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NONCENTRAL LIMIT THEOREMS FOR QUADRATIC FORMS IN RANDOM VARIABLES HAVING LONG-RANGE DEPENDENCE

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We study the weak convergence in $D[0, 1]$ of the quadratic form $\sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} a_{j-k} H_m(X_j) H_m(X_k)$, adequately normalized. Here a_s, $-\infty < s < \infty$ is a symmetric sequence satisfying $\sum |a_s| < \infty$, H_m is the mth Hermite polynomial and $\{X_j\}$, $j \geq 1$, is a normalized Gaussian sequence with covariances $r_k \sim k^{-D}L(k)$ as $k \to \infty$, where $0 < D < 1$ and L is slowly varying. We prove that, for all $m \geq 1$, the limit is Brownian motion when $\frac{1}{2} < D < 1$ and it is the non-Gaussian Rosenblatt process when $0 < D < \frac{1}{2}$.

1. Introduction. Dobrushin and Major (1979), Taqqu (1979a) and Breuer and Major (1983) have studied the weak convergence of the stochastic process $\sum_{j=1}^{[Nt]} H_m(X_j)$, $0 \leq t \leq 1$. Here, H_m is the mth Hermite polynomial and the sequence X_j, $j \geq 1$, is Gaussian with mean 0 and covariances $r_k = EX_j X_{j+k}$ that behave like $k^{-D}L(k)$ as $k \to \infty$, where $0 < D < 1$ and L is slowly varying. The sequence $\{X_j\}$ exhibits a long-range dependence because $\sum_{k=-\infty}^{+\infty} r_k = \infty$. It was shown that when $D > 1/m$, $\sum_{j=1}^{[Nt]} H_m(X_j)$, adequately normalized, converges to Brownian motion. But when $0 < D < 1/m$, the limit depends on m. It is non-Gaussian when $m \geq 2$. When $m = 2$, the limit is the Rosenblatt process defined in Section 2.

We study here the weak convergence in $D[0, 1]$ of the quadratic form

$$\sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} a_{j-k} H_m(X_j) H_m(X_k),$$

where a_s, $-\infty < s < \infty$ is a sequence satisfying $a_{-s} = a_s$ and $\sum_{s=-\infty}^{+\infty} |a_s| < \infty$.

We prove that when $0 < D < \frac{1}{2}$, this quadratic form, adequately normalized, converges weakly, *for all* $m \geq 1$, to $CR(t)$ where $R(t)$ is the Rosenblatt process and $C = m!m(\sum_{s=-\infty}^{+\infty} a_s r_s^{m-1})$ is a constant. On the other hand, when $\frac{1}{2} < D < 1$ the quadratic form converges weakly, *for all* $m \geq 1$, to Brownian motion. Thus the limiting process is either the Rosenblatt process or Brownian motion, depending on whether $0 < D < \frac{1}{2}$ or $\frac{1}{2} < D < 1$. The fact that we deal with quadratic forms causes the case of general m to behave like the case $m = 2$.

When $0 < D < \frac{1}{2}$, $m = 1$ and $\sum_{s=-\infty}^{+\infty} a_s = 0$, convergence to $m!m(\sum_{s=-\infty}^{+\infty} a_s r_s^{m-1})R(t)$ means convergence to 0. Using a result of Fox and Taqqu (1983) we show that when further conditions are imposed on the sequences r_s and a_s, the quadratic form $\sum_{i=1}^{N} \sum_{j=1}^{N} a_{i-j} X_i X_j$ can be renormalized so that the limiting distribution is Gaussian.

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The main results are stated in Section 2. The proofs utilize the Wiener-Itô-Dobrushin representation of $H_m(X_j)$ and the corresponding diagram formula. Preliminary lemmas are established in Sections 3 and 4. The results stated in Section 2 are proven in Sections 5, 6 and 7.

2. Limit theorems. Let X_j, $j \geq 1$, be a stationary Gaussian sequence satisfying $EX_j = 0$ and $EX_j^2 = 1$ and suppose that there is a constant $0 < D < 1$ and a slowly varying function L such that

$$r_k = EX_jX_{j+k} \sim k^{-D}L(k) \quad \text{as} \quad k \to \infty. \tag{2.1}$$

(We write $a_k \sim b_k$ if $a_k/b_k \to 1$.) Recall that L is slowly varying at ∞ (at 0) if it is nonnegative and if $L(xt)/L(x) \to 1$ as $x \to \infty$ ($x \to 0$), for all $t > 0$.

Let m be a positive integer and consider the random variables $H_m(X_j)$, $j \geq 1$, where H_m is the mth Hermite polynomial with leading coefficient 1. In particular, $H_1(X) = X$, $H_2(X) = X^2 - 1$ and $H_3(X) = X^3 - 3X$. We indicated in the introduction that when $0 < D < 1/m$, the weak limit in $D[0, 1]$ of the stochastic process $\sum_{j=1}^{[Nt]} H_m(X_j)$, $0 \leq t \leq 1$, adequately normalized, is different for different values of m. In the case $m = 2$,

$$\frac{1}{N^{1-D}L(N)} \sum_{j=1}^{[Nt]} H_2(X_j) \Rightarrow R(t)$$

where $R(t)$ is the Rosenblatt process (Taqqu, 1975). The process $R(t)$ admits the following representation in terms of Wiener-Itô-Dobrushin integrals:

$$R(t) = \frac{1}{2\Gamma(D)\cos(D\pi/2)} \int''_{\mathbb{R}^2} \frac{e^{i(x_1+x_2)t} - 1}{i(x_1 + x_2)} |x_1|^{(D-1)/2} |x_2|^{(D-1)/2} \, dW(x_1)\, dW(x_2)$$

where W is a complex-valued Gaussian white noise measure on $\mathbb{R}^1$ and where $\int''$ means that no integration is performed on the diagonals $x_1 = \pm x_2$. (See Dobrushin, 1979; Taqqu, 1979b; or Major, 1981.) The finite-dimensional distributions of $R(t)$ are determined by

$$E \exp\left(i \sum_{j=1}^{p} u_j R(t_j)\right)$$
$$= \exp\left\{\frac{1}{2} \sum_{k=2}^{\infty} \frac{(2i)^k}{k} \sum_{s_1, s_2, \cdots, s_k \in \{1,2,\cdots,p\}} u_{s_1} u_{s_2} \cdots u_{s_p} \right.$$
$$\cdot \int_0^{t_{s_1}} dx_1 \int_0^{t_{s_2}} dx_2 \cdots \int_0^{t_{s_k}} dx_k$$
$$\left. \cdot |x_1 - x_2|^{-D} |x_2 - x_3|^{-D} \cdots |x_{k-1} - x_k|^{-D} |x_k - x_1|^{-D} \right\}.$$

(This is the corrected form of formula (6.1) of Taqqu (1975).)

Let a_s, $-\infty < s < \infty$, be a sequence of constants satisfying $a_{-s} = a_s$ and

$$\sum_{s=-\infty}^{+\infty} |a_s| < \infty.$$

THEOREM 1. *Suppose $0 < D < 1/2$. Then the stochastic process*

$$Z_N(t) = \frac{1}{N^{1-D}L(N)} \{\textstyle\sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} a_{j-k} H_m(X_j) H_m(X_k) - E \sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} a_{j-k} H_m(X_j) H_m(X_k)\}$$

converges weakly in $D[0, 1]$ to

$$m!m(\textstyle\sum_{s=-\infty}^{+\infty} a_s r_s^{m-1}) R(t)$$

as $N \to \infty$.

The next theorem shows that when $1/2 < D < 1$, the quadratic form, adequately normalized, converges to Brownian motion.

