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A LIMIT THEOREM FOR NONNEGATIVE ADDITIVE FUNCTIONALS OF STORAGE PROCESSES

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We consider a storage process $X(t)$ having a compound Poisson process as input and general release rules, and a nonnegative additive functional $Z(t) = \int_0^t f(X(s))\,ds$. Under the situation that the input rate is equal to the maximal output rate, it is shown for a suitable class of functions of f that an appropriate normalization of the process $Z(t)$ converges weakly to a process which is represented as a constant times the local time of a Bessel process at zero.

1. Introduction. Let us consider a storage process $X(t)$ which is defined as a unique solution of the following equation:

$$X(t) = X(0) - \int_0^t r(X(s))\,ds + A(t). \tag{1}$$

Here $A(t)$ is assumed to be an increasing compound Poisson process, i.e.,

$$A(t) = \textstyle\sum_{i=1}^{N(t)} S_i \tag{2}$$

where $N(t)$ is a Poisson process of parameter λ and $\{S_i,\ i = 1, 2, \cdots\}$, is a sequence of independent and identically distributed random variables independent of $N(t)$ with a common distribution function $F(x)$ for which the second moment is finite and $F(0) = 0$. We assume that $r(\cdot)$ is a nondecreasing nonnegative function defined on $[0, \infty)$ and $r(0) = 0$. Then equation (1) has a unique nonnegative solution (Çinlar and Pinsky [1]).

Let us define the following notations:

$$\bar{r} = \sup_{x\ge 0} r(x) \quad (= \lim_{x\to\infty} r(x)),$$

$$\mu = ES_i, \quad \sigma_2 = \int_0^\infty y^2\,dF(y), \quad k = \sqrt{\lambda\sigma_2}.$$

Then it was shown in Çinlar and Pinsky [2] that as $t \to \infty$, $X(t)$ has a limiting distribution when $\lambda\mu < \bar{r}$, $X(t) \to \infty$ a.s. when $\lambda\mu > \bar{r}$ and that $X(t) \to \infty$ in probability when $\lambda\mu = \bar{r}$.

The purpose of this paper is to give some functional type limit theorems for nonnegative additive functionals of the process $X(t)$ under the last situation, i.e.,

$$\lambda\mu = \bar{r}. \tag{A1}$$

Let $f(x) \ge 0$ a bounded measurable function defined on $R = (-\infty, \infty)$, and we will

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consider the weak convergence problem for a sequence of processes defined by

$$Z_n(f)(t) = \frac{1}{k\sqrt{n}} \int_0^{nt} f(X(s))\, ds, \quad n = 1, 2, \cdots \tag{3}$$

in the following two cases:

Case I. $x[\bar{r} - r(x)] = 0$ for all $x \geq 0$, i.e., $r(x) = \bar{r}$ for $x > 0$.

Case II. $\lim_{x\to\infty} x[\bar{r} - r(x)] = c > 0$.

For Case II, it will be shown in Corollary 2 of Theorem 1 that for any bounded nonnegative measurable function with compact support, $Z_n(f)$ converges weakly to a null process. For Case I we restrict the class of f to functions satisfying condition (A3) and show that $Z_n(f)$ converges weakly to a process $Z(t) = T(f)L_0(t)$ where $L_0(t)$ is the local time of a reflecting Brownian motion at zero and $T(f)$ is a constant depending on f (Theorem 1). In Corollary 1 of Theorem 1, $f \in C_k^\infty$ (the set of infinitely differentiable nonnegative functions with compact supports) is shown to satisfy condition (A3) and $T(f)$ is calculated explicitly using the Fourier transform of f. Finally in Corollary 3 of Theorem 1, for any continuous function with compact support the weak limit of $Z_n(f)$ is shown to be the process $Z(t) = T(f)L_0(t)$ where $T(f)$ is the limit of $T(f_m)$, $f_m \in C_k^\infty$, $m = 1, 2, \cdots$, and $\{f_m\}$ are such that there exists a compact set K satisfying $\operatorname{supp}(f_m) \cup \operatorname{supp}(f) \subset K$ and $\sup_x |f_m(x) - f(x)| \to 0$ ($\operatorname{supp}(f)$ is the support of f).

Some unsolved problems are:

1. to extend our results for Case I to functions such as $f(x) = I_{(\alpha,\beta)}(x)$, (in connection with this, see Proposition 1 and Remark 3),

2. to investigate the case where $\lim_{x\to\infty} x[\bar{r} - r(x)] = 0$. (Case I is a special example of this case),

3. to find, in Case II, suitably normalized processes for which the weak limits are not degenerate on the null process.

There are some general results on limit theorems for the occupation time of Markov processes (see, for example, Bingham [3], Kasahara [4, 5]). As suggested by Bingham in [3], the applicability of his result yields the weak convergence theorem for occupation times of queues in heavy traffic obtained in Whitt [14], it may be possible to get our theorem from these general results. Here we try, however, to get our result directly by following the approach of stochastic calculus used in Ikeda and Watanabe [6, Chapter III, Section 4.4] and Papanicolaou, Stroock and Varadhan [7, Section 3.5]. In this approach we essentially use the fact that the processes $X_n(t) = (1/k\sqrt{n})X(nt)$, $n = 1, 2, \cdots$ converge weakly to a Bessel process with index $1 + 2c/k^2$ where $c = \lim_{x\to\infty} x[\bar{r} - r(x)]$ is assumed to exist (Yamada [9], Rosenkrantz [15]).

Let $D([0, T], R^d)$ be the space of all right continuous R^d-valued functions in $[0, T]$ with limits from the left, and let us assume that the space $D([0, T], R^d)$ is endowed with Skorohod's J_1 topology. All the processes appearing in (1) and (2) are assumed to be defined on a probability space $(\Omega, \mathscr{F}, P)$ and to be realized in the space $D([0, \infty), R^1)$ with the extension of Skorohod's J_1 topology (Lindval [8]). The notation $X_n \Rightarrow X$ denotes the weak convergence of the distributions of the processes X_n to a distribution of X. We also denote by $Z_n \to_P Z$ the convergence of the corresponding random variables in probability.

