x, Y = y) \quad \text{is } RR_2 \text{ in} \quad x, y,$$

$$P(X = x, Y \in I, Z = z) = P(X = x, Z = z) \quad \text{is } RR_2 \text{ in} \quad x, z,$$

$$P(X \in I, Y = y, Z = z) = P(Y = y, Z = z) \quad \text{is } RR_2 \text{ in} \quad y, z.$$

Case 2, $0 \in I$, $1 \notin I$:

$$P(X = x, Y = y, Z \in I) = P(X = x, Y = y, Z = 0) \quad \text{is } RR_2 \text{ in} \quad x, y,$$

$$P(X = x, Y \in I, Z = z) = P(X = x, Y = 0, Z = z) \quad \text{is } RR_2 \text{ in} \quad x, z,$$

$$P(X \in I, Y = y, Z = z) = P(X = 0, Y = y, Z = z) \quad \text{is } RR_2 \text{ in} \quad y, z.$$

Case 3, $0 \notin I$, $1 \in I$: can be checked similarly as in case 2.

Case 4, $0 \notin I$, $1 \notin I$: trivial.

Now, let $\phi(z) = e^z$ be a PF_2 function, then

$$\int P(X = x, Y = y, Z = z)\phi(z)\, dm(z)$$
$$= P(X = x, Y = y, Z = 0) + eP(X = x, Y = y, Z = 1) \quad \text{is } TP_2 \text{ in } \quad x, y,$$

hence X, Y, Z is not S-MRR_2.

PROPOSITION 2.2. *Let $f(x_1, \cdots, x_n)$ be an S-MRR_2 function. Then for any PF_2 function ϕ, the integral $\int f(x_1, \cdots, x_n)\phi(x_i)\, dx_i$ is an S-MRR_2 function in the variables $x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$.*

PROOF. This follows from the definition.

Define, for $s > 0$,

$$\gamma^{(s)}(t) = \begin{cases} (-t)^{s-1}/\Gamma(s), & t \le 0 \\ 0, & t > 0. \end{cases}$$

Then for $s \ge 1$, $\gamma^{(s)}(x - y)$ is TP_2 in x and y (see Karlin, 1968).

Let $X_1, \cdots, X_n$ be random variables with joint distribution function F. Let $\gamma^{(s)}(t)$ be defined as above. For $k_i > 0$, define the n fold integral $\Psi_{k_1,\cdots,k_n}(x_1, \cdots, x_n)$ as

$$\Psi_{k_1,\cdots,k_n}(x_1, \cdots, x_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \gamma^{(k_1)}(x_1 - t_1) \cdots \gamma^{(k_n)}(x_n - t_n)\, dF(t_1, \cdots, t_n)$$

and define $\Psi_{0,\cdots,0}(x_1, \cdots, x_n) = f(x_1, \cdots, x_n)$ if the joint density f exists.

Also define $\Psi_{0,\cdots,0}, \Psi_{0,\cdots,k_{i+1},\cdots,k_n}(x_1, \cdots, x_n)$ to be the $(n - i)$-fold integral

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \gamma^{(k_{i+1})}(x_{k_{i+1}} - t_{i+1}) \cdots \gamma^{(k_n)}(x_{k_n} - t_n) g(x_1, \cdots, x_i)\, dF(t_{i+1}, \cdots, t_n \mid x_1, \cdots, x_i)$$

where g is the joint density of $X_1, \cdots, X_i$, and $F(t_{i+1}, \cdots, t_n \mid x_1, \cdots, x_i)$ is the conditional distribution of $X_{i+1}, \cdots, X_n$ given $X_1 = x_1, \cdots, X_i = x_i$, for $k_{i+1} > 0, \cdots, k_n > 0$. Similarly, we can define $\Psi_{k_1,\cdots,k_n}(x_1, \cdots, x_n)$ with any subset of $\{k_1, \cdots, k_n\}$ consisting of zeros.

With the above concepts, we can introduce the following definition.

DEFINITION 2.3. For $k_1, \cdots, k_n \ge 0$, the positive random vector $(X_1, \cdots, X_n)$ is said to be *dependent by reverse regular rule of order 2 with degree* $(k_1, \cdots, k_n)$, denoted by DRR$(k_1, \cdots, k_n)$, if $\Psi_{k_1,\cdots,k_n}(x_1, \cdots, x_n)$ is an S-MRR_2 function.