THEOREM 2. *Suppose $1/2 < D < 1$. Then the stochastic process*

$$Z_N(t) = \frac{1}{\sqrt{N}} \{\textstyle\sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} a_{j-k} H_m(X_j) H_m(X_k) - E \sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} a_{j-k} H_m(X_j) H_m(X_k)\}$$

converges weakly in $D[0, 1]$ to $\sigma_m B(t)$ where $B(t)$ is standard Brownian motion and where

$$\sigma_m^2 = (m!)^2 \sum_{n=1}^{m} \binom{m}{n}^2 \sum_{s_1=-\infty}^{+\infty} \sum_{s_2=-\infty}^{+\infty} a_{s_1} a_{s_2} (r_{s_1} r_{s_2})^{m-n} \cdot \sum_{q=0}^{n} \binom{n}{q}^2 \sum_{k=-\infty}^{+\infty} r_k^q r_{k+s_1-s_2}^q r_{k+s_1}^{n-q} r_{k-s_2}^{n-q}.$$

REMARK 1. In fact,

$$\sigma_m B(t) = m! \sum_{n=1}^{m} \frac{1}{n!} \binom{m}{n} [\textstyle\sum_{s=-\infty}^{+\infty} a_s r_s^{m-n} Z(n, s, t)]$$

where $\{Z(n, s, t), 1 \le n \le m, -\infty < s < \infty\}$ is a collection of dependent Brownian motions satisfying $EZ(n, s, t) = 0$ and

$$EZ(n_1, s_1, t_1) Z(n_2, s_2, t_2) = \begin{cases} 0 & \text{if } n_1 \neq n_2 \\ \min(t_1, t_2)(n!)^2 \sum_{q=0}^{n} \binom{n}{q}^2 \sum_{k=-\infty}^{+\infty} r_k^q r_{k+s_1-s_2}^q r_{k+s_1}^{n-q} r_{k-s_2}^{n-q} & \text{if } n_1 = n_2 = n. \end{cases}$$

REMARK 2. When $m = 1$, $\sigma_1 B(t) = \sum_{s=-\infty}^{+\infty} a_s Z(1, s, t)$ and

$$\sigma_1^2 = 2 \sum_{k=-\infty}^{+\infty} \sum_{s_1=-\infty}^{+\infty} \sum_{s_2=-\infty}^{+\infty} a_{s_1} a_{s_2} r_k r_{k+s_1-s_2}. \tag{2.2}$$

It is possible to obtain convergence to a normal distribution even when $0 < D < 1/2$. Indeed, set $t = 1$, $m = 1$ and suppose that the conditions of Theorem 1 are satisfied and that, in addition,

$$\sum_{s=-\infty}^{+\infty} a_s = 0. \tag{2.3}$$

Then, according to Theorem 1,

$$\frac{1}{N^{1-D}L(N)} \{\sum_{j=1}^N \sum_{k=1}^N a_{j-k}X_jX_k - E \sum_{j=1}^N \sum_{k=1}^N a_{j-k}X_jX_k\} \to 0$$

in probability as $N \to \infty$. Thus the possibility exists that if further conditions are placed on the sequence a_s, a normalization smaller than $N^{1-D}L(N)$ might lead to a nondegenerate limit distribution.

In particular, we could assume

$$a_s \sim s^{-\gamma}L_1(s) \quad \text{as} \quad s \to \infty, \tag{2.4}$$

where $\gamma > 1$ and $L_1(s)$ is a slowly varying function. If further restrictions are imposed, Relations (2.1), (2.3) and (2.4) become equivalent to

$$f(x) \sim |x|^{D-1}L_2(x) \quad \text{as} \quad x \to 0$$

and

$$|g(x)| \sim |x|^{\gamma-1}L_3(x) \quad \text{as} \quad x \to 0,$$

where L_2 and L_3 are slowly varying at 0 and f and g are defined by

$$r_k = \int_{-\pi}^{\pi} e^{ikx}f(x)\,dx$$

and

$$a_s = \int_{-\pi}^{\pi} e^{isx}g(x)\,dx.$$

Now say that a sequence $\{u_k, k \geq 1\}$ has *bounded variation* if $\sum_{k=1}^{\infty} |u_k - u_{k+1}| < \infty$ and that it is *quasi-monotonically convergent to zero* if $u_k \to 0$ and if for some constants $c \geq 0$ and $k_0(c) > 0$,

$$u_{k+1} \leq u_k(1 + (c/k)) \quad \text{for all} \quad k \geq k_0(c).$$

(This last definition assumes that the u_k are positive for large k. An analogous definition applies if the u_k are negative for large k.)

THEOREM 3. *Suppose that*

(1) $r_k \sim k^{-D}L(k)$ *with* $0 < D < 1/2$, *the sequence* $\{r_k\}$ *has bounded variation and it is quasi-monotonically convergent to* 0.

(2) $|a_k| \sim k^{-\gamma}L_1(k)$ *with* $1 < \gamma < 3$, *satisfying*

  (i) $a_k = a_{-k}$

  (ii) a_k *is positive for large* k (*or negative for large* k)

  (iii) $\sum_{k=-\infty}^{\infty} a_k = 0$.

(3) $D + \gamma > 3/2$.

Then

$$Z_N = \frac{1}{\sqrt{N}} \{\sum_{j=1}^N \sum_{k=1}^N a_{j-k}X_jX_k - E \sum_{j=1}^N \sum_{k=1}^N a_{j-k}X_jX_k\}$$

tends in distribution as $N \to \infty$ *to a normal random variable with mean* 0 *and variance* $16\pi^3 \int_{-\pi}^{\pi} [f(x)g(x)]^2\,dx$.

EXAMPLE. Fractional Brownian motion $B_\alpha(t)$ with index $0 < \alpha < 1$ is a Gaussian process with stationary increments, mean zero and satisfying $EB_\alpha^2(t) = |t|^{2\alpha}$. Its increments $B_\alpha(k) - B_\alpha(k-1)$, $-\infty < k < \infty$, have covariances

$$u_k = \tfrac{1}{2}\{(k+1)^{2\alpha} - 2k^{2\alpha} + |k-1|^{2\alpha}\}, \quad k \geq 0$$

which satisfy $u_k \sim \alpha(2\alpha - 1)k^{2\alpha-2}$ as $k \to \infty$ when $\alpha \neq \tfrac{1}{2}$ and also $\sum_{k=-\infty}^{+\infty} u_k = 0$ when $\alpha < \tfrac{1}{2}$. Examples for the sequences $\{r_k\}$ and $\{a_k\}$ of Theorem 3 can be obtained by setting $r_k = u_k$ with $D = 2 - 2\alpha$ and $\tfrac{3}{4} < \alpha < 1$, and by setting $a_k = u_k$ with $\gamma = 2 - 2\alpha$ and $0 < \alpha < \tfrac{1}{2}$.

3. Applications of the diagram formula. Let X_j, $j \geq 1$, be a stationary Gaussian sequence as defined in Section 2. In order to prove Theorems 1 and 2, we will make use of the spectral representation of the sequence X_j. Let G be the Borel measure on $[-\pi, \pi]$ satisfying

$$r_k = \int_{-\pi}^{\pi} e^{ikx}\, dG(x), \quad -\infty < k < \infty.$$

We have the representation

$$X_j = \int_{-\pi}^{\pi} e^{ijx}\, dZ_G(x),$$

where Z_G is the random spectral measure determined by the sequence X_j. (See Major, 1981, for example.) According to Theorem 4.2 of Major (1981)

$$H_m(X_j) = \int_{[-\pi,\pi]^m}'' e^{ij(x_1+\cdots+x_m)}\, dZ_G(x_1) \cdots dZ_G(x_m), \tag{3.1}$$

where the integral is a multiple Wiener-Itô-Dobrushin integral in the sense of Dobrushin (1979).