2. Preliminaries. Under assumption (A1) we can rewrite equation (1) as

$$X(t) = X(0) + \int_0^t [\bar{r} - r(X(s))]\, ds + A(t) - \lambda\mu t.$$

Let us define processes X_n, B_n, M_n, $n = 1, 2, \cdots$ as follows:

$$X_n(t) = \frac{1}{k\sqrt{n}} X(nt)$$

$$B_n(t) = \frac{1}{k\sqrt{n}} \int_0^{nt} [\bar{r} - r(X(s))]\, ds \tag{4}$$

$$M_n(t) = \frac{1}{k\sqrt{n}} (A(nt) - \lambda\mu nt).$$

Then we have

$$X_n(t) = X_n(0) + B_n(t) + M_n(t). \tag{5}$$

Note that $M_n(t)$ is an $\mathscr{F}_{nt}$-martingale and $\mathscr{F}_t = \sigma(X(0), A(s), s \le t)$ is a sub-σ-field of $\mathscr{F}$ generated by $X(0)$ and $A(s)$, $0 \le s \le t$. The quadratic variation $\langle M_n \rangle (t)$ is given by

$$\langle M_n \rangle (t) = \lambda n(1/k^2 n) ES_i^2 \cdot t = t.$$

Now let us assume condition (A2):

$$\lim_{x\to\infty} x[\bar{r} - r(x)] = c < \infty. \tag{A2}$$

Then we have the following result.

LEMMA 1. *Under* (A1-2), $(X_n, B_n, M_n) \Rightarrow (Y, V, W)$ *in* $D([0, T], R^3)$, T *arbitrary, where* Y *is a Bessel process with index* $1 + 2c/k^2$, W *a Wiener process, and* V *a continuous increasing process. Furthermore we have*

$$Y(t) = V(t) + W(t). \tag{6}$$

When $c = 0$, *with probability one* $V(t)$ *increases only on the set of* t *when* $Y(t) = 0$; *that is, equation* (6) *is the Skorohod equation for the reflecting Brownian motion* $Y(t)$. *When* $c > 0$, *with probability one* $V(t)$ *is strictly increasing.*

PROOF. The fact that $(X_n, B_n, W_n) \Rightarrow (Y, V, W)$ on $D([0, T], R^3)$, T arbitrary, was proved in Yamada [9]. Equation (6) is a direct consequence of this result and (5). Now let us consider the case where $c = 0$. Since $(X_n, B_n, M_n) \Rightarrow (Y, V, W)$, we may assume, due to Skorohod's representation theorem [10], that $P(A) = 1$ where $A = \{\omega;\ X_n(t) \to Y(t)$ and $B_n(t) \to V(t)$ uniformly on each finite t-interval in $[0, T]\}$. Take arbitrary $\omega \in A$ and suppose that $Y(\tilde{t}, \omega) > 0$. Then there exist $t' > \tilde{t}$ such that $\inf_{\tilde{t}\le t\le t'} Y(t, \omega) > 0$ and

$\inf_{\tilde{t}\le t\le t'} X_n(t, \omega) \ge \alpha > 0$ for an α and for sufficiently large n. Then

$$B_n(t', \omega) - B_n(\tilde{t}, \omega) = \frac{1}{k}\int_{\tilde{t}}^{t'} \sqrt{n}[\bar{r} - r(k\sqrt{n}X_n(s))]\, ds$$
$$\le \frac{1}{k^2}\frac{1}{\alpha}\int_{\tilde{t}}^{t'} k\sqrt{n}X_n(s)[\bar{r} - r(k\sqrt{n}X_n(s))]\, ds$$

for sufficiently large n, and the last term converges to zero since $X_n(s) \to Y(s) > 0$ ($\tilde{t} \le s \le t'$) and $k\sqrt{n}X_n(s)\,[\bar{r} - r(k\sqrt{n}X_n(s))] \to c = 0$ for $s \in [\tilde{t}, t']$. Then since

$$V(t', \omega) - V(\tilde{t}, \omega)$$
$$= B_n(t', \omega) - B_n(\tilde{t}, \omega) + (V(t', \omega) - B_n(t', \omega)) - (V(\tilde{t}, \omega) - B_n(\tilde{t}, \omega)),$$

we have $V(t', \omega) = V(\tilde{t}, \omega)$. That is $V(\cdot, \omega)$ does not increase at $\tilde{t}$. Next let us consider the case where $c > 0$. Take any $s < t$. Then since $Y(t)$ is positive for a.e. t, there exists an interval $[s_1, t_1]$ such that $Y(u) > 0$ for all $u \in [s_1, t_1] \subset [s, t]$. Thus for a $\beta > 0$ and for all n, $\sup_{u\in[s_1,t_1]} X_n(u, \omega) < \beta$. Then

$$B_n(t, \omega) - B_n(s, \omega) \ge \frac{1}{k^2}\frac{1}{\beta}\int_{s_1}^{t_1} k\sqrt{n}X_n(s)[\bar{r} - r(k\sqrt{n}X_n(s))]\, ds$$

for all n. Taking the limit in the above, we have

$$V(t, \omega) - V(s, \omega) \ge c(t_1 - s_1)/(k^2\beta) > 0.$$

Thus $V(t)$ is strictly increasing with probability one. □

Since we have shown that the limit process $Y(t)$ of $X_n(t)$, which is a Bessel process of index $1 + 2c/k^2$, is a continuous semimartingale, following Jacod [11] we shall define its local time as follows:

DEFINITION. The local time of the process $Y(t)$ at a, which we denote by L_a, is the unique continuous increasing process with $L_a(0) = 0$ satisfying

$$L_a(t) = |Y(t) - a| - |Y(0) - a| - \int_0^t \operatorname{sig}(Y(s) - a)\, dY(s)$$

$$\left(= |Y(t) - a| - |Y(0) - a| - \int_0^t \operatorname{sig}(Y(s) - a)\, dV(s) - \int_0^t \operatorname{sig}(Y(s) - a)\, dW(s)\right) \tag{7}$$

where $\operatorname{sig}(x) = 1$ if $x > 0$, 0 if $x = 0$, -1 if $x < 0$.