Since $\Psi_{0,\cdots,0}(x_1, \cdots, x_n) = f(x_1, \cdots, x_n)$ if the joint density exists, the condition that a random vector $(X_1, \cdots, X_n)$ is DRR$(0, \cdots, 0)$ is equivalent to the condition that the joint density is S-MRR$_2$ (or RR$_2$ in $n = 2$ case). For example, the multinomial distribution, multivariate hypergeometric distribution, etc., are DRR$(0, \cdots, 0)$, (see Block et al., 1982, and Karlin-Rinott, 1980). Also a random vector $\mathbf{X}$ is DRR$(1, \cdots, 1)$ means that $\bar{F}(x_1, \cdots, x_n)$ is an S-MRR$_2$ function when $n \geq 3$, or is RR$_2$ when $n = 2$. For the bivariate case, from the remarks of Block et al. (1982) we can see that DRR(0, 0) implies DRR(1, 1). For the multivariate case we have from Proposition 2.7 that if f is an S-MRR$_2$ density, then the joint survival function $\bar{F}$ is again S-MRR$_2$. To prove this and more general results, we need several lemmas.

LEMMA 2.4. *The function $1_{(x,y)}(u)$ is TP$_2$ in pairs of x, y, u, where (x, y) is any interval in R' and*

$$1_{(x,\, y)}(u) = \begin{cases} 1 & \text{if } \ u \in (x, y) \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. This can be easily checked.

LEMMA 2.5. *For any PF$_2$ function ϕ, the integrals $\int_w^a (w - x)^k \phi(x)\, dx$ and $\int_b^w (w - x)^k \phi(x)\, dx$ are both PF$_2$ functions in w, for any $k \geq 0$, and any extended real numbers a, b.*

PROOF. We have

$$\int_{w-s}^{a} (w - x - s)^k \phi(x)\, dx = \int_{w}^{a+s} (w - u)^k \phi(u - s)\, du$$
$$= \int_{-\infty}^{\infty} (w - u)^k \phi(u - s) 1_{(w,a+s)}(u)\, du.$$

Now, since $(w - u)^k$ is TP$_2$ in w, u, $\phi(u - s)$ is TP$_2$ in pairs of s, u, w, and $1_{(w,a+s)}(u)$ is TP$_2$ in pairs of s, u, w, the integral is TP$_2$ in w, s by Theorem 5.1, page 123, Karlin. Similarly $\int_b^{w-s} (w - s - x)^k \phi(x)\, dx$ is TP$_2$ in w, s.

NOTE. On page 193, Karlin (1968) has deduced by applying the composition lemma that if $f(x)$ is PF$_r$, and if f vanishes for nonpositive x, then

$$\int_u^{\infty} (\xi - u)^{\alpha} f(\xi)\, d\xi$$

is PF$_r$ provided either α is an integer or $\alpha \geq r - 2$. That result turns out to be a special case of the above lemma when $r = 2$.

LEMMA 2.6. *Let $f(w_1, \cdots, w_n)$ be an S-MRR$_2$ function, then*

$$\int f(w_1, \cdots, w_n) \gamma^{(k)}(x - w_j)\, dw_j$$

is an S-MRR$_2$ *function in $w_1, \cdots, w_{j-1}, x, w_{j+1}, \cdots, w_n$, for any $k \geq 0$ and any $j \in \{1, 2, \cdots, n\}$.*

PROOF. For any PF_2 function ϕ,

$$\int \left(\int f(w_1, \cdots, w_n)\gamma^{(k)}(x - w_j)\, dw_j \right) \phi(x)\, dx = \int \left(\int_{-\infty}^{w_j} \frac{(w_j - x)^{k-1}}{\Gamma(k)} \phi(x)\, dx \right) f(w_1, \cdots, w_n)\, dw_j.$$

The inner integral is PF_2 in w_j by the previous lemma, hence the outer integral is MRR_2 in $w_1, \cdots, w_{j-1}, w_{j+1}, \cdots, w_n$. Now we prove by induction that

$$\int \left(\int f(w_1, \cdots, w_n)\gamma^{(k)}(x - w_j)\, dw_j \right) \phi(w_\ell)\, dw_\ell$$

is MRR_2 in the variables $w_1, \cdots, w_{j-1}, x, w_{j+1}, \cdots, w_{\ell-1}, w_{\ell+1}, \cdots, w_n$.