The proofs of Theorems 1 and 2 involve the "diagram formula" for multiple Wiener-Itô-Dobrushin integrals, which can be found, for example, as Theorem 5.3 of Major (1981). We state the diagram formula below as Proposition 3.1. First we need to introduce some notation. Let $\overline{h}_G^n$ be the space of functions $h: [-\pi, \pi]^n \to C$ satisfying

$$h(-x_1, \cdots, -x_n) = \overline{h(x_1, \cdots, x_n)}$$

and

$$\int_{[-\pi,\pi]^n} |h(x_1, \cdots, x_n)|^2\, dG(x_1) \cdots dG(x_n) < \infty.$$

For $h \in \overline{h}_G^n$, we define

$$I_n(h) = \int_{[-\pi,\pi]^n}'' h(x_1, \cdots, x_n)\, dZ_G(x_1) \cdots dZ_G(x_n).$$

If c is a constant put $I_0(c) = c$.

Let $n_1, \cdots, n_p$ be given positive integers. The diagram formula is useful in evaluating products of the form $\prod_{k=1}^{p} I_{n_k}(h_k)$, where $h_k \in \bar{h}_G^{n_k}$. Put $N_0 = 0$ and $N_k = n_1 + \cdots + n_k$, $k = 1, \cdots, p$. Introduce the set of "vertices":

$$V = \{(1, 1), (1, 2), \cdots, (1, n_1), (2, 1), \cdots, (2, n_2), \cdots, (p, 1), \cdots, (p, n_p)\}.$$

To each vertex $v \in V$ we associate an integer denoting the position at which v appears in the above list. Thus the position of (1, 1) is 1, the position of (1, 2) is 2, and so on. The position of the last vertex (p, n_p) is N_p. A *diagram* γ of order $(n_1, \cdots, n_p)$ is an undirected graph on the vertices V such that each vertex is met by at most one edge and such that if vertices (j_1, k_1) and (j_2, k_2) are joined by an edge it follows that $j_1 \neq j_2$. The diagram γ then has N_p vertices and may have $0, 1, 2, \cdots$ edges. Let $\Gamma(n_1, \cdots, n_p)$ denote the set of all diagrams of order $(n_1, \cdots, n_p)$. For each diagram $\gamma \in \Gamma(n_1, \cdots, n_p)$, let $|\gamma|$ denote the number of edges in γ.

For a fixed $\gamma \in \Gamma(n_1, \cdots, n_p)$, there are $n_\gamma = N_p - 2|\gamma|$ vertices which are met by no edges in γ. Denote by $\tau_1 < \tau_2 < \cdots < \tau_{n_\gamma}$ the positions of these vertices. Let $\sigma_1 < \sigma_2 < \cdots < \sigma_{|\gamma|}$ be the positions of the vertices which are connected by edges in γ to vertices with larger positions. Let $\delta_1, \cdots, \delta_{|\gamma|}$ be the positions of the vertices which are connected to the vertices with positions $\sigma_1, \cdots, \sigma_{|\gamma|}$, respectively. Then we have

$$\{1, \cdots, N_p\} = \{\tau_1, \cdots, \tau_{n_\gamma}, \sigma_1, \cdots, \sigma_{|\gamma|}, \delta_1, \cdots, \delta_{|\gamma|}\}.$$

Now suppose that functions $h_1 \in \bar{h}_G^{n_1}, h_2 \in \bar{h}_G^{n_2}, \cdots, h_p \in \bar{h}_G^{n_p}$ are given and define

$$h(x_1, \cdots, x_{N_p}) = h_1(x_1, \cdots, x_{n_1}) h_2(x_{N_1+1}, \cdots, x_{N_2}) \cdots h_p(x_{N_{p-1}+1}, \cdots, x_{N_p}).$$

Then, for each diagram $\gamma \in \Gamma(n_1, \cdots, n_p)$, perform the two following operations:

(1) introduce new variables $y_1, \cdots, y_{n_\gamma}$ and $z_1, \cdots, z_{|\gamma|}$ and let them appear as arguments of h by setting $y_j = x_{\tau_j}$, $z_j = x_{\sigma_j}$, and $-z_j = x_{\delta_j}$. The new function is denoted $h(y_1, \cdots, y_{n_\gamma}, z_1, \cdots, z_{|\gamma|})$.

(2) using G as integrator, integrate out the variables z_j so as to obtain

$$h_\gamma(y_1, \cdots, y_{n_\gamma}) = \int_{\mathbb{R}^{|\gamma|}} h(y_1, \cdots, y_{n_\gamma}, z_1, \cdots, z_{|\gamma|})\, dG(z_1) \cdots dG(z_{|\gamma|}).$$

PROPOSITION 3.1. *The Diagram Formula.*

$$\prod_{k=1}^{p} I_{n_k}(h_k) = \sum_{\gamma \in \Gamma(n_1, \cdots, n_p)} I_{n_\gamma}(h_\gamma).$$

A diagram $\gamma \in \Gamma(n_1, \cdots, n_p)$ is called *complete* if each vertex is met by an edge. If γ is complete then $|\gamma| = \frac{1}{2} N_p$ and $n_\gamma = 0$. Let $\Gamma_0(n_1, \cdots, n_p)$ be the set of complete diagrams of order $(n_1, \cdots, n_p)$. Since $I_{n_\gamma}(h_\gamma)$ is the constant h_γ when $n_\gamma = 0$ and is an integral with mean 0 when $n_\gamma \geq 1$, the following corollary to Proposition 3.1 holds.

PROPOSITION 3.2.

$$E[\prod_{k=1}^{p} I_{n_k}(h_k)] = \sum_{\gamma \in \Gamma_0(n_1, \cdots, n_p)} h_\gamma.$$

A complete diagram $\gamma \in \Gamma_0(n_1, \cdots, n_p)$ is called *regular* if, whenever (j, i_1) is joined to (k_1, ℓ_1) and (j, i_2) is joined to (k_2, ℓ_2), it follows that $k_1 = k_2$. Let $\Gamma_1(n_1, \cdots, n_p)$ be the set of regular diagrams of order $(n_1, \cdots, n_p)$. That set is empty if p is odd or if $n_1, \cdots, n_p$ are not pairwise equal.

The following proposition is about moments of Gaussian random variables whose covariances are identical to those of Wiener-Itô-Dobrushin integrals.

PROPOSITION 3.3. *Let $Z_1, \cdots, Z_p$ be a jointly Gaussian collection of random variables having mean 0 and satisfying*

$$EZ_jZ_k = EI_{n_j}(h_j)I_{n_k}(h_k), \quad 1 \le j, \quad k \le p.$$

Then

$$E(Z_1 \cdots Z_p) = \sum_{\gamma \in \Gamma_1(n_1, \cdots, n_p)} h_\gamma.$$

PROOF. We may suppose that p is even and $n_1, \cdots, n_p$ are pairwise equal (otherwise $EZ_1 \cdots Z_p = 0$ and the proposition trivially holds). Let $\Gamma' = \Gamma_0(\ell_1, \cdots, \ell_p)$, where $\ell_1 = \ell_2 = \cdots = \ell_p = 1$. We consider a diagram $g \in \Gamma'$ as a graph on the vertices $\{1, \cdots, p\}$ in which each vertex has degree 1. Choose $g \in \Gamma'$ such that one has $n_j = n_k$ for each edge $(j, k) \in g$.