The local time L_a defined in this way has the following property.

LEMMA 2. *The local time L_a of the Bessel process $Y(t)$ is the unique continuous* *process of bounded variation with* $L_a(0) = 0$ *such that*

$$L_a(t) - |Y(t) - a| + |Y(0) - a| + \int_0^t \operatorname{sig}(Y(s) - a)\, dV(s) \tag{8}$$

$$= \textit{local martingale.}$$

L_0 equals V when $c = 0$ and is a null process when $c > 0$.

PROOF. Let L_a' be another continuous process of bounded variation with $L_a'(0) = 0$ satisfying (8). Then

$$L_a(t) - L_a'(t) = \text{local martingale}$$

and this implies $L_a' = L_a$. By the definition of $L_0(t)$, and since $Y(t) > 0$ for a.e. t with probability one, we have

$$\begin{aligned} L_0(t) &= Y(t) - \int_0^t \operatorname{sig}(Y(s))\, dV(s) - \int_0^t \operatorname{sig}(Y(s))\, dW(s) \\ &= \int_0^t (1 - \operatorname{sig}(Y(s)))\, dV(s) + \int_0^t (1 - \operatorname{sig}(Y(s)))\, dW(s) \qquad (9) \\ &= \int_0^t I_{\{Y(s)=0\}}\, dV(s). \end{aligned}$$

Then since

$$\int_0^t I_{\{Y(s)=0\}}\, dV(s) = V(t)$$

when $c = 0$ by Lemma 1, we have that $L_0 = V$. When $c > 0$, in view of Lemma 1 $V(t)$ is continuous and strictly increasing. Hence, for a.e. t, $Y(V^{-1}(t)) > 0$ since $Y(t) > 0$ for a.e. t. Here

$$V^{-1}(t) = \inf\{s;\ V(s) > t\}.$$

Then

$$L_0(t) = \int_0^{V(t)} I_{\{Y(V^{-1}(s))=0\}}\, ds = 0. \quad \square$$

A direct consequence of Lemma 1 and 2 is the following:

PROPOSITION 1. *Assume* (A1) *and let* $f(x) = I_{\{a\}}(x)$. *Then in Case I,* $Z_n(f) \Rightarrow Z(f)$ *in* $D([0, T], R)$, T *arbitrary, where*

$$Z(f)(t) = \begin{cases} (1/\lambda\mu) L_0(t), & a = 0 \\ 0, & a \neq 0. \end{cases}$$

PROOF. We note that when $a = 0$,

$$\frac{1}{k\sqrt{n}} \int_0^{nt} f(X(s))\, ds = \frac{1}{\bar{r}\, k\sqrt{n}} \int_0^{nt} [\bar{r} - r(X(s))]\, ds = \frac{1}{\bar{r}} B_n(t).$$

In view of Lemma 1 and 2, $B_n(\cdot) \Rightarrow L_0(\cdot)$ in $D([0, T], R)$, T arbitrary. The case $a \neq 0$ is trivial. □

REMARK 1. Let us consider a queueing process $M/G/1$ where customers arrive according to a Poisson process $N(t)$ with parameter λ and service times $\{S_i,\ i = 1, 2, \cdots\}$ form a sequence of independent and identically distributed random variables independent of $\{N(t),\ t \geq 0\}$ with a common distribution function $F(x)$. The virtual waiting time is defined as the solution $X(t)$ of equation (1) where $r(x) = 1$ $(x > 0)$. So $I(t) = \int_0^t I_{\{0\}}(X(s))\, ds$ is the idle time up to time t, and Proposition 1 asserts $Z_n(f)(\cdot) = I(n\ \cdot)/k\sqrt{n} \Rightarrow L_0(\cdot)$ under the condition $1 = \bar{r} = \lambda\mu$ where $f(x) = I_{\{0\}}(x)$. This result is also easily obtained by noting that $I(t)$ can be represented as

$$I(t) = -\inf_{0 \leq s \leq t}(A(s) - s)$$

when $X(0) = 0$ and the processes $(A(nt) - nt)/k\sqrt{n}$ converge weakly to a Wiener process, and then by using the continuous mapping theorems in Billingsley [16, I, Theorem 5.1] (Whitt [14]). Here we should also note that $-\inf_{0 \leq s \leq t} W(s)$ is the local time of a reflecting Brownian motion at zero where $W(t)$ is a Brownian motion with $W(0) = 0$ (Ikeda and Watanabe [6, III, Corollary of Theorem 4.2]). □

Regarding f for which processes $\{Z_n\}$ were defined in (3), we shall impose in Theorem 1 the following assumption (A3):

(A3) $f\colon R \to R^+ = [0, \infty)$ is a bounded measurable function with a compact support. Furthermore there exists a bounded measurable function $G''(x)\colon R \to R$ such that

$$\int_0^\infty \int_0^y uG''(x + y - u)\, du\, dF(y) = f(x)$$

and $G''(x) \to 0$ as $x \to \infty$.

Let us define functions G and G' by

$$G'(x) = \int_0^x G''(y)\, dy, \quad G(x) = \int_0^x G'(y)\, dy.$$

With respect to condition (A3) we have

LEMMA 3. *Suppose that* $\int_0^\infty y^3\, dF(y) < \infty$. *Then under condition* (A3) $G(x)$ *has the following properties*:

(i) $\lim_{x\to\infty} G'(x) = 2/\sigma_2(D + \bar{f})$ *where*

$$D = \int_0^\infty \int_0^y uG'(y - u)\, du\, dF(y) \quad \textit{and} \quad \bar{f} = \int_0^\infty f(x)\, dx.$$

(ii) $\lim_{\lambda\to\infty}(1/\lambda)G(\lambda x) = 2/\sigma_2(D + \bar{f})x$ *uniformly on every compact set of* x.