For the $n = 3$ case, we assume, without loss of generality, that $\ell = 1, j = 3$. We show that

$$\int \int f(w_1, w_2, w_3)\gamma^{(k)}(x - w_3)\, dw_3 \phi(w_1)\, dw_1$$

is RR_2 in w_2, x. Rewrite

$$\int_x^{\infty} \left(\int f(w_1, w_2, w_3)\phi(w_1)\, dw_1 \right) \frac{(w_3 - x)^{k-1}}{\Gamma(k)}\, dw_3 = \int g(x, w_3)h(w_2, w_3)\, dw_3,$$

where

$$g(x, w_3) = 1_{(x,\infty)}(w_3)((w_3 - x)^{k-1}/\Gamma(k)),$$

and

$$h(w_2, w_3) = \int f(w_1, w_2, w_3)\phi(w_1)\, dw_1.$$

Notice that $g(x, w_3)$ is TP_2 in x, w_3, while $h(w_2, w_3)$ is RR_2 in w_2, w_3. Thus by the basic composition formula of Karlin (1968, Lemma 1.1, page 99)

$$\int g(x, w_3)h(w_2, w_3)\, dw_3 \quad \text{is } RR_2 \text{ in } \quad w_2, x.$$

Now, assume the lemma is true for case of $n - 1$ variables, then for

$$\int \left(\int f(w_1, \cdots, w_n)\gamma^{(k)}(x - w_j)\, dw_j \right) \phi(w_\ell)\, dw_\ell = \int \left(\int f(w_1, \cdots, w_n)\phi(w_\ell)\, dw_\ell \right) \gamma^{(k)}(x - w_j)\, dw_j$$

the inner integral is S-MRR_2 in $w_1, \cdots, w_{\ell-1}, w_{\ell+1}, \cdots, w_n$, hence the result is true for any $n \geq 3$ by induction. By Proposition 2.2 the above holds true if we integrate several PF_2 functions. Thus the proof is complete.

Now we have the following result.

Proposition 2.7. *If* $(X_1, \cdots, X_n)$ *is* $DRR(k_1, \cdots, k_n)$, *then it is* $DRR(s_1, \cdots, s_n)$ *for any integers* $s_i \geq k_i$, $i = 1, \cdots, n$.

Proof. We have

$$\Psi_{s_1,\cdots,s_n}(x_1, \cdots, x_n) = \int \cdots \int \Psi_{k_1,\cdots,k_n}(w_1, \cdots, w_n)\gamma^{(s_1-k_1)}(x_1 - w_1) \cdots \gamma^{(s_n-k_n)}(x_n - w_n)\, dw_1 \cdots dw_n.$$

For the bivariate case, the result follows by using the basic composition formula twice. For multivariate case, it follows from the previous lemma.

The DRR families are also closed under linear transformations.

Proposition 2.8. *If* $(X_1, \cdots, X_n)$ *is* $DRR(k_1, \cdots, k_n)$, *then the linear transformation vector* $(a_1X_1 + b_1, \cdots, a_nX_n + b_n)$ *is* $DRR(k_1, \cdots, k_n)$, *for any* $a_i \geq 0$, $i = 1, \cdots, n$ *and any* $b_1, \cdots, b_n$ *real.*

Proof. See Lee (1982).

The joint distribution of two independent sets of DRR random variables is again DRR.

Proposition 2.9. *Let* $(X_1, \cdots, X_m)$ *be independent of* $(Y_1, \cdots, Y_n)$. *If* $(X_1, \cdots, X_m)$ *is* $DRR(k_1, \cdots, k_m)$ *and* $(Y_1, \cdots, Y_n)$ *is* $DRR(\ell_1, \cdots, \ell_n)$, *then* $(X_1, \cdots, X_m, Y_1, \cdots, Y_n)$ *is* $DRR(k_1, \cdots, k_m, \ell_1, \cdots, \ell_n)$.