A graph g of this type can be used to construct a diagram $\gamma \in \Gamma_1(n_1, \cdots, n_p)$ as follows. For each edge $(j, k) \in g$, construct a complete diagram γ_{jk} on the vertices

$$(j, 1), \cdots, (j, n_j), \quad (k, 1), \cdots, (k, n_j).$$

In this way we obtain a diagram $\gamma \in \Gamma_1(n_1, \cdots, n_p)$ such that

$$h_\gamma = \prod_{(j,k)\in g} h^{jk}_{\gamma_{jk}},$$

where

$$h^{jk}(x_1, \cdots, x_{2n_j}) = h_j(x_1, \cdots, x_{n_j})h_k(x_{n_j+1}, \cdots, x_{2n_j})$$

and

$$h^{jk}_{\gamma_{jk}} = \int_{\mathbb{R}^{n_j}} h^j(z_1, \cdots, z_{n_j})h^k(-z_1, \cdots, -z_{n_j})\, dG(z_1) \cdots dG(z_{n_j}).$$

Therefore

$$\begin{aligned}\sum_{\gamma\in\Gamma_1(n_1,\cdots,n_p)} h_\gamma &= \sum_{g\in\Gamma'} \prod_{(j,k)\in g} \sum_{\gamma_{jk}\in\Gamma_0(n_j,n_k)} h^{jk}_{\gamma_{jk}} \\ &= \sum_{g\in\Gamma'} \prod_{(j,k)\in g} E[I_{n_j}(h_j)I_{n_k}(h_k)]\end{aligned}$$

by Proposition 3.2. This last expression equals

$$\sum_{g\in\Gamma'} \prod_{(j,k)\in g} EZ_jZ_k = E(Z_1 \cdots Z_p)$$

since $Z_1, \cdots, Z_p$ are jointly Gaussian. This establishes Proposition 3.3. □

LEMMA 3.4.

$$H_m(X_j)H_m(X_k) = m!r_{j-k}^m + \sum_{n=1}^m \left[(m-n)!\binom{m}{n}^2 r_{j-k}^{m-n}K_n(j,k)\right]$$

where

$$(3.2)\quad K_n(j,k) = \int_{[-\pi,\pi]^{2n}}'' e^{ij(x_1+\cdots+x_n)+ik(x_{n+1}+\cdots+x_{2n})}\,dZ_G(x_1)\cdots dZ_G(x_{2n}).$$

PROOF. In order to apply Proposition 3.1, define

$$h_1(x_1, \cdots, x_m) = e^{ij(x_1+\cdots+x_m)}$$

and

$$h_2(x_1, \cdots, x_m) = e^{ik(x_1+\cdots+x_m)}.$$

For each diagram $\gamma \in \Gamma(m, m)$, define h_γ as above and let $n = \frac{1}{2}n_\gamma$. It is clear that

$$I_{2n}(h_\gamma) = \int_{\mathbb{R}^{2n}} e^{ij(x_1+\cdots+x_n)+ik(x_{m+1}+\cdots+x_{m+n})}$$

$$\cdot\, dZ_G(x_1)\cdots dZ_G(x_n)\,dZ_G(x_{m+1})\cdots dZ_G(x_{m+n})$$

$$\cdot \int_{\mathbb{R}^{m-n}} e^{i(j-k)(z_1+\cdots+z_{m-n})}\,dG(z_1)\cdots dG(z_{m-n})$$

$$= \begin{cases} r_{j-k}^m & \text{if } n = 0 \\ r_{j-k}^{m-n}K_n(j,k) & \text{if } n = 1, \cdots, m. \end{cases}$$

Since the number of diagrams $\gamma \in \Gamma(m, m)$ satisfying $|\gamma| = m - n$ is $(m-n)!\binom{m}{n}^2$, the statement of Lemma 3.4 follows from (3.1). □

Let

$$\mu_{N,m} = E\sum_{j=1}^N \sum_{k=1}^N a_{j-k}H_m(X_j)H_m(X_k).$$

LEMMA 3.5.

$$\sum_{j=1}^N \sum_{k=1}^N a_{j-k}H_m(X_j)H_m(X_k) - \mu_{N,m}$$

$$= \sum_{n=1}^m (m-n)!\binom{m}{n}^2 \sum_{|s|<N} a_s r_s^{m-n} W(n, s, N),$$

where

$$(3.3)\qquad W(n, s, N) = \sum_{j=1}^{N-|s|} K_n(j, j+|s|).$$

PROOF. The result follows from Lemma 3.4 because

$$\sum_{j=1}^{N} \sum_{k=1}^{N} a_{j-k} H_m(X_j) H_m(X_k)$$

$$= \sum_{j=1}^{N} \sum_{k=1}^{N} a_{j-k} \left\{ m! r_{j-k}^{m} + \sum_{n=1}^{m} (m-n)! \binom{m}{n}^2 r_{j-k}^{m-n} K_n(j, k) \right\}$$

$$= \mu_{N,m} + \sum_{n=1}^{m} (m-n)! \binom{m}{n}^2 \left[\sum_{j=1}^{N} \sum_{k=1}^{N} a_{j-k} r_{j-k}^{m-n} K_n(j, k) \right]$$

$$= \mu_{N,m} + \sum_{n=1}^{m} (m-n)! \binom{m}{n}^2 \sum_{|s|<N} a_s r_s^{m-n} W(n, s, N). \quad \square$$

LEMMA 3.6.

$$E[K_n(j, j+s) K_n(k, k+t)] = \sum_{q=0}^{n} (n!)^2 \binom{n}{q}^2 r_{j-k}^{q} r_{j-k+s-t}^{q} r_{j-k+s}^{n-q} r_{j-k-t}^{n-q}.$$

PROOF. In order to use Proposition 3.2, define

$$h_1(x_1, \cdots, x_{2n}) = \exp(ij(x_1 + \cdots + x_n) + i(j+s)(x_{n+1} + \cdots + x_{2n}))$$

and

$$h_2(x_1, \cdots, x_{2n}) = \exp(ik(x_1 + \cdots + x_n) + i(k+t)(x_{n+1} + \cdots + x_{2n})).$$

For each $\gamma \in \Gamma(2n, 2n)$, define h_γ as above. Then we have

$$EK_n(j, j+s) K_n(k, k+t) = \sum_{\gamma \in \Gamma_0(2n,2n)} h_\gamma.$$

Fix a diagram $\gamma \in \Gamma_0(2n, 2n)$. Let q be the number of edges in γ which connect a vertex of the form $(1, j)$ to a vertex of the form $(2, k)$, where $1 \le j, k \le n$. The number of edges connecting other pairs of vertices are then determined as follows. There are $n - q$ edges from $(1, j)$ to $(2, k)$, where $1 \le j \le n$ and $n + 1 \le k \le 2n$. There are also $n - q$ edges from $(1, j)$ to $(2, k)$, where $n + 1 \le j \le 2n$ and $1 \le k \le n$. Thus there are q edges from $(1, j)$ to $(2, k)$, where $n + 1 \le j, k \le 2n$. Therefore we conclude

$$h_\gamma = r_{j-k}^{q} r_{j-k+s-t}^{q} r_{j-k+s}^{n-q} r_{j-k-t}^{n-q}.$$

Since the number of diagrams of the above form is

$$\binom{n}{q}^4 [q!(n-q)!]^2 = \binom{n}{q}^2 (n!)^2,$$

the result of Lemma 3.6 follows. $\square$

4. Preliminary lemmas. Define $K_n(j, k)$ as in (3.2) and $W(n, s, N)$ as in (3.3).

LEMMA 4.1.

(a) *If* $0 < D < 1/2$ *then for* $n = 1, \cdots, m$

$$\sup_{0 \le M < N} \sup_{s \ge 0} \frac{E[\sum_{j=M+1}^N K_n(j, j+s)]^2}{(N-M)^{2-2D} L^2(N-M)} < \infty.$$

(b) *If* $0 < D < 1/2$ *then for* $n = 2, \cdots, m$

$$\lim_{N\to\infty} \sup_{|s|<N} \frac{E[W(n, s, N)]^2}{N^{2-2D} L^2(N)} = 0.$$

(c) *If* $1/2 < D < 1$ *then for* $n = 1, \cdots, m$

$$\sup_{0 \le M < N} \sup_{s \ge 0} \frac{E[\sum_{j=M+1}^N K_n(j, j+s)]^4}{N^2} < \infty.$$