PROOF. (i) We have

$$\int_0^\infty \int_0^y u \int_0^z G''(x + y - u)\, dx\, du\, dF(y) = \int_0^z f(x)\, dx, \quad z \geq 0,$$

so

$$\int_0^\infty \int_0^y u(G'(z + y - u) - G'(y - u))\, du\, dF(y) = \int_0^z f(x)\, dx.$$

Application of the mean value theorem yields

$$\begin{aligned} &\int_0^\infty \int_0^y uG'(z)\, du\, dF(y) \\ &+ \int_0^\infty \int_0^y uG''(z + \theta(z, y - u)(y - u))(y - u)\, du\, dF(y) - D \\ &= \int_0^z f(x)\, dx \end{aligned} \tag{10}$$

where $0 \leq \theta(z, y - u) \leq 1$ and

$$D = \int_0^\infty \int_0^y uG'(y - u)\, du\, dF(y).$$

Letting z tend to infinity in (10), we get the conclusion by the bounded convergence theorem.

(ii) This is trivial from (i). □

For a locally square integrable martingale M, let $\langle M \rangle\ (t)$ be the quadratic process of M. Then we have

LEMMA 4. (Lenglart [12]). *Let* $M_n (n = 1, 2, \cdots)$ *be locally square integrable* $\mathscr{G}_t^n$*-martingales which are right continuous with left limits on a probability space* $(\Omega, \mathscr{G}, P)$ *with a reference family* $\{\mathscr{G}_t^n\}$ *which is right continuous and* $\mathscr{G}_0^n$ *contains all P-null sets. Then for an arbitrary* $T > 0$, $\langle M_n \rangle\ (T) \to_P 0$ *implies* $\sup_{0\leq t\leq T} | M_n(t) | \to_P 0$.

REMARK 2. In [12], M_n are defined with respect to a single reference family. But as far as T is nonrandom, it is easy from the discussion in [12] to see that the above is true.

LEMMA 5. (Yamada [9]). *For each* $T > 0$ *there exists a constant* $K(T)$, *not depending on* n, *such that*

$$E[(X_n(t) - X_n(s))^2 | \mathscr{F}_{ns}] \leq K(t - s), \quad 0 \leq s < t \leq T.$$

Application of Itô's formula yields the above result easily. For the detailed discussion see the proof of (ii) in Lemma 1 in [9] where $\bar{r}_n - \rho_n = 0$ for all n in our case.

3. Main result. Let $Z_n(f)(t)$, $n = 1, 2, \cdots$, be the processes defined in (3) of Section 1, and in this section we assume that $\int_0^\infty y^3\, dF(y) < \infty$. Then we have

THEOREM 1. *Assume condition* (A1). *Then for a function f satisfying condition* (A3), *the following result holds*:

CASE I. $r(x) = \bar{r}$ *for all* $x > 0$. *Then* $Z_n(f) \Rightarrow Z(f)$ *in* $D([0, T], R)$, T *arbitrary, where the process* $Z(f)$ *is defined as*

$$Z(f)(t) = T(f)L_0(t), \quad T(f) = R(f) + (2/k^2)\bar{f}.$$

Here $L_0(t)$ *is the local time at zero of a reflecting Brownian motion, and* $R(f)$ *is defined by*

$$R(f) = \frac{2}{k^2}\int_0^\infty \int_0^y u \int_0^{y-u} G''(x)\, dx\, du\, dF(y).$$

CASE II. $\lim_{x\to\infty} x[\bar{r} - r(x)] = c > 0$. *Then* $Z_n(f) \Rightarrow Z(f)$ *in* $D([0, T], R)$, T *arbitrary, where* $Z(f)$ *is a null process.*

PROOF. Let $N_A(ds\, dx)$ be the counting measure of the point process $A(t)$ and $\hat{N}_A(ds\, dx)$ be its compensator ([6, pages 43, 60]). Then $\hat{N}_A(ds, dx) = \lambda\, ds\, dF(x)$, and $X(t)$ can be written as

$$X(t) = X(0) + \int_0^t [\bar{r} - r(X(s))]\, ds + \int_0^t \int_0^\infty x\tilde{N}_A(ds\, dx)$$

where the last integral is defined as

$$\int_0^t \int_0^\infty x\tilde{N}_A(ds\, dx) = \int_0^t \int_0^\infty xN_A(ds\, dx) - \int_0^t \int_0^\infty x\hat{N}_A(ds\, dx)$$

$$(= A(t) - \lambda\mu t)$$

and is a square integrable $\mathscr{F}_t$-martingale. Now putting

$$B(t) = \int_0^t [\bar{r} - r(X(s))]\, ds,$$

and applying generalized Ito's formula for the process $X(t)$ ([6, page 66]), we have

$$\begin{aligned} G(X(t)) = G(X(0)) &+ \int_0^t G'(X(s))\, dB(s) \\ &+ \int_0^t \int_0^\infty \{G(X(s-) + x) - G(X(s-)) - G'(X(s-))x\}\hat{N}_A(ds\, dx) \\ &+ \int_0^t \int_0^\infty \{G(X(s-) + x) - G(X(s-))\}\tilde{N}_A(ds\, dx) \end{aligned}$$

where the last integral is defined as a stochastic integral with respect to the point process $A(t)$ ([6, Section 3, Chapter II]). Rewriting the above yields that

$$G(X(t)) = G(X(0)) + \int_0^t G'(X(s))\, dB(s) + \int_0^t G'(X(s-))\, dM(s) + \lambda \int_0^t f(X(s))\, ds + N(t)$$

where

$$M(t) = \int_0^t \int_0^\infty x\tilde{N}_A(ds\, dx) \quad (= A(t) - \lambda\mu t)$$

and

$$N(t) = \int_0^t \int_0^\infty \{G(X(s-) + x) - G(X(s-)) - G'(X(s-))x\}\tilde{N}_A(ds\, dx).$$

Note that $N(t)$ is a locally square integrable $\mathscr{F}_t$-martingale and its quadratic process $\langle N \rangle\ (t)$ is given by

$$\langle N \rangle\ (t) = \int_0^t \int_0^\infty \{G(X(s-) + x) - G(X(s-)) - G'(X(s-))x\}^2\lambda\, ds\, dF(x).$$

