The Annals of Probability
1985, Vol. 13, No. 2, 363–378

ON THE CONVERGENCE OF DIFFUSION PROCESSES CONDITIONED TO REMAIN IN A BOUNDED REGION FOR LARGE TIME TO LIMITING POSITIVE RECURRENT DIFFUSION PROCESSES

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Let $X(t)$ be a diffusion process on R^d with generator $L = (1/2)\nabla \cdot a\nabla + b\nabla$ and let $\{P_x\}$, $x \in R^d$, be the corresponding measures on paths. Pick $0 < t < T < \infty$ and consider the process on the time interval $[0, t]$ conditioned to remain in a certain open, connected bounded region G up to time T. We obtain a new process $Y^T(s)$, $0 \le s \le t$. Let $\tau_G = \inf\{s: X(s) \notin G\}$. With certain hypotheses on $P_x(\tau_G > s)$ (which are always satisfied if $a^{-1}b$ is a gradient function), we show that $Y^T(s)$ is an inhomogeneous diffusion process and that as $T \to \infty$, $Y^T(s)$, $0 \le s \le t$ converges to a limiting homogeneous positive recurrent diffusion $Y(s)$, $0 \le s \le t$, with state space G. Since t is arbitrary, we actually obtain a limiting process $Y(s)$, $0 \le s < \infty$. The generator of the limiting process may be written in the form $L_G = (1/2)\nabla \cdot a\nabla + b\nabla + a(\nabla g_0/g_0)\nabla - a\nabla h_{g_0}\nabla$ where g_0 is the square root of the density of a measure μ_0 which minimizes the I-function for the process, over all $\gamma \in \mathscr{P}(\bar{G})$, the set of probability measures on $\bar{G}$. The function h_{g_0} appears in the explicit calculation of $I(\mu_0)$ and solves a certain variational equation. The invariant measure for the process is μ_0.

1. Introduction. Let $X(s)$ be a diffusion process on R^d with generator $L = (1/2)\nabla \cdot a\nabla + b\nabla$ where a is a positive $d \times d$ matrix with coefficients $a_{ij} \in C^1(R^d)$ and b is a continuous d-vector, and let $\{P_x, x \in R^d\}$ be the collection of measures on $C([0, \infty), R^d)$ which define the process. Let $G \subset R^d$ be a bounded open connected set and put $\tau_G = \{\inf s \ge 0: X(s) \notin G\}$. In this paper, we will consider the diffusion process conditioned to remain in G for large time. For $0 < t < T < \infty$, we define a conditioned process up to time t, starting from $x \in G$ at time zero by $Q_{x,0}^{T,t}(Y^T(\cdot), 0 \le s \le t) = P_x(X(\cdot), 0 \le s \le t \mid \tau_G > T)$. We will show that $Y^T(\cdot)$ is an inhomogeneous diffusion, and that as $T \to \infty$, $Y^T(\cdot)$ converges to a limiting homogeneous, positive recurrent diffusion on G. Since t is arbitrary, the limiting process is actually defined for all $t \ge 0$.

In order to motivate the results and to state them in a completely probabalistic manner, we describe briefly the I-function theory for large deviations of Markov processes. Let $\omega = z(\cdot)$ be a path of a strong Feller process with generator $\mathscr{L}$ on a domain D of R^d and consider for $B \subset D$, $L_t(\omega, B) = (1/t) \int_0^t \chi_{(B)}(z(s))\, ds$. $L_t(\omega, B)$ measures the proportion of time up to t that the process is in B. Hence $L_t(\omega, \cdot) \in \mathscr{P}(D)$, the space of probability measures on D; it is the occupation

Received December 1983; revised May 1984.
AMS 1980 *subject classification.* Primary 60J60 F.
Key words and phrases. Conditioned diffusion processes, I-function, convergence of diffusion processes, invariant measures.