PROOF. We can choose a constant C_1 and a nonincreasing sequence b_k so that $|r_k| \le b_k$ and $b_k \sim C_1 k^{-D} L(k)$. (See Seneta, 1976, page 20.) Note that for all $0 \le M < N$ and $s \ge 0$

$$\sum_{j=M+1}^N \sum_{k=M+1}^N b_{j-k+s}\, b_{j-k-s} \le (N-M) \sum_{|k|<N-M} b_k^2, \tag{4.1}$$

because for each $j = M+1, \cdots, N$

$$\begin{aligned}\sum_{k=M+1}^N b_{j-k+s} b_{j-k-s} &\le \{\sum_{k=M+1}^N b_{j-k+s}^2 \sum_{\ell=M+1}^N b_{j-\ell-s}^2\}^{1/2} \\ &\le \sum_{|k|<N-M} b_k^2.\end{aligned}$$

(a) Because of Lemma 3.6, Part a will follow from the relation

$$\sup_{0\le M<N} \sup_{s \ge 0} \frac{\sum_{j=M+1}^N \sum_{k=M+1}^N r_{j-k}^{2q} r_{j-k+s}^{n-q} r_{j-k-s}^{n-q}}{(N-M)^{2-2D} L^2(M-N)} < \infty, \quad q = 0, \cdots, m.$$

First suppose that $q = 0$. Then the above sum is majorized by

$$\sum_{j=M+1}^N \sum_{k=M+1}^N b_{j-k+s} b_{j-k-s},$$

which by (4.1) is majorized by

$$(N-M) \sum_{|k|<N-M} b_k^2 \le C_2 (N-M)^{2-2D} L^2(N-M)$$

for some constant C_2. On the other hand, if $q \ge 1$ the sum is majorized by $\sum_{j=M+1}^N \sum_{k=M+1}^N b_{j-k}^2$ and (4.1) is again applicable.

(b) Because of Lemma 3.6, Part b will follow from the relation

$$\lim_{N\to\infty} \sup_{|s|<N} \frac{\sum_{j=1}^{N-s} \sum_{k=1}^{N-s} r_{j-k}^{2q} r_{j-k+s}^{n-q} r_{j-k-s}^{n-q}}{N^{2-2D} L^2(N)} = 0, \quad q = 0, \cdots, n. \tag{4.2}$$

To prove (4.2), we consider three cases. When $q \ge 2$, the sum in (4.2) is majorized by

$$\sum_{j=1}^N \sum_{k=1}^N r_{j-k}^{2q} \le N \sum_{k=-N}^N r_k^{2q} = o(N^{2-2D} L^2(N)).$$

When $n - q \geq 2$, the sum in (4.2) is majorized by

$$\sum_{j=1}^{N} \sum_{k=1}^{N} b_{j-k+s}^{n-q} b_{j-k-s}^{n-q} \leq N \sum_{k=-N}^{N} b_k^{2(n-q)} = o(N^{2-2D}L^2(N)),$$

where we have used (4.1).

Recall that in Part b we suppose $n \geq 2$. Therefore, if neither $q \geq 2$ nor $n - q \geq 2$ we are in the case $n = 2$ and $q = 1$. In this case, the inner sum in (4.2) is majorized by

$$\sum_{k=1}^{N-s} | r_{j-k} r_{j-k} r_{j-k+s} r_{j-k-s} | \leq [\sum_{k=1}^{N-s} r_{j-k}^4 \sum_{k=1}^{N-s} r_{j-k}^4 \sum_{k=1}^{N-s} r_{j-k+s}^4 \sum_{k=1}^{N-s} r_{j-k-s}^4]^{1/4}.$$

Each of these sums in brackets is at most $\sum_{k=-N}^{N} r_k^4$. Therefore the double sum in (4.2) is majorized by $N \sum_{k=-N}^{N} r_k^4 = o(N^{2-2D}L^2(N))$.

(c) Part c can be established by adapting the proof of the proposition in Section II of Breuer and Major (1983). That proof was applicable to

$$E[\sum_{j=1}^{N} K_n(j, j)]^4 = E[\sum_{j=1}^{N} H_{2m}(X_j)]^4.$$

There is no difficulty in applying the same method to $E[\sum_{j=M+1}^{N} K_n(j, j + s)]^4$. □

LEMMA 4.2. *Given a collection of random variables $Y(s, N)$, $N \geq 1$, $|s| < N$, define*

$$S(k, N) = \sum_{|s|<k} Y(s, N)$$

and

$$T(k, N) = S(N, N) - S(k, N) = \sum_{k \leq |s| < N} Y(s, N).$$

Suppose that there exist random variables S_k, $k \geq 1$, and S such that

(1) *For each k, $S(k, N)$ tends to S_k in distribution as $N \to \infty$.*
(2) *Sy tends to S in distribution as $k \to \infty$.*
(3) *$T(k, N)$ tends to 0 in probability as N and k tend to infinity.*

Then $S(N, N)$ tends to S in distribution as $N \to \infty$.

PROOF. Let x be a continuity point of the distribution of S. According to Condition 3 of Lemma 4.2, we can choose a sequence $\{k_n\}$ such that $P\{| T(k_n, N) | \geq \frac{1}{2}n\} \leq \frac{1}{2}n$ for each $n \geq 1$, $N \geq k_n$. We can also find δ_n satisfying $\frac{1}{2}n \leq \delta_n \leq 1/n$, such that $x + \delta_n$ is a continuity point of S_{k_n}. Thus $\delta_n \to 0$ and $P\{| T(k_n, N) | \geq \delta_n\} \leq \delta_n$. For each $n \geq 1$ and $N \geq k_n$, we have $P\{S(N, N) \leq x\} = P\{S(k_n, N) + T(k_n, N) \leq x\} \leq P\{S(k_n, N) \leq x + \delta_n\} + P\{| T(k_n, N) | \geq \delta_n\} \leq P\{S(k_n, N) \leq x + \delta_n\} + \delta_n$. Hence, by Conditions 1 and 2,

$$\begin{aligned}\limsup_{N\to\infty} P\{S(N, N) \leq x\} &\leq P\{S_{k_n} \leq x + \delta_n\} + \delta_n \\ &\leq \limsup_{n\to\infty} P\{S_{k_n} - \delta_n \leq x\} + \limsup_{n\to\infty} \delta_n \\ &= P\{S \leq x\}.\end{aligned}$$

A similar argument shows that $\limsup_{N\to\infty} P\{S(N, N) > x\} \leq P\{S > x\}$. This implies that $\liminf_{N\to\infty} P\{S(N, N) \leq x\} \geq P\{S \leq x\}$, which completes the proof of the lemma. □

5. Proof of Theorem 1.

5.1 *Convergence of the finite-dimensional distributions.* We show that the finite-dimensional distributions of $Z_N(t)$ converge to those of

$$m!m(\sum_s a_s r_s^{m-1})R(t).$$

According to Lemma 3.5,

$$\sum_{j=1}^N \sum_{k=1}^N a_{j-k} H_m(X_j) H_m(X_k) - \mu_{N,m} = \sum_{n=1}^m (m-n)! \binom{m}{n}^2 V(n, N),$$

where

$$V(n, N) = \sum_{|s|<N} a_s r_s^{m-n} W(n, s, N).$$

Since $\sum |a_s| < \infty$, Lemma 4.1b implies that $\sup_{M=1,\cdots,N} E[V(n, M)]^2 = o(N^{2-2D}L^2(N))$ as $N \to \infty$, for $n = 2, \cdots, m$. Therefore it suffices to show that the finite-dimensional distributions of $V(1, [Nt])/N^{1-D}L(N)$ converge to those of $m!m(\sum a_s r_s^{m-1})R(t)$. This will follow from Lemma 4.2 if we prove that the conditions of that lemma are satisfied when

$$Y(s, N) = \frac{\sum_{j=1}^p d_j a_s r_s^{m-1} W(1, s, [Nt_j])}{N^{1-D}L(N)},$$

$$S_k = (\sum_{|s|<k} a_s r_s^{m-1}) \sum_{j=1}^p d_j R(t_j),$$

and

$$S = (\sum_{s=-\infty}^{\infty} a_s r_s^{m-1}) \sum_{j=1}^p d_j R(t_j),$$

where $d_1, \cdots, d_p$ and $0 \le t_1, \cdots, t_p \le 1$ are fixed. It is clear that Condition 2 of Lemma 4.2 is satisfied. We shall now verify Condition 3 of that lemma.