([6, page 62]). Then we have

$$\begin{aligned}\frac{1}{k\sqrt{n}} G(k\sqrt{n}\, X_n(t)) &= \frac{1}{k\sqrt{n}} G(k\sqrt{n}\, X_n(0)) + \int_0^t G'(k\sqrt{n}\, X_n(s))\, dB_n(s) \\ &\quad + \int_0^t G'(k\sqrt{n}\, X_n(s-))\, dM_n(s) + \frac{\lambda}{k\sqrt{n}} \int_0^{nt} f(X(s))\, ds + N_n(t)\end{aligned}$$

where $N_n(t) = N(nt)/k\sqrt{n}$ and B_n, M_n were defined in (4) in Section 2. Then noting equation (5), the above can be written as

$$\begin{aligned}&\frac{1}{k\sqrt{n}} G(k\sqrt{n}\, X_n(t)) - \frac{2D}{\sigma_2} X_n(t) \\ (11) \qquad &= \frac{1}{k\sqrt{n}} G(k\sqrt{n}\, X_n(0)) - \frac{2D}{\sigma_2} X_n(0) + \int_0^t \left(G'(k\sqrt{n}\, X_n(s)) - \frac{2D}{\sigma_2}\right) dB_n(s) \\ &\quad + \int_0^t \left(G'(k\sqrt{n}\, X_n(s-)) - \frac{2D}{\sigma_2}\right) dM_n(s) + \frac{\lambda}{k\sqrt{n}} \int_0^{nt} f(X(s))\, ds + N_n(t).\end{aligned}$$

Let us define three dimensional processes H_n, $n = 1, 2, \cdots$, by

$$H_n(t) = \left(\frac{1}{k\sqrt{n}} G(k\sqrt{n}\, X_n(t)) - \frac{2D}{\sigma_2} X_n(t), \int_0^t \left(G'\ (k\sqrt{n}\, X_n(s)) - \frac{2D}{\sigma_2}\right) dB_n(s), \right.$$
$$\left. \int_0^t \left(G''(k\sqrt{n}\, X_n(s)) - \frac{2D}{\sigma_2}\right) dM_n(s)\right).$$

Then we shall show the following: (A) $H_n \Rightarrow H$ in $D([0, T], R^3)$, T arbitrary, where the process H is defined in Case I, as

$$H(t) = ((2\bar{f}/\sigma_2)Y(t), -\lambda R(f)V(t), H_3(t)),$$

$H_3(t)$ is a continuous martingale and, in Case II,

$$H(t) = \left((2\bar{f}/\sigma_2)Y(t), \int_0^t \frac{2\bar{f}}{\sigma_2}\,\mathrm{sig}(Y(s))\,dV(s), H_3(t)\right).$$

(B) $\sup_{0\le t\le T}|N_n(t)| \to_P 0$ as $n \to \infty$ where T is an arbitrary nonrandom number.

Once (A) and (B) are proved, this implies that the process $Z_n(f)$ converges weakly in $D([0, T], R^1)$, T arbitrary, to a process $Z(f)$ where, in Case I,

$$Z(f)(t) = \frac{2\bar{f}}{\lambda\sigma_2}\,Y(t) + R(f)V(t) - \frac{1}{\lambda}\,H_3(t)$$

and, in Case II,

$$Z(f)(t) = \frac{2\bar{f}}{\lambda\sigma_2}\,Y(t) - \frac{2\bar{f}}{\lambda\sigma_2}\int_0^t \mathrm{sig}(Y(s))\,dV(s) - \frac{1}{\lambda}\,H_3(t).$$

Note that in Case I,

$$\int_0^t \mathrm{sig}(Y(s))\,dV(s) = 0$$

(Lemma 1), and so

$$Z(f)(t) - R(f)V(t) = \frac{2\bar{f}}{\lambda\sigma_2}\,Y(t) - \frac{2\bar{f}}{\lambda\sigma_2}\int_0^t \mathrm{sig}(Y(s))\,dV(s) - \frac{1}{\lambda}\,H_3(t).$$

Since $f \ge 0$, the limit process $Z(f)(t)$ is increasing, and so $Z(f)(t) - R(f)V(t)$ is a continuous process of bounded variation. Thus by Lemma 2

$$Z(f)(t) - R(f)V(t) = \frac{2\bar{f}}{\lambda\sigma_2}\,L_0(t)$$

where $L_0(t)$ is the local time at zero of a reflecting Brownian motion. Furthermore in Case I, $V(t) = L_0(t)$ (Lemma 2), and hence

$$Z(f)(t) = \left(R(f) + \frac{2\bar{f}}{\lambda\sigma_2}\right)L_0(t).$$

Thus we have shown the conclusion for Case I. Similary in Case II,

$$Z(f)(t) = \frac{2\bar{f}}{\lambda\sigma_2}\,L_0(t)$$

where $L_0(t)$ is the local time at zero of the process $Y(t)$. But in view of Lemma 2, in Case II $L_0(t)$ is a null process. So we get the desired conclusion for Case II. Now let us show (A) and (B). To see (B) first, by Lemma 4, it suffices to show that $\langle N_n \rangle (T) \to_P 0$. We have

$$\langle N_n\rangle(T) = \frac{1}{k^2 n} \langle N\rangle(nT)$$
$$= \frac{\lambda}{k^2} \int_0^T \int_0^\infty \{G(k\sqrt{n}\, X_n(s) + y) - G(k\sqrt{n}\, X_n(s)) - G'(k\sqrt{n}\, X_n(s))y\}^2\, dF(y)\, ds.$$

Since, with probability one, $k\sqrt{n}\, X_n(s) \to \infty$ as $n \to \infty$ for a.e. s, we have with probability one

$$G(k\sqrt{n}\, X_n(s) + y) - G(k\sqrt{n}\, X_n(s)) - G'(k\sqrt{n}\, X_n(s))y \to 0 \quad (n \to \infty)$$

for a.e. s. While we have

$$|\, G(k\sqrt{n}\, X_n(s) + y) - G(k\sqrt{n}\, X_n(s)) - G'(k\sqrt{n}\, X_n(s))y\,|^2 \le Ky^2,$$

K a constant. Thus by the bounded convergence theorem, we have that $\langle N_n\rangle(T) \to 0$ with probability one.