Proposition 2.10. *For* $n \geq 3$, *assume* $\mathbf{X} = (X_1, \cdots, X_n)$ *is* $DRR(k_1, \cdots, k_n)$ *with* $k_i = 0$ *or* 1 *for some* $1 \leq i \leq n$, *then* $\mathbf{X}^{(i)} = (X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n)$ *is* $DRR(k_1, \cdots, k_{i-1}, k_{i+1}, \cdots, k_n)$.

Proof. Without loss of generality assume that $i = 1$. First, assume that $k_1 = 0$ and $k_j > 0$ for $j = 2, \cdots, n$. Let g be the density function of X_1, then

$$\Psi_{0,k_2,\cdots,k_n}(x_1, \cdots, x_n) = \int_0^\infty \cdots \int_0^\infty \gamma^{(k_2)}(x_2 - t_2) \cdots \gamma^{(k_n)}(x_n - t_n)g(x_1)\, dF(t_2, \cdots, t_n \mid x_1)$$

is S-MRR$_2$. Let $\phi_1(x_1) \equiv 1$, and $\{\phi_\nu\}_2^n$ be a sequence of PF$_2$ functions. Then by definition

$$\int_0^\infty \cdots \int_0^\infty \Psi_{k_2,\cdots,k_n}(x_2, \cdots, x_n)\phi_2(x_{j_2}) \cdots \phi_{n-\ell}(x_{j_{n-\ell}})\, dx_{j_2} \cdots dx_{j_{n-\ell}}$$
$$= \int_0^\infty \cdots \int_0^\infty \int_0^\infty \Psi_{0,k_2,\cdots,k_n}(x_1, \cdots, x_n)\phi_1(x_1)\phi_2(x_{j_2}) \cdots \phi_{n-\ell}(x_{j_{n-\ell}})\, dx_1\, dx_{j_2} \cdots dx_{j_{n-\ell}}$$

is MRR$_2$ in $x_{\nu_1}, \cdots, x_{\nu_\ell}$, where $(\nu_1, \cdots, \nu_\ell)$ and $(1, j_2, \cdots, j_{n-\ell})$ are complementary sets of indices. Hence we have $(X_2, \cdots, X_n)$ is DRR$(k_2, \cdots, k_n)$.

Now, assume $k_1 = 1$ and $k_j > 0$ for $j = 2, \cdots, n$, then $\Psi_{1,k_2,\cdots,k_n}(x_1, \cdots, x_n) = \int_0^\infty \cdots \int_0^\infty \int_{x_1}^\infty \gamma^{(k_2)}(x_2 - t_2) \cdots \gamma^{(k_n)}(x_n - t_n)\, dF(t_1, t_2, \cdots, t_n)$ is S-MRR_2. Thus $\Psi_{k_2,\cdots,k_n}(x_2, \cdots, x_n) = \lim_{x_1 \to 0} \Psi_{1,k_2,\cdots,k_n}(x_1, \cdots, x_n)$ is S-MRR_2. i.e. $(X_2, \cdots, X_n)$ is $\mathrm{RR}_2(k_2, \cdots, k_n)$.

Shaked (1977) mentioned that if (X_1, X_2) is DTP(m, n) and also DRR(m, n), then X_1, X_2 are independent. We generalize this result as follows.

PROPOSITION 2.11. *Let $(X_1, \cdots, X_n)$ be DTP$(k_1, \cdots, k_n)$ and also be DRR$(k_1, \cdots, k_n)$ with nonnegative integer k_i for $i = 1, 2, \cdots, n$, then $X_1, \cdots, X_n$ are independent.*

PROOF. Because of Proposition 2.4 in Lee (1985) and Proposition 2.7, we may assume that $k_i \geq 1$ for $i = 1, 2, \cdots, n$. By the assumption of both DRR and DTP, we have the partial differential equation

$$\frac{\partial^2}{\partial x_i \partial x_j} \log \Psi_{k_1,\cdots,k_n}(x_1, \cdots, x_n) = 0,$$

for any $i \neq j$, $i, j \in \{1, 2, \cdots, n\}$. Therefore $(\partial/\partial x_i)\log \Psi_{k_1,\cdots,k_n}(x_1, \cdots, x_n) = f_i(x_i)$ for some univariate function f_i, $i = 1, \cdots, n$, and the solution for $\log \Psi_{k_1,\cdots,k_n}$ must be of the form

$$\log \Psi_{k_1,\cdots,k_n}(x_1, \cdots, x_n) = \textstyle\sum_{i=1}^n g_i(x_i) + C,$$

where $g_i(x_i) = \int f_i(x_i)\, dx_i$, and C some constant.