We can choose a constant C_3 so that $M^{1-D}L(M) \le C_3 N^{1-D}L(N)$ for all $1 \le M \le N$. To see this, note that there is a slowly varying function L_0 such that $N^{1-D}L_0(N)$ is nondecreasing and $L_0(N) \sim L(N)$ as $N \to \infty$ (Seneta, 1976, page 20). Thus there are constants C_3' and C_3'' so that $N^{1-D}L(N) \le C_3' N^{1-D}L_0(N) \le C_3'' N^{1-D}L(N)$ for all $N \ge 1$. Hence $M^{1-D}L(M) \le C_3' M^{1-D}L_0(M) \le C_3' N^{1-D}L_0(N) \le C_3' C_3'' N^{1-D}L(N)$.

According to Lemma 4.1a there is a constant C_4 so that

$$\frac{E[W(1, s, [Nt])]^2}{N^{2-2D}L^2(N)} \le \frac{C_4 [Nt]^{2-2D} L^2([Nt])}{N^{2-2D}L^2(N)} \le C_4 C_3^2$$

for $N \ge 1$, $0 \le s < N$, $0 \le t \le 1$. Thus if $T(k, N) = \sum_{k \le |s| < N} Y(s, N)$, then

$$E[T(k, N)]^2$$

$$= \sum_{i=1}^p \sum_{j=1}^p \sum_{k \le |s| < N} \sum_{k \le |t| < N} d_i d_j a_s r_s^{m-1} a_t r_t^{m-1} \frac{E[W(1, s, [Nt_i]) W(1, t, [Nt_j])]}{N^{2-2D}L^2(N)}$$

$$\le C_4 C_3^2 (\sum_{j=1}^p |d_j|)^2 (\sum_{k \le |s| < N} |a_s|)^2,$$

which tends to 0 as N and k tend to infinity because $\sum |a_s| < \infty$. Condition 3 of Lemma 4.2 is thus satisfied.

It remains to show that Condition 1 of Lemma 4.2 holds. We note that if x_k, $-\infty < k < \infty$, satisfies $x_{-k} = x_k$ and $x_k \sim k^{-\gamma}L_1(k)$, where $0 < \gamma < 1$ and $L_1(k)$ is slowly varying at ∞, then for any s_1, s_2, u_1, u_2 we have

$$\lim_{N\to\infty} \frac{\sum_{j=1}^{[Nu_1]-s_1} \sum_{k=1}^{[Nu_2]-s_2} x_{j-k}}{N^{2-\gamma}L_1(N)} = \int_0^{u_1} \int_0^{u_2} |x - y|^{-\gamma}\, dx\, dy.$$

Since $r_k r_{k+s-t} \sim k^{-2D}L^2(k)$ and $r_{k+s}r_{k-t} \sim k^{-2D}L^2(k)$, Lemma 3.6 implies that for each i, j, s_1, s_2

$$\lim_{N\to\infty} \frac{EW(1, s_1, [Nt_i])W(1, s_2, [Nt_j])}{N^{2-2D}L^2(N)} = \int_0^{t_1} \int_0^{t_2} |x - y|^{-2D}\, dx\, dy.$$

Hence

$$\lim_{N\to\infty} E[(1/N^{1-D}L(N)) \textstyle\sum_{j=1}^{p} d_j(W(1, s_j, [Nt_j]) - W(1, 0, [Nt_j]))]^2 = 0$$

for any choice of $d_1, \cdots, d_p$ and $s_1, \cdots, s_p$. Therefore the limiting distribution of

$$(1/N^{1-D}L(N))(W(1, s_1, [Nt_1]), \cdots, W(1, s_p, [Nt_p]))$$

is the same as that of

$$(1/N^{1-D}L(N))(W(1, 0, [Nt]), \cdots, W(1, 0, [Nt_p])).$$

According to Lemma 3.4,

$$W(1, 0, [Nt_j]) = \textstyle\sum_{j=1}^{[Nt_j]} K_1(j, j) = \sum_{j=1}^{[Nt_j]} (X_j^2 - 1).$$

As observed in Section 2, the finite-dimensional distributions of $\sum_{j=1}^{[Nt]} (X_j^2 - 1)$ converge to those of $R(t)$. Therefore for any $s_1, \cdots, s_p$

$$(1/N^{1-D}L(N))(W(1, s_1, [Nt_1]), \cdots, W(1, s_p, [Nt_p]))$$

tends in distribution to $(R(t_1), \cdots, R(t_p))$, establishing Condition 1 of Lemma 4.2. □

5.2 *Tightness.* We now show that the sequence $Z_N(t)$ is tight in $D[0, 1]$. Choose $0 \le t_1 < t_2 < t_3 \le 1$. According to Lemma 4.1a, there is a constant C_5 such that

$$E[\textstyle\sum_{j=[Nt_1]+1}^{[Nt_2]} K_n(j, j + s)]^2 \le C_5([Nt_2] - [Nt_1])^{2-2D}L^2([Nt_2] - [Nt_1])$$

for $s \ge 0$ and

$$\begin{aligned} E[\textstyle\sum_{j=1}^{[Nt_2]-s} K_n(j, j + s)]^2 &\le C_5([Nt_2] - s)^{2-2D}L^2([Nt_2] - s) \\ &\le C_5C_3^2([Nt_2] - [Nt_1])^{2-2D}L^2([Nt_2] - [Nt_1]) \end{aligned}$$

for $[Nt_1] \le s < [Nt_2]$, where as before, C_3 is a constant satisfying $M^{1-D}L(M) \le$

$C_3N^{1-D}L(N)$ for $1 \le M \le N$. Since

$$N^{1-D}L(N)[Z_{[Nt_2]} - Z_{[Nt_1]}]$$
$$= \sum_{n=1}^{m} (m-n)!\binom{n}{m}^2 a_s r_s^{m-n} \sum_{|s|<[Nt_1]} \sum_{j=[Nt_1]-s+1}^{[Nt_2]-s} K_n(j, j+s)$$
$$+ \sum_{n=1}^{m} (m-n)!\binom{n}{m}^2 a_s r_s^{m-n} \sum_{[Nt_1]\le|s|<[Nt_2]} \sum_{j=1}^{[Nt_2]-s} K_n(j, j+s),$$

it follows that

$$E[Z_{[Nt_2]} - Z_{[Nt_1]}]^2 \le \frac{C_6([Nt_2]-[Nt_1])^{2-2D}L^2([Nt_2]-[Nt_1])}{N^{2-2D}L^2(N)}$$
$$\le C_6C_7\left(\frac{[Nt_2]-[Nt_1]}{N}\right)^{2-2D-\varepsilon}$$

where ε is chosen so that $2 - 2D - \varepsilon > 1$ and C_7 is a constant satisfying $M^\varepsilon L(M) \le C_7N^\varepsilon L(N)$ for $1 \le M \le N$. Therefore

$$E\,|\,(Z_{[Nt_2]} - Z_{[Nt_1]})(Z_{[Nt_3]} - Z_{[nt_2]})\,|$$
$$\le C_8\left(\frac{[Nt_2]-[Nt_1]}{N}\right)^{1-D-(\varepsilon/2)}\left(\frac{[Nt_3]-[Nt_2]}{N}\right)^{1-D-(\varepsilon/2)}.$$

If $t_3 - t_1 \ge 1/N$, it follows that

$$E\,|\,(Z_{[Nt_2]} - Z_{[Nt_1]})(Z_{[Nt_3]} - Z_{[Nt_2]})\,| \le 2^{2-2D-\varepsilon}C_8(t_3-t_1)^{2-2D-\varepsilon}. \tag{5.1}$$

Relation (5.1) also holds when $t_3 - t_1 < 1/N$ because, in that case, the left-hand side of (5.1) equals 0. Tightness follows from Theorem 15.6 of Billingsley (1968). The proof of Theorem 1 is now complete. □