Next to see (A), it suffices to show that (i) $\{H_n\}$ is tight in $D([0, T], R^3)$, and (ii) any weak limit of $\{H_n\}$ is identified as H. Let $H_n = (H_n^1(t), H_n^2(t), H_n^3(t))$. Then we can show that

$$E[(H_n^i(t) - H_n^i(s))^2 \,|\, \mathscr{F}_{ns}] \le K(t - s), \quad t > s, \quad i = 1, 2, 3 \tag{12}$$

where K is a constant not depending on n. Then this implies also that

$$E[\,|\, H_n(t) - H_n(s)\,|^2 \,|\, \mathscr{F}_{ns}] \le K'(t - s), \quad t > s,$$

where K', a constant, does not depend on n, and from this we at once obtain tightness of $\{H_n\}$ by checking a sufficient condition for tightness in Varadhan [13, page 51]. To see (12), first we note that

$$\left\{\frac{1}{k\sqrt{n}}\, G(k\sqrt{n}\, X_n(t)) - \frac{1}{k\sqrt{n}}\, G(k\sqrt{n}\, X_n(s))\right\}^2$$
$$= \frac{1}{k^2 n} \{G'(\xi_n)k\sqrt{n}(X_n(t) - X_n(s))\}^2 \le K(X_n(t) - X_n(s))^2, \quad K \text{ a constant},$$

where $\min(k\sqrt{n}\, X_n(t), k\sqrt{n}\, X_n(s)) \le \xi_n \le \max(k\sqrt{n}\, X_n(t), k\sqrt{n}\, X_n(s))$, and we used the boundedness of $G'(\cdot)$. Then by Lemma 5, we have

$$E[(H_n^1(t) - H_n^1(s))^2 \,|\, \mathscr{F}_{ns}] \le K(t - s), \quad s < t,$$

where K does not depend on n. Next we have

$$(H_n^2(t) - H_n^2(s))^2 \le K(B_n(t) - B_n(s))^2$$

where K does not depend on n. Then equation (5) and the fact that

$$E[(M_n(t) - M_n(s))^2 \,|\, \mathscr{F}_{ns}] = t - s$$

imply that $E[(H_n^2(t) - H_n^2(s))^2 | \mathscr{F}_{ns}] \leq K(t-s)$. Finally we have

$$\begin{aligned} &E[(H_n^3(t) - H_n^3(s))^2 | \mathscr{F}_{ns}] \\ &= E\left[\int_s^t \left(G'(k\sqrt{n}\, X_n(u)) - \frac{2D}{\sigma_2}\right)^2 d\,\langle M_n\rangle(u) \,|\, \mathscr{F}_{ns}\right] \\ &\leq K[\langle M_n\rangle(t) - \langle M_n\rangle(s) \,|\, \mathscr{F}_{ns}] = K(t-s) \end{aligned}$$

where K is a constant not depending on n. Thus we have shown (12). To identify any weak limit of $\{H_n\}$ as H, since $(X_n, B_n) \Rightarrow (Y, V)$ in $D([0, T], R^2)$ we may assume, in view of Skorohod representation theorem [10], that with probability one, $X_n(t) \to Y(t)$ and $B_n(t) \to V(t)$ uniformly on every compact set of t in $[0, T]$. Then by Lemma 1 and 3 we easily obtain that with probability one,

$$H_n^1(t) = \frac{1}{k\sqrt{n}} G(k\sqrt{n}\, X_n(t)) - \frac{2D}{\sigma_2} X_n(t) \to \frac{2\bar{f}}{\sigma_2} Y(t) = H_1(t)$$

uniformly on every compact set of t in $[0, T]$. Then this implies $H_n^1 \Rightarrow H_1$ in $D([0, T], R^1)$.

Next we show that with probability one,

$$\int_0^t \left(G'(k\sqrt{n}\, X_n(s)) - \frac{2D}{\sigma_2}\right) dB_n(s) \to -\lambda R(f) V(t)$$

for each t in Case I, and

$$\int_0^t \left(G'(k\sqrt{n}\, X_n(s)) - \frac{2D}{\sigma_2}\right) dB_n(s) \to \int_0^t \frac{2\bar{f}}{\sigma_2} \operatorname{sig}(Y(s))\, dV(s)$$

for each t in Case II. Indeed in Case I, if we define the process $B_n^{-1}(t)$ by

$$B_n^{-1}(t) = \inf\{s: B_n(s) > t\},$$

then it is not hard to see $X_n(B_n^{-1}(t)) = 0$ for all t. (Draw the graph of $B_n(t)$ and $X_n(t)$.) Then

$$\begin{aligned} &\int_0^t \left(G'(k\sqrt{n}\, X_n(s)) - \frac{2D}{\sigma_2}\right) dB_n(s) \\ &= \int_0^{B_n(t)} \left(G'(k\sqrt{n}\, X_n(B_n^{-1}(s))) - \frac{2D}{\sigma_2}\right) ds \\ &= -\lambda R(f) B_n(t) \to -\lambda R(f) V(t) \quad \text{for each } t \end{aligned}$$

with probability one. In Case II, since $V(\cdot)$ is continuous and strictly increasing and since $B_n(t)$ converges to $V(t)$ uniformly in t-compact set (Lemma 1), this implies $B_n^{-1}(t) \to V^{-1}(t)$ for each t. Then uniform convergence of λX_n to Y implies $X_n(B_n^{-1}(t)) \to Y(V^{-1}(t))$ for each t. Then for each s

$$I_{\{Y(V^{-1}(s))>0\}}\left(G'(k\sqrt{n}\, X_n(B_n^{-1}(s))) - \frac{2D}{\sigma_2}\right) \to \frac{2\bar{f}}{\sigma_2} \operatorname{sig}(Y(V^{-1}(s)))$$

from Lemma 3 (i). Thus

$$\int_0^t \left(G'(k\sqrt{n}\, X_n(s)) - \frac{2D}{\sigma_2}\right) dB_n(s)$$

$$= \int_0^{B_n(t)} \left(G'(k\sqrt{n}\, X_n(B_n^{-1}(s))) - \frac{2D}{\sigma_2}\right) ds$$

$$= \int_0^{B_n(t)} I_{\{Y(V^{-1}(s))>0\}}\left(G'(k\sqrt{n}\, X_n(B_n^{-1}(s))) - \frac{2D}{\sigma_2}\right) ds$$

$$\to \int_0^{V(t)} \frac{2\bar{f}}{\sigma_2} \operatorname{sig}(Y(V^{-1}(s)))\, ds = \int_0^t \frac{2\bar{f}}{\sigma_2} \operatorname{sig}(Y(s))\, dV(s).$$

In the above we used the fact that $Y(V^{-1}(t)) > 0$ for a.e. t (see the proof of Lemma 2). This completes Case II.