measure for the particular path ω. For $\mu \in \mathscr{P}(D)$, define the I-function for the process by $I(\mu) = -\inf_{u \in \mathscr{D}^+} \int_D (\mathscr{L}u/u)\, d\mu$ where $\mathscr{D}^+ = \{u \in \mathscr{D}: u \geq c > 0\}$ and $\mathscr{D}$ is the domain of the generator $\mathscr{L}$. It is easy to check that $I(\mu) \geq 0$ (pick $u =$ const.) and lower semicontinuous under the weak topology on $\mathscr{P}(D)$. Let P_x be the measure on paths induced by the process starting from $x \in D$. Under a suitable transitivity condition, Donsker and Varadhan [3, 4] have proved that for open sets $U \subset \mathscr{P}(D)$,

$$\liminf_{t\to\infty} (1/t)\log P_x(L_t(\omega, \cdot) \in U) \geq -\inf_{\mu \in U} I(\mu) \tag{1.1}$$

and for compact sets $C \subset \mathscr{P}(D)$,

$$\limsup_{t\to\infty} (1/t)\log P_x(L_t(\omega, \cdot) \in C) \leq -\inf_{\mu \in C} I(\mu), \quad \text{for all} \quad x \in D. \tag{1.2}$$

Furthermore, the following propositions hold.

PROPOSITION 1. *$I(\mu) = 0$ if and only if μ is invariant for the process.* (Use Lemmas 2.5 and 3.1 in [3].)

Now consider a compact D so that closed sets in $\mathscr{P}(D)$ are compact. Let $B_\varepsilon(\mu) \subset \mathscr{P}(D)$ be the open ε neighborhood (with respect to some suitable metric) around μ. Then (1.1), (1.2), Proposition 1 and the lower semicontinuity of $I(\cdot)$ give us

PROPOSITION 2. *If μ is invariant for the process, then* $\liminf_{t\to\infty}(1/t)\log P_x(L_t(\omega, \cdot) \in B_\varepsilon(\mu)) = 0$ *for all $\varepsilon > 0$ and all $x \in D$. If μ is not invariant for the process, then for each $x \in D$,* $\limsup_{t\to\infty}(1/t)\log P_x(L_t(\omega, \cdot) \in B_\varepsilon(\mu)) < 0$ *for sufficiently small $\varepsilon > 0$.*

In the context of the diffusion processes above, the following proposition also holds [5, Theorem 2.2].

PROPOSITION 3. *Let $G \subset R^d$ be a bounded open connected set. Then*

$$\lim_{t\to\infty} (1/t)\log P_x(\tau_G > t) = -\inf_{\{\mu \in \mathscr{P}(R^d): \operatorname{supp}\mu \subset \bar{G}\}} I(\mu), \tag{1.3}$$

for $x \in G$.

To motivate our main theorem, we apply the above proposition to the conditioned process $Y^{T,t}(\cdot)$. Note that since $\mathscr{P}(\bar{G})$ is compact and since $I(\cdot)$ is lower semicontinuous, the infimum on the right hand side of (1.3) is actually attained. Suppose the infimum occurs uniquely at some measure μ_0. Then, for any other measure $\mu \in \mathscr{P}(R^d)$, supported in $\bar{G}$, pick $\varepsilon > 0$ small enough so that $\mu_0 \notin$

$\bar{B}_\varepsilon(\mu) \subset \mathscr{P}(R^d)$. Thus $\inf_{\{\gamma\in\bar{B}_\varepsilon(\mu):\text{supp}\gamma\subset\bar{G}\}} I(\gamma) > I(\mu_0)$ and we have

$$\begin{aligned}
&\limsup_{T\to\infty}(1/T)\log Q_X^{T,T}(L_T(\omega,\cdot)\in B_\varepsilon(\mu)\cap\mathscr{P}(\bar{G}))\\
&\quad=\limsup_{T\to\infty}(1/T)\log P_x(L_T(\omega,\cdot)\in B_\varepsilon(\mu)\mid\tau_G>T)\\
&\quad=\limsup_{T\to\infty}(1/T)\log\frac{P_x(L_T(\omega,\cdot)\in B_\varepsilon(\mu),\ \tau_G>T)}{P_x(\tau_G>T)} \qquad (1.4)\\
&\quad=\limsup_{T\to\infty}(1/T)\log\frac{P_x(L_T(\omega,\cdot)\in B_\varepsilon(\mu)\cap\mathscr{P}(\bar{G}))}{P_x(\tau_G>T)}\\
&\quad\le -\inf_{\{\gamma\in\bar{B}_\varepsilon(\mu):\text{supp}\gamma\subset\bar{G}\}} I(\gamma)+I(\mu_0)<0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\liminf_{T\to\infty}(1/T)\log Q_x^{T,T}(L_T(\omega,\cdot)\in B_\varepsilon(\mu_0)\cap\mathscr{P}(\bar{G}))\\
&\quad=\liminf_{T\to\infty}(1/T)\log\frac{P_x(L_T(\omega,\cdot)\in B_\varepsilon(\mu_0),\ \tau_G>T)}{P_x(\tau_G>T)} \qquad (1.5)\\
&\quad=\liminf_{T\to\infty}(1/T)\log P_x(L_T(\omega,\cdot)\in B_\varepsilon(\mu_0),\ \tau_G>T)+I(\mu_0).
\end{aligned}$$