6. Proof of Theorem 2. To show that the finite-dimensional distributions of $Z_N(t)$ converge to those of $\sigma_m B(t)$, it is sufficient to prove that the conditions of Lemma 4.2 are satisfied when

$$Y(s, N) = \frac{1}{\sqrt{N}} \sum_{j=1}^{p} d_j \sum_{n=1}^{m} (m-n)!\binom{m}{n}^2 a_s r_s^{m-n} W(n, s, [Nt_j]),$$

$$T(k, N) = \sum_{k\le|s|<N} Y(s, N) = 2\sum_{s=k}^{N-1} Y(s, N),$$

$$S_k = \sum_{j=1}^{p} d_j \sum_{|s|<k} \sum_{n=1}^{m} (m-n)!\binom{m}{n}^2 a_s r_s^{m-n} Z(n, s, t_j),$$

and

$$S = \sum_{j=1}^{p} d_j \sum_{s=-\infty}^{\infty} \sum_{n=1}^{m} (m-n)!\binom{m}{n}^2 a_s r_s^{m-n} Z(n, s, t_j),$$

where $d_1, \cdots, d_p$ and $0 \le t_1, \cdots, t_p \le 1$ are fixed. The processes W are defined in (3.3) and the processes Z are defined in Remark 1 of Section 2.

Condition 2 of Lemma 4.2 is trivially satisfied. To verify Condition 3, note that by Lemma 4.1c, there is a constant C_9 such that

$$\frac{E[\sum_{j=M+1}^{N} K_n(j, j+s)]^2}{N} < C_9$$

for $n = 1, \cdots, m$, $0 \le M < N$, $s \ge 0$. Also, note that by Proposition 3.2, $EW(n_1, s_1, N_1)W(n_2, s_2, N_2) = 0$ for $n_1 \ne n_2$ (the set of complete diagrams $\Gamma_0(2n_1, 2n_2)$ is empty when $n_1 \ne n_2$). Therefore

$$\begin{aligned} E[T(k, N)]^2 &= 4 \textstyle\sum_{s_1=k}^{N-1} \sum_{s_2=k}^{N-1} E[Y(s_1, N)Y(s_2, N)] \\ &= 4 \textstyle\sum_{s_1=k}^{N-1} \sum_{s_2=k}^{N-1} \sum_{j_1=1}^{p} \sum_{j_2=1}^{p} \sum_{n=1}^{m} \left[(m-n)!\binom{m}{n}^2\right]^2 a_{s_1}a_{s_2}r_{s_1}^{m-n}r_{s_2}^{m-n}d_{j_1}d_{j_2} \\ &\quad \cdot N^{-1}E[W(n, s_1, [Nt_{j_1}])W(n, s_2, [Nt_{j_2}])] \\ &\le 4C_9[\textstyle\sum_{j=1}^{p} d_j]^2 \sum_{n=1}^{m} \left[(m-n)!\binom{m}{n}^2\right]^2 [\sum_{s=k}^{N-1} |a_s|]^2. \end{aligned}$$

Since this tends to 0 as k and N tend to ∞, it follows that Condition 3 of Lemma 4.2 is satisfied.

In order to verify that Condition 1 of Lemma 4.2 is satisfied, it is enough to show that for any $0 \le t_1, \cdots, t_p \le 1$ and integers $s_1, \cdots, s_p \ge 0$, $1 \le n_1, \cdots, n_p \le m$, the random vector

$$(1/\sqrt{N})(W(n_1, s_1, [Nt_1]), \cdots, W(n_p, s_p[Nt_p]))$$

converges in distribution to

$$(Z(n_1, s_1, t_1), \cdots, Z(n_p, s_p, t_p)).$$

This will follow if we show

$$\lim_{N\to\infty} \nu_N = E[Z(n_1, s_1, t_1) \cdots Z(n_p, s_p, t_p)] \tag{6.1}$$

where

$$\begin{aligned} \nu_N &= (1/N^{p/2})E[W(n_1, s_1, [Nt_1]) \cdots W(n_p, s_p, [Nt_p])] \\ &= (1/N^{p/2}) \textstyle\sum_{j_1=1}^{[Nt_1]-s_1} \cdots \sum_{j_p=1}^{[Nt_p]-s_p} E[K_{n_1}(j_1, j_1+s_1) \cdots K_{n_p}(j_p, j_p+s_p)]. \end{aligned}$$

We use Proposition 3.2 to evaluate this last expectation. Let

$$\Gamma_0 = \Gamma_0(2n_1, \cdots, 2n_p), \quad \Gamma_1 = \Gamma_1(2n_1, \cdots, 2n_p), \quad \text{and} \quad \Gamma_2 = \Gamma_0/\Gamma_1.$$

If indices $j_1, \cdots, j_p$ are fixed, introduce

$$h_k(x_1, \cdots, x_{2n_k}) = \exp(ij_k(x_1 + \cdots + x_{n_k}) + i(j_k + s_k)(x_{n_k+1} + \cdots + x_{2n_k})),$$
$$k = 1, \cdots, p,$$

so that $K_{n_k}(j_k, j_k + s_k) = I_{2n_k}(h_k)$. Define the function h and the constants h_γ,

$\gamma \in \Gamma_0$, as indicated prior to Proposition 3.1. According to Proposition 3.2,

$$\nu_N = \frac{1}{N^{p/2}} \sum_{j_1=1}^{[Nt_1]-s_1} \cdots \sum_{j_p=1}^{[Nt_p]-s_p} \sum_{\gamma \in \Gamma_0} h_\gamma$$

$$= \frac{1}{N^{p/2}} \sum_{j_1=1}^{[Nt_1]-s_1} \cdots \sum_{j_p=1}^{[Nt_p]-s_p} \sum_{\gamma \in \Gamma_1} h_\gamma$$

$$+ \frac{1}{N^{p/2}} \sum_{j_1=1}^{[Nt_1]-s_1} \cdots \sum_{j_p=1}^{[Nt_p]-s_p} \sum_{\gamma \in \Gamma_2} h_\gamma.$$

We can express h_γ in terms of the covariances r_k. To do so, let $E(\gamma)$ be the edge set of the diagram $\gamma \in \Gamma_0$. If $e \in E(\gamma)$ joins (k, i_1) to (ℓ, i_2), where $k > \ell$, define

$$d(e) = k, \quad f(e) = \ell,$$

and

$$s(e) = \begin{cases} 0 & \text{if } \; 1 \le i_1 \le n_k, \quad 1 \le i_2 \le n_\ell \\ s_k & \text{if } \; n_k + 1 \le i_1 \le 2n_k, \quad 1 \le i_2 \le n_\ell \\ -s_\ell & \text{if } \; 1 \le i_1 \le n_k, \quad n_\ell + 1 \le i_2 \le 2n_\ell \\ s_k - s_\ell & \text{if } \; n_k + 1 \le i_1 \le 2n_k, \quad n_\ell + 1 \le i_2 \le 2n_\ell. \end{cases}$$

With these definitions

$$h_\gamma = \prod_{e \in E(\gamma)} r(j_{d(e)} - j_{f(e)} + s(e)),$$

with $r(k)$ denoting r_k.