Finally we will show that $H_3^n \Rightarrow H_3$ in $D([0, T], R^1)$ where $H_3(t)$ is a continuous martingale. Let $\hat{H}_3$ be any weak limit of $\{H_n^3\}$. Then for a subsequence of $\{H_n^3\}$, $H_{n'}^3 \Rightarrow \hat{H}_3$ in $D([0, T], R^1)$. We note that $\hat{H}_3$ is a continuous martingale. Indeed since $\{(H_n^1, H_n^2, H_{n'}^3)\}$ is tight in $D([0, T], R^3)$ and since $H_n^1 \Rightarrow H_1$ and $H_n^2 \Rightarrow H_2$ in $D([0, T], R^1)$, we may assume that $(H_{n'}^1, H_{n'}^2, H_{n'}^3) \Rightarrow (H_1, H_2, \hat{H}_3)$. Then by (B) and from (11), we have that $\lambda Z_{n'}(f) \Rightarrow \hat{Z} \equiv H_1 - H_2 - \hat{H}_3$. But since the process $\lambda Z_n(f)(= \lambda Z_n(f)(t))$ is continuous, its weak limit $\hat{Z}$ is also continuous. Then since H_1 and H_2 are continuous, $\hat{H}_3$ must also be continuous. To see that $\hat{H}_3$ is a martingale, it suffices to note that $\{H_n^3\}$ are martingales and that

$$\sup_n E[H_n^3(t)]^2 = \sup_n E \int_0^t \left(G'(k\sqrt{n}\, X_n(s)) - \frac{2D}{\sigma_2}\right)^2 ds < \infty.$$

Now let $\tilde{H}_3$ be another weak limit of $\{H_n^3\}$. Then as in the case of $\hat{H}_3$, for a subsequence of $\{H_n^3\}$, $\lambda Z_{n''} \Rightarrow \tilde{Z} \equiv H_1 - H_2 - \tilde{H}_3$. Then since any weak limit of $\{\lambda Z_n\}(\lambda > 0)$ is continuous and increasing, $\hat{Z}(t) - \tilde{Z}(t) = \tilde{H}_3(t) - \hat{H}_3(t)$ is a continuous martingale of bounded variation. More precisely, there exists a probability space with a reference family which is an extension of probability spaces with reference families supporting $(\hat{Z}, H_3)$ and $(\tilde{Z}, \tilde{H}_3)$ and these processes can be regarded as those defined on the extended probability space with the reference family (see [6, page 89] for the notion of extension of probability space with a reference family). Then as such processes it holds that $\hat{Z}(t) - \tilde{Z}(t) = \tilde{H}_3(t) - \hat{H}_3(t)$ and this implies that $\tilde{H}_3 = \hat{H}_3$ on the extended probability space, and hence any weak limits of $\{H_n^3\}$ have the same probability law on $D([0, T], R^1)$. Thus $\{H_n^3\}$ converge weakly to a continuous martingale H_3 in $D([0, T], R^1)$. □

In the sequel we seek a class of f for which condition (A3) holds and then extend Theorem 1.

COROLLARY 1. *$f \in C_K^\infty$ satisfies condition (A3), and G'' is given by*

$$G''(x) = \mathrm{Re}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\frac{\tilde{f}(\xi)}{H(\xi)}\,d\xi\right)$$

where

$$\tilde{f}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\,dx \quad \text{(Fourier transform)}$$

$$H(\xi) = \int_0^\infty \int_0^y u e^{-i\xi(y-u)}\,du\,dF(y)$$

$$= \frac{1}{\xi^2}(-iu\xi + 1 - \tilde{F}(\xi))$$

$$\tilde{F}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x}\,dF(x).$$

Thus in Theorem 1, if $f \in C_K^\infty$, then $R(f)$ is explicitly calculated as

$$R(f) = \frac{2}{\lambda\sigma_2}\int_0^\infty \int_0^y u \int_0^{y-u} \mathrm{Re}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\frac{\tilde{f}(\xi)}{H(\xi)}\,d\xi\right)dx\,du\,dF(y).$$

PROOF. We can easily show that $H(\xi) \neq 0$ for all ξ. On the other hand if $f \in C_K^\infty$, then $|\tilde{f}(\xi)| = o(|\xi|^{-n})$ as $|\xi| \to \infty$ for any $n \geq 1$. Thus

$$\frac{|\tilde{f}(\xi)|}{|H(\xi)|} = o(|\xi|^{-k}) \quad \text{as} \quad |\xi| \to \infty \quad \text{for any} \quad k \geq 1$$

and this implies

$$\int_{-\infty}^{\infty} \frac{|\tilde{f}(\xi)|}{|H(\xi)|}\,d\xi < \infty.$$

Hence

$$\hat{G}''(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi}\frac{\tilde{f}(\xi)}{H(\xi)}\,d\xi$$

is well-defined, and it is easy to check that

$$\int_0^\infty \int_0^y u\hat{G}''(x + y - u)\,du\,dF(y) = f(x).$$

Since $f(x)$ is real-valued, we have

$$\int_0^\infty \int_0^y uG''(x + y - u)\,du\,dF(y) = f(x).$$

Clearly $G''(x)$ is bounded, and $G''(x) \to 0$ as $x \to \infty$ by the Riemann Lebesgue theorem. □