(1.6) If $\liminf_{T\to\infty}(1/T)\log P_x(L_T(\omega,\cdot)\in B_\varepsilon(\mu_0),\ \tau_G>T)=-I(\mu_0)$,

then (1.5) becomes

$$\liminf_{T\to\infty}(1/T)\log Q_x^{T,T}(L_T(\omega,\cdot)\in B_\varepsilon(\mu_0)\cap\mathscr{P}(\bar{G}))=0. \qquad (1.7)$$

Unfortunately, we don't quite have the technical machinery to prove (1.6), although, if we take a δ-neighborhood G_δ of G, then, with τ_{G_δ} replacing τ_G, and with "≥" replacing "=", (1.6) is true [4, Theorem 8.1].

Comparing (1.4) and (1.5) (or 1.7) to Proposition 2, one is led to wonder whether as $T\to\infty$, the conditioned process, $Q^{T,T}$, converges to a limiting positive recurrent process on G with μ_0 as the invariant measure. Indeed this will more or less be the case ("more or less" because we will actually consider $Q^{T,t}$ as $T\to\infty$).

Let $\tilde{L}\equiv \frac{1}{2}\nabla\cdot a\nabla - b\nabla - \nabla\cdot b$ be the formal adjoint to L. The Krein-Rutman theory of positive operators provides us with the following theorem [10].

KR. *The operators $-L$ and $-\tilde{L}$ with the Dirichlet boundary condition on ∂G have a common simple real eigenvalue λ_0 at the bottom of their respective spectra. The corresponding eigenfunctions, φ_0 and $\tilde{\varphi}_0$, are positive on G and vanish on ∂G. We may assume that φ_0 and $\tilde{\varphi}_0$ have been normalized so that $\int_G \varphi_0\tilde{\varphi}_0(x)\,dx=1$.*

We now propose 3 hypotheses which will be assumed only when explicitly stated. The main theorem will require Hypotheses 1 and 2.

HYPOTHESIS 1. *$P_x(\tau_G>t)\in C^2(G)$ as a function of x.*

HYPOTHESIS 2. *$P_x(\tau_G > t) = C_1\varphi_0\exp(-\lambda_0 t) + o(\exp(-\lambda_0 t))$ as $t \to \infty$ and $\nabla P_x(\tau_G > t) = C_1\nabla\varphi_0\exp(-\lambda_0 t) + o(\exp(-\lambda_0 t))$ as $t \to \infty$ with $o(\exp(-\lambda_0 t))$ uniform on compact subsets of G.*

HYPOTHESIS 3. *$P_x(X(t) \in dy, \tau_G > t)$ has a density of the form $C_2\exp(-\lambda_0 t)\varphi_0(x)\tilde{\varphi}_0(y) + o(\exp(-\lambda_0 t))$ as $t \to \infty$ uniformly for x in compact subsets of G.*

These hypotheses are discussed in the appendix. At this point in the exposition, we will content ourselves with mentioning that Hypotheses 1 and 2 hold if $a^{-1}b(= \nabla Q)$ is a gradient function. In particular, this is always the case in one dimension. If, furthermore, $Q \in C^2(\bar{G})$, or equivalently, $b \in C^1(\bar{G})$, then Hypothesis 3 holds.

Before we can state the main result of this paper, we must give the following explicit representation of the I-function which we obtained in [7]. For $\mu \in \mathscr{P}(R^d)$ with support in $\bar{G}$,

$$
\begin{aligned}
I(\mu) &= \frac{1}{2}\int_G \left(\frac{\nabla g}{g} - a^{-1}b\right)a\left(\frac{\nabla g}{g} - a^{-1}b\right)g^2\,dx \\
&\quad -\inf_{h\in C^2(R^d)}\int_G (\nabla h - a^{-1}b)a(\nabla h - a^{-1}b)g^2\,dx, \\
&\qquad \text{if } \mu \text{ has a density } \varphi \text{ with } \varphi^{1/2} \equiv g \in W_1^2(G). \\
I(\mu) &= \infty, \quad \text{otherwise.}
\end{aligned}
\tag{1.8}
$$