Hence $\Psi_{k_1,\cdots,k_n}(x_1, \cdots, x_n) = \prod_{i=1}^n h_i(x_i)$, for some functions $h_i(x_i)$, $i = 1, \cdots, n$. Notice that the partial derivatives of $\Psi_{k_1,\cdots,k_n}(\mathbf{x})$ of this form are still the product of some univariate functions. Now,

$$\begin{aligned}\bar{F}(x_1, \cdots, x_n) &= \Psi_{1,\cdots,1}(x_1, \cdots, x_n) \\ &= (-1)^{\sum_{i=1}^n k_i - n} \frac{\partial^{\sum_{i=1}^n k_i - n}}{\partial x_1^{k_1 - 1} \cdots \partial x_n^{k_n - 1}} \Psi_{k_1,\cdots,k_n}(x_1, \cdots, x_n)\end{aligned}$$

if $k_i > 1$ for all $i = 1, \cdots, n$; otherwise take proper partial derivatives. Thus $\bar{F}(x_1, \cdots, x_n) = \prod_{i=1}^n u_i(x_i)$, for some functions u_i, $i = 1, \cdots, n$. By the boundary condition, we have $u_i(x_i) = \bar{F}_i(x_i)$, for $i = 1, \cdots, n$, i.e., $X_1, \cdots, X_n$ are independent.

3. Relationships with other dependence concepts. In this section we discuss the relationships of the DRR families with the other families of negative dependence distributions considered by Brindley and Thompson (1972) and Ebrahimi and Ghosh (1981). We also give some results involving reliability theory.

Brindley and Thompson (1972) defined a random vector $(X_1, \cdots, X_n)$ to be *right corner set decreasing* (RCSD) if

$$P(X_1 > x_1', \cdots, X_n > x_n' \mid X_1 > x_1, \cdots, X_n > x_n) \quad \text{is decreasing}$$

in $\{x_i\colon x_i \geq x_i'\}$ for every choice of $x_1', \cdots, x_n'$. As in the RCSI situation, these authors gave an equivalent condition for the property RCSD. That is, $X_1, \cdots, X_n$ are RCSD if and only if

$$P(\mathbf{X}_K > \mathbf{x}_{K'} \mid \mathbf{X}_K > \mathbf{x}_K, \mathbf{X}_{\bar{K}} > \mathbf{x}_{\bar{K}}) \quad \text{is decreasing}$$

in $x_{\bar{K}}$ for all sets $K \subset \{1, 2, \cdots, n\}$. See Lee (1985) for the definition of $\mathbf{X}_K$ and $\mathbf{X}_{\bar{K}}$.

The next proposition gives the relationship between RCSD and DRR$(1, \cdots, 1)$.

PROPOSITION 3.1. *Let $(X_1, \cdots, X_n)$ be a positive random vector. If $(X_1, \cdots, X_n)$ is DRR$(1, \cdots, 1)$, then $(X_1, \cdots, X_n)$ is RCSD. For the bivariate case, the converse holds.*

PROOF. $(X_1, \cdots, X_n)$ is DRR$(1, \cdots, 1)$ implies that the survival function $\bar{F}$ is MRR_2 according to Karlin and Rinott (1980). Hence

$$\bar{F}(\mathbf{x} \vee \mathbf{y})\bar{F}(\mathbf{x} \wedge \mathbf{y}) \le \bar{F}(\mathbf{x})\bar{F}(\mathbf{y}) \quad \text{for any} \quad \mathbf{x}, \mathbf{y},$$

and this inequality is equivalent to the condition that

$$P(\mathbf{X}_K > \mathbf{x}_{K'} \mid \mathbf{X}_K > \mathbf{x}_K, \mathbf{X}_{\bar{K}} > \mathbf{x}_{\bar{K}}) \quad \text{is decreasing}$$

in $\mathbf{x}_{\bar{K}}$ for all sets K. For $n = 2$, the converse can be checked from the definition.