COROLLARY 2. *For any bounded measurable nonnegative function f with compact support, in Case* II $Z_n(f)$ *converges weakly to a null process in* $D([0, T], R)$, T *arbitrary.*

PROOF. For any f with the above property, there exists a function $g \in C_K^\infty$ such that $f(x) \leq g(x)$ for all x. Then

$$0 \leq Z_n(f)(t) \leq Z_n(g)(t).$$

But in view of Theorem 1 (Case II) and Corollary 1, $Z_n(g) \Rightarrow 0$ (a null process is denoted by 0) in $D([0, T], R)$, T arbitrary. Thus $Z_n(f) \Rightarrow 0$ in $D([0, T], R)$, T arbitrary. □

COROLLARY 3. *Let f be a nonnegative continuous function with compact support. Then for Case* I $Z_n(f) \Rightarrow Z(f)$ *in* $D([0, T], R)$, T *arbitrary with* $Z(f)(t) = T(f)L_0(t)$. *Here* $T(f)$ *can be determined as* $T(f) = \lim_{m\to\infty} T(f_m)$, *for any sequence of* $f_m \in C_K^\infty$ *such that* $\operatorname{supp}(f_m) \cup \operatorname{supp}(f) \subset K$ *(a compact support) and*

$$\sup_x |f_m(x) - f(x)| \to 0 \quad as \quad m \to \infty.$$

PROOF. (suggested by Y. Kasahara). Let f_m, $m = 1, 2, \cdots, \in C_K^\infty$ be such that $\operatorname{supp}(f_m) \cup \operatorname{supp}(f) \subset K$ (a compact set) and $\sup_x |f_m(x) - f(x)| \to 0$ as $m \to \infty$. Then $\lim_{x\to\infty} T(f_m)$ exists and does not depend on the choice of $\{f_m\}$. Indeed let $g \in C_K^\infty$ be such that $0 \leq g(x) \leq 1$ for $x \in R$ and $g(x) = 1$ on K. Since $\sup_x |f_m(x) - f(x)| \to 0$ as $m \to \infty$, for an arbitrary $\varepsilon > 0$

$$\sup_x |f_m(x) - f_n(x)| < \varepsilon$$

for sufficiently large m and n. Then by the definition of g,

$$-\varepsilon g(x) \leq f_m(x) - f_n(x) \leq \varepsilon g(x), \quad x \in R.$$

Then for sufficiently large m and n,

$$-\varepsilon T(g) \leq T(f_m) - T(f_n) \leq \varepsilon T(g).$$

(Note that $T(f)$, $f \in C_K^\infty$, is linear and nonnegative, i.e., $T(f) \geq 0$ if $f \geq 0$.) Thus $\{T(f_m)\}$ is a Cauchy sequence and has a limit $T(f) = \lim_{m\to\infty} T(f_m)$. Next we will show that $Z_n(f) \Rightarrow Z(f)$ in $D([0, T], R)$, T arbitrary, where $Z(f)(t) = T(f)L_0(t)$. For an arbitrary $\varepsilon > 0$, take m so large that $\sup_x |f_m(x) - f(x)| < \varepsilon$ and $|T(f_m) - T(f)| < \varepsilon$. Then we have

$$\begin{aligned}
&|Z_n(f)(t) - Z(f)(t)| \\
&\quad \leq |Z_n(f)(t) - Z_n(f_m)(t)| + |Z_n(f_m)(t) - T(f_m)L_0(t)| \\
&\qquad + |T(f_m)L_0(t) - T(f)L_0(t)| \\
&\quad \leq \varepsilon Z_n(g)(t) + |Z_n(f_m)(t) - T(f_m)L_0(t)| + \varepsilon L_0(t) \equiv S_n(t).
\end{aligned}$$

The process $S_n(t)$ converges weakly to the process $\varepsilon(T(g) + 1)L_0(t)$. But ε was arbitrary, hence the process $|Z_n(f)(t) - Z(f)(t)|$ converges weakly to a null process, which was the desired result. □

REMARK 3. From the proof of Corollary 3, we can say that there exists a measure μ on R such that

$$T(f) = \int_0^\infty f(x)\mu(dx)$$

for any $f \in C_K^\infty$. (Note that $T(f) = 0$ if $f(x) = 0$ for $x \geq 0$. Hence $\mu((-\infty, 0)) = 0$.) Thus for any bounded measurable function f with compact support $T(f)$ can be defined as the right-hand side integral in the above. Then in view of Proposition 1 we conjecture that

$$T(1_{\{a\}}(\cdot)) = \begin{cases} 1/\lambda\mu, & a = 0 \\ 0, & a \neq 0, \end{cases}$$

i.e., the measure μ has a mass on $\{0\}$. Once this is verified it is not difficult to show that for functions such as $f(x) = I_{[\alpha,\beta]}(x)$ or $f(x) = I_{(\alpha,\beta)}(x)$, $Z_n(f) \Rightarrow Z(f)$ where $Z(f)(t) = T(f)L_0(t)$.

REMARK 4. As was mentioned in Remark 3, there exists an appropriate measure μ on R such that $Z_n(f) \Rightarrow Z(f)$ where $Z(f)(t) = T(f)L_0(t)$, $T(f) = \int_0^\infty f(x)\mu(dx)$ and $L_0(t)$ is the local time at zero of a Bessel process. While it is known that

$$\frac{1}{\sqrt{n}} \int_0^{n.} f(B(s))\, ds \Rightarrow \bar{f} \cdot L_0^B(\cdot)$$

where $B(t)$ is a Brownian motion with $B(0) = 0$, f a bounded measurable function with a compact support, $L_0^B(\cdot)$ the local time of $B(t)$ at zero, and $\bar{f} = \int_{-\infty}^\infty f(x)\,dx$ (Ikeda and Watanabe [6, III, Theorem 4.4]). In the above both the limit processes are constants times local time processes at zero of a Brownian motion or a Bessel process, and these constants are the integration of f with respect to some measures on R. In Kasahara [4, Theorem 1] the situation was clarified under which such measures appear for general Markov processes. However, the direct application of his result to our problem seems to be difficult.

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