Furthermore, there exists a unique $h_g \in W_1^2(D, d\mu)$ at which the infimum above is attained. In fact, $h = h_g$ is the unique solution to the variational equation

$$
\int_G (\nabla h a \nabla q - \nabla q \cdot b)g^2\,dx = 0, \quad \text{for all} \quad q \in C^1(G). \tag{1.9}
$$

($W_1^2(D, d\mu)$ is the Sobolev space of functions with one generalized $L_2(D, d\mu)$ derivative and $W_1^2(D) \equiv W_1^2(D, dx)$ where dx is Lebesgue measure.) We now state our

THEOREM. *Let $X(s)$, $0 \le s < \infty$ be a homogeneous diffusion process on R^d with generator $L = \frac{1}{2}\nabla \cdot a\nabla + b\nabla$ where a is a positive $d \times d$ matrix with coefficients $a_{ij} \in C^1(R^d)$ and b is a continuous d-vector. Assume that the process satisfies Hypotheses 1 and 2. Let $\{P_x\}$, $x \in R^d$ be the associated probability measures on $C([0, \infty), R^d)$ and put $0 \le t < T < \infty$. Define a new process $Y^T(s)$, $0 \le s \le t$ on G by*

$$
Q_{x,0}^{T,t}(Y^T(s),\ 0 \le s \le t) = P_x(X(s),\ 0 \le s \le t \mid \tau_G > T).
$$

Then $Y^T(s)$, $0 \le s \le t$ is an inhomogeneous diffusion process with generator

$$
L_x^T = \frac{1}{2}\nabla \cdot a\nabla + b\nabla + \frac{a\nabla_x P_x(\tau_G > T - s)}{P_x(\tau_G > T - s)}
$$

and, as $T \to \infty$, $\{Y^T(s),\ 0 \le s \le t\}$ converges to a limiting homogeneous *diffusion* $\{Y(s), 0 \le s \le t\}$ *on* G *(that is, for each* $x \in G$*, there exists a measure* Q_x^t *on* $C([0, t], G)$ *such that* $Q_{x,0}^{T,t} \Rightarrow Q_x^t$ *as* $T \to \infty$*) with generator* $L_G = \frac{1}{2}\nabla \cdot a\nabla + b\nabla + (a\nabla g_0/g_0)\nabla - a\nabla h_{g_0}\nabla$ *where* g_0 *is the square root of the density of a certain measure* μ_0 *at which* $\inf_{\{\mu\in\mathscr{P}(R^d):\text{supp}\mu\subset\bar{G}\}} I(\mu)$ *is attained. Since* t *is arbitrary, we actually obtain a limiting process* $Y(s)$*,* $0 \le s < \infty$ *(and measure* Q_x *on* $C([0, \infty), G)$*) with generator* L_G*. The measure* μ_0 *with density* g_0^2 *is invariant for the limiting process. Furthermore,* $\varphi_0 = g_0\exp(-h_{g_0})$ *and* $\tilde{\varphi}_0 = g_0\exp(h_{g_0})$ *where* φ_0 *and* $\tilde{\varphi}_0$ *are as in Hypotheses* 2 *and* 3*. Thus, in particular, the density of* u_0 *may also be written as* $\varphi_0\tilde{\varphi}_0$*.*

Note that the invariant measure for the limiting process, μ_0 with density $\varphi_0\tilde{\varphi}_0$, can be obtained from the following double limit:

$$\mu_0(\cdot) = \lim_{t\to\infty}\lim_{T\to\infty}P_x(X(t) \in \cdot \mid \tau_G > T).$$

Under Hypotheses 2 and 3, the following proposition is immediate.

PROPOSITION 1.10. $\gamma_0(\cdot) = \lim_{T\to\infty}P_x(X(T) \in \cdot \mid \tau_G > T)$ *exists and has density* $\tilde{\varphi}_0/\int_G\tilde{\varphi}_0\,dy$*.*

Hence $\gamma_0 \ne \mu_0$, and in particular, since $\varphi_0 = \tilde{\varphi}_0 = 0$ on ∂G, we see that μ_0 gives less measure to small neighborhoods of ∂G than does γ_0. The intuition for this is easy. Let A be a small neighborhood of ∂G. Then $P_x(X(T) \in A \mid \tau_G > T)$ is rather small because paths ending up at time T far away from ∂G are relatively more likely to have remained in G for all time up to T than are paths which end up at time T in A. However, for $t < T$, $P_x(X(t) \in A \mid \tau_G > T)$ is even smaller, because paths ending up at time t far away from ∂G are all the more relatively likely to have remained in G up to time t and to continue to remain in G up to time T than are paths which end up in A at time t.

It is interesting to compare our result to a similar result for periodic irreducible Markov chains $\{X_n\}$ with a discrete state space. See [2] for the finite case and [8] for the countably infinite case. For the finite case, let 0 be an absorbing state, let $1, 2, \cdots, s$ be the s other states and let $\tau_0 = \inf\{n \ge 0: X_n = 0\}$. Write the transition matrix in the form

$$P = \{P_{ij}\} = \begin{pmatrix} 1 & 0 \\ A & R \end{pmatrix}$$

where R is $s \times s$, A is $s \times 1$ and 0 is the $1 \times s$ vector of zeroes. By the Perron-Frobenius Theorem, the largest left and right eigenvalues of R are real, simple and equal to one another, and the corresponding left $1 \times s$ and right $s \times 1$ eigenvectors, V and W respectively, may be picked with all positive entries. Assume V and W are normalized so that $V \cdot W = 1$ (analogous to $\int_G \varphi_0\tilde{\varphi}_0\,dx = 1$). Then for $i, j \ne 0$ and $n \le m$,

$$P_i(X_n = j \mid \tau_0 > m) = \frac{P_i(X_n = j)P_j(\tau_0 > m - n)}{P_i(\tau_0 > m)} = \frac{R_{ij}^n e_j R^{m-n}\tilde{1}}{e_i R^m \tilde{1}}, \tag{1.11}$$

where $\tilde{1}$ is the $s \times 1$ vector of ones and e_j is the standard $1 \times s$ unit vector in the j direction. It can be shown that

$$R^m = \lambda_0^m WV + O(m) \quad \text{as} \quad m \to \infty. \tag{1.12}$$

Thus,

$$Q_{ij}^n \equiv \lim_{m\to\infty} P_i(X_n = j \mid \tau_0 > m)$$
$$= \lambda_0^{-n} \frac{W_j}{W_i} R_{ij}^n.$$

One easily checks that $\{Q_{ij}^n\}_{n=1}^\infty$ are stochastic matrices which satisfy the Chapman-Kolmogorov equations. Thus, the inhomogeneous process obtained by conditioning $\{X_j\}_{j=1}^n$ on not being absorbed by time m, converges as $m \to \infty$ to a limiting process with state space $\{1, 2, \ldots s\}$ with transition matrix $Q_{ij} = \lambda^{-1}(W_j/W_i)R_{ij}$. Furthermore, letting μ_0 be the $1 \times s$ vector with ith component $V_i W_i$, we have $\sum_{i=1}^s (\mu_0)_i = \sum_{i=1}^s V_i W_i = 1$ and $(\mu_0 Q)_j = \sum_{i=1}^s V_i W_i \lambda^{-1}(W_j/W_i)R_{ij} = V_j W_j = (\mu_0)_j$ so μ_0 is invariant for the limiting process, analogous to $\varphi_0\tilde{\varphi}_0$ being invariant for the limiting diffusion.

To obtain the finite Markov chain result corresponding to Proposition 1.10, we use (1.11) and (1.12) to obtain

$$\lim_{m\to\infty} P_i(X_m = j \mid \tau_0 > m) = \lim_{m\to\infty} \frac{R_{ij}^m e_j \cdot \tilde{1}}{e_i R^m \cdot \tilde{1}} = \frac{V_j}{\sum_{i=1}^s V_i},$$

analogous to the limiting probability measure $(\tilde{\varphi}_0/\int_G \tilde{\varphi}_0 \, dy)$ in the diffusion case.

In the countable state space case, let 0 be the absorbing state and let $\{1, 2, \cdots\}$ be the other states. A couple of additional conditions are required. The principal one is that there exists a positive constant r such that for any $i \neq 0$, $P_i(\tau_0 \geq n)r^n$ converges to a finite nonzero limit as $n \to \infty$. This is analogous to Hypothesis 2.

In Section 2 we prove our theorem and in Section 3 we give several interesting examples to illustrate the theory.

2. Proof of Theorem.

A note on notation: We will usually use the generic $E_{x,s}(\cdot)$ for expectations. It will be clear which process we are referring to because inside the parentheses will be expressions involving $X(\cdot)$ or $Y^T(\cdot)$, etc. For homogeneous processes starting from $t = 0$, we will write E_x for $E_{x,0}$. When considering the stopping time τ_G, we will write $P_x(\tau_G > s)$ or $Q_{x,0}^{T,t}(\tau_G > s)$, etc. We will let $E(f, A) \equiv E(\chi_A f)$ for f a measurable real valued function defined on the sample path space and A a Borel set of the sample path space. We will also write $E(A) \equiv E(\chi_A)$ for A as above.

We begin with a lemma which identifies the conditioned process as in the statement of the theorem.

LEMMA 2.1. *Assume Hypothesis* 1 *holds. Then the process* $Y^T(\cdot)$, $0 \leq s \leq t$ *is* *an inhomogeneous strong Feller diffusion process on G with generator*

$$L_s^T = \frac{1}{2}\nabla \cdot a\nabla + b\nabla + a\,\frac{\nabla P_x(\tau_G > T - s)}{P_x(\tau_G > T - s)}\,\nabla.$$

PROOF. It is trivial to check that $Y^T(\cdot)$ is Markovian. Since $X(\cdot)$ is a strong Feller process, so is $Y^T(\cdot)$. To show that $Y^T(\cdot)$ is a diffusion with generator L_s^T, we must verify that the following three conditions are satisfied for $s < t$ and $x \in G$.

(i) $\lim_{h\to 0}(1/h)E_{x,s}(|Y^T(s+h) - x| > \varepsilon) = 0$

(ii) $\lim_{h\to 0}(1/h)E_{x,s}(Y_i^T(s+h) - x_i,\ |Y^T(s+h) - x| < \varepsilon)$

$$= \frac{1}{2}(\nabla \cdot a)_i(x) + b_i(x) + \frac{(a(x)\nabla P_x(\tau_G > T - s))_i}{P_x(\tau_G > T - s)}.$$

(iii) $\lim_{h\to 0}(1/h)E_{x,s}((Y_i^T(s+h) - x_i)(Y_j^T(s+h) - x_j),\ |Y^T(s+h) - x| < \varepsilon)$

$= a_{ij}(x)$

To obtain (i), we write

$$\begin{aligned}\limsup_{h\to 0}\ (1/h)E_{x,s}(|Y^T(s+h) - x| > \varepsilon) &= \limsup_{h\to 0}(1/h)E_x(|X(h) - x| > \varepsilon \mid \tau_G > T - s)\\ &\le \limsup_{h\to 0}\frac{E_x(|X(h) - x| > \varepsilon)}{h\,P_x(\tau_G > T - s)} = 0\end{aligned}$$

since $X(\cdot)$ is a diffusion.

To prove (ii), write

$$\begin{aligned}&E_{x,s}(Y_i^T(s+h) - x_i,\ |Y^T(s+h) - x| < \varepsilon)\\ &= \frac{E_x(X_i(h) - x_i,\ |X(h) - x| < \varepsilon,\ \tau_G > T - s)}{P_x(\tau_G > T - s)}\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}&E_x(X_i(h) - x_i,\ |X(h) - x| < \varepsilon,\ \tau_G > T - s)\\ &= E_x(E(X_i(h) - x_i,\ |X(h) - X| < \varepsilon,\ \tau_G > T - s \mid X(h)))\\ &= E_x((X_i(h) - x_i)P_{X(h)}(\tau_G > T - s - h),\ |X(h) - x| < \varepsilon,\ \tau_G > h)\\ &= E_x((X_i(h) - x_i)P_{X(h)}(\tau_G > T - s - h),\ |X(h) - x| < \varepsilon)\\ &\quad - E_x((X_i(h) - x_i)P_{X(h)}(\tau_G > T - s - h),\ |X(h) - x| < \varepsilon,\ \tau_G < h)\\ &= E_x((X_i(h) - x_i)P_{X(h)}(\tau_G > T - s),\ |X(h) - x| < \varepsilon)\\ &\quad + E_x((X_i(h) - x_i)(P_{X(h)}(\tau_G > T - s - h)\\ &\quad - P_{X(h)}(\tau_G > T - s)),\ |X(h) - x| < \varepsilon)\\ &\quad - E_x((X_i(h) - x_i)P_{X(h)}(\tau_G > T - s - h),\ |X(h) - x| < \varepsilon,\ \tau_G < h).\