Random vectors with DRR properties can also be characterized by some reliability functions as was in the DTP cases. See Lee (1985).

PROPOSITION 3.2. *Let X be a positive absolutely continuous random vector.*
(1) *If $\mathbf{X}$ is DRR$(0, \cdots, 0, 1)$, then $r(x_n \mid \mathbf{X}^{(n)} = \mathbf{x}^{(n)})$ is increasing in $\mathbf{x}^{(n)} \in S_{\mathbf{X}^{(n)}}$, for any x_n.*
(2) *If $\mathbf{X}$ is DRR$(0, \cdots, 0, 2)$, then $m(x_n \mid \mathbf{X}^{(n)} = \mathbf{x}^{(n)})$ is decreasing in $\mathbf{x}^{(n)} \in S_{\mathbf{X}^{(n)}}$, for any x_n.*
(3) *If $\mathbf{X}$ is DRR$(0, \cdots, 0, m)$, for some $m > 1$ then*

$$\frac{E[(X_n - x_n)^{m-1} \mid X_n > x_n, \mathbf{X}^{(n)} = \mathbf{x}^{(n)}]}{E[(X_n - x_n)^{m-2} \mid X_n > x_n, \mathbf{X}^{(n)} = \mathbf{x}^{(n)}]}$$

is decreasing in $\mathbf{x}^{(n)} \in S_{\mathbf{X}^{(n)}}$ for any x_n.
(4) *If X is DRR$(1, \cdots, 1)$, then $r(x_j \mid \mathbf{X}^{(j)} > \mathbf{x}^{(j)})$ is increasing in $\mathbf{x}^{(j)}$, for any x_j, $j = 1, \cdots, n$.*

For $n = 2$ case, the converses of the above four assertions are also true.

PROOF. These follow since S-MRR_2 functions are RR_2 in pairs, and if a twice differentiable function $g(\mathbf{x})$ is RR_2 in pairs then $\partial^2/(\partial x_i \partial x_j)\log g(\mathbf{x}) \le 0$ for any $i, j = 1, \cdots, n$. The bivariate case can easily be checked.

Ebrahimi and Ghosh (1981) discussed the following notions of negative dependence.

DEFINITION 3.3. (Ebrahimi and Ghosh, 1981). A sequence of random variables $\{X_1, \cdots, X_n\}$ is said to be *right tail decreasing in sequence* (RTDS) if for all real x_{i+1}, $i = 1, 2, \cdots, n-1$.

$$P[X_{i+1} > x_{i+1} \mid \cap_{j=1}^{i} (X_j > x_j)] \text{ is decreasing in } x_1, \cdots, x_i.$$

DEFINITION 3.4. (Ebrahimi and Ghosh, 1981). The random variables $X_1, \cdots, X_n$ are said to be *conditionally decreasing in sequence* (CDS) if for $i = 2, 3, \cdots, n$, and all real numbers x_i,

$$P(X_i > x_i \mid X_1 = x_1, \cdots, X_{i-1} = x_{i-1}) \text{ is decreasing}$$

in $x_1, \cdots, x_{i-1}$.

PROPOSITION 3.5. *Let* $\mathbf{X}$ *be a positive random vector.*
(1) *If* $\mathbf{X}$ *is DRR*$(0, \cdots, 0, 1)$ *then* $\mathbf{X}$ *is CDS.*
(2) *If* $\mathbf{X}$ *is DRR*$(1, \cdots, 1)$, *then* $\mathbf{X}$ *is RTDS.*

PROOF. The proof follows from Proposition 2.10 and Proposition 3.2, and by similar arguments as in the DTP situations.

Even though many of the properties and theorems are similar to the DTP case, the construction of DTP distributions does not carry through in the DRR case. Namely, if (X, W) is DRR$(m, 0)$ and (Y, W) is DRR$(n, 0)$, such that X and Y are conditionally independent given $W = w$, then by the basic composition formula, (X, Y) is DTP(m, n) instead of DRR.

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