The conditions of Theorem 2 allow us to fix $\varepsilon > 0$ satisfying $-D + \varepsilon < -\frac{1}{2}$. If $C > 0$ is given, define

$$r'_k = r'(k) = \begin{cases} 1 & k = 0 \\ C|k|^{-D+\varepsilon} & k \ne 0. \end{cases}$$

We can choose C so that $|r_{k+s}| \le r'_k$ for all $-\infty < k < \infty$ and $|s| \le \max(|s_1|, |s_2|, \cdots, |s_p|)$. Therefore

$$\left| \frac{1}{N^{p/2}} \sum_{j_1=1}^{[Nt_1]-s_1} \cdots \sum_{j_p=1}^{[Nt_p]-s_p} \sum_{\gamma \in \Gamma_2} h_\gamma \right|$$

$$= \left| \frac{1}{N^{p/2}} \sum_{j_1=1}^{[Nt_1]-s_1} \cdots \sum_{j_p=1}^{[Nt_p]-s_p} \sum_{\gamma \in \Gamma_2} \prod_{e \in E(\gamma)} r(j_{d(e)} - j_{f(e)} + s(e)) \right|$$

$$\le \frac{1}{N^{p/2}} \sum_{j_1=1}^{N} \cdots \sum_{j_p=1}^{N} \sum_{\gamma \in \Gamma_2} \prod_{e \in E(\gamma)} r'(j_{d(e)} - j_{f(e)}).$$

Since $-D + \varepsilon < -\frac{1}{2}$, the proposition on page 433 of Breuer and Major (1983) implies that this last expression tends to 0 as $N \to \infty$. Hence

$$\lim_{N \to \infty} \nu_N = \lim_{N \to \infty} \frac{1}{N^{p/2}} \sum_{j_1=1}^{[Nt_1]-s_1} \cdots \sum_{j_p=1}^{[Nt_p]-s_p} \sum_{\gamma \in \Gamma_1} h_\gamma.$$

By Proposition 3.3, the quantities $\sum_{\gamma \in \Gamma_1} h_\gamma$ are moments of jointly Gaussian random variables $K_n^G(j, k)$ having the same covariances as $K_n(j, k)$. Therefore the limiting random variables $Z(n, s, t)$ are Gaussian with covariance

$$\begin{aligned} &EZ(n_1, s_1, t_1)Z(n_2, s_2, t_2) \\ &\quad = \lim_{N\to\infty}(1/N)EW(n_1, s_1, [Nt_1])\,W(n_2, s_2, [Nt_2]) \\ &\quad = \lim_{N\to\infty}(1/N) \textstyle\sum_{j_1=1}^{[Nt_1]-s_1} \sum_{j_2=1}^{[Nt_2]-s_2} EK_n(j_1, j_1 + s_1)K_{\ell}(j_2, j_2 + s_2) \\ &\quad = \begin{cases} 0 & \text{if } n_1 \neq n_2 \\ \min(t_1, t_2) \sum_{q=0}^{n} (n!)^2\binom{n}{q}^2 \sum_{k=-\infty}^{\infty} r_k^q r_{k+s_1-s_2}^q r_{k+s_1}^{n-q} r_{k-s_2}^{n-q} & \text{if } n_1 = n_2 = n, \end{cases} \end{aligned}$$

where we have used Lemma 3.6 and the elementary fact that if $\{x_k\}$ is a sequence satisfying $\sum_{k=-\infty}^{\infty} |x_k| < \infty$ then

$$\lim_{N\to\infty} \frac{\sum_{j=1}^{[Nt_1]-s_1} \sum_{k=1}^{[Nt_2]-s_2} x_{j-k}}{N} = \min(t_1, t_2) \textstyle\sum_{k=-\infty}^{\infty} x_k.$$

The sequence $Z_N(t)$, $N \geq 1$, is tight in $D[0, 1]$ because there is a constant C such that for any $0 \leq t_1 < t_2 < t_3 \leq 1$,

$$E(Z_N(t_2) - Z_N(t_1))^2(Z_N(t_3) - Z_N(t_1))^2 \leq C\,|t_3 - t_1|^2.$$

The existence of such a constant is established by using Lemma 4.1c and proceeding as in the proof of tightness for Theorem 1. □

7. Proof of Theorem 3. Set

$$C(\alpha) = \frac{\pi}{2\Gamma(\alpha)\cos(\alpha\pi/2)}.$$

We will use the following three propositions.

PROPOSITION 7.1 (Yong, 1974, Theorem III-12). *Let $\{u_k, k \geq 1\}$ be of bounded variation and quasi-monotonically convergent to zero. Let $0 < \alpha < 1$. Then*

$$u_k \sim k^{-\alpha}L(k)$$

as $k \to \infty$, if and only if

$$\textstyle\sum_{k=1}^{\infty} u_k \cos(kx) \sim C(\alpha)x^{\alpha-1}L(1/x)$$

as $x \to +0$.

PROPOSITION 7.2 (Yong, 1974, Theorem III-27). *For $1 < \alpha < 3$,*

$$\textstyle\sum_{k=1}^{\infty} k^{-\alpha}L(k)\cos kx - \sum_{k=1}^{\infty} k^{-\alpha}L(k) \sim C(\alpha)x^{\alpha-1}L(1/x) \qquad (7.1)$$

as $x \to +0$.

For any sequence a_k and for any integer K, $\sum_{k=1}^{K} a_k\cos kx - \sum_{k=1}^{K} a_k = O(x^2)$

as $x \to 0$. One can therefore replace (7.1) by

(7.2) $$\sum_{k=1}^{\infty} a_k \cos kx - \sum_{k=1}^{\infty} a_k \sim C(\alpha)(\lim_{k\to\infty} \operatorname{sgn} a_k) x^{\alpha-1} L(1/x)$$

as $x \to +0$, where $1 < \alpha < 3$, $|a_k| \sim k^{-\alpha} L(k)$ as $k \to \infty$ and where a_k is positive for all large k or negative for all large k. If, moreover, $a_k = a_{-k}$ and $\sum_{k=-\infty}^{+\infty} a_k = 0$, then

(7.3) $$\left|\sum_{k=-\infty}^{+\infty} a_k e^{ikx}\right| = 2\left|\sum_{k=1}^{\infty} a_k \cos kx - \sum_{k=1}^{\infty} a_k\right| \sim 2C(\alpha) x^{\alpha-1} L(1/x)$$

as $x \to 0$.

In the next proposition, $\{X_k\}$ is a mean 0, stationary Gaussian sequence with spectral density $f(x) \sim x^{-\alpha} L_2(x)$ as $x \to 0$; also, $\{a_k\}$ is a symmetric sequence with $a_k = \int_{-\pi}^{\pi} e^{ikx} g(x)\,dx$ and $|g(x)| \sim x^{-\beta} L_3(x)$ as $x \to 0$. The functions L_2 and L_3 are slowly varying at 0. Furthermore, we suppose that f and g are bounded in the interval $[\delta, \pi]$ for all $\delta > 0$ and that their discontinuities have Lebesgue measure zero. (The function g need not be nonnegative.)

PROPOSITION 7.3. (Fox and Taqqu, 1983, Theorem 3). *If* $\alpha < 1$, $\beta < 1$ *and* $\alpha + \beta < \frac{1}{2}$, *then*

$$(1/\sqrt{N})\left\{\sum_{j=1}^{N} \sum_{k=1}^{N} a_{j-k} X_j X_k - E \sum_{j=1}^{N} \sum_{k=1}^{N} a_{j-k} X_j X_k\right\}$$

tends in distribution as $N \to \infty$ *to a normal random variable with mean* 0 *and variance* $16\pi^3 \int_{-\pi}^{\pi} (f(x) g(x))^2\,dx$.

To prove Theorem 3, one needs only to verify that the conditions of Proposition 7.3 are satisfied. First apply Proposition 7.1 to $\{r_k\}$ with $\alpha = D \in (0, \frac{1}{2})$ and Relation (7.3) to $\{a_k\}$ with $\alpha = \gamma \in (1, 3)$. Note that $\alpha \in (\frac{1}{2}, 1)$, $\beta \in (-2, 0)$ and that $\alpha + \beta = 2 - D - \gamma < \frac{1}{2}$ by Assumption 3 of Theorem 3. It remains to verify that f and g are continuous on $[\delta, \pi]$. The continuity of f follows from the assumption that the r_k have bounded variation and the continuity of g from $\sum |a_k| < \infty$. This completes the proof of Theorem 3. □

REMARK. The conditions of Theorem 3 were used to verify the assumptions of Proposition 7.3. It may be possible to find weaker conditions that achieve the same purpose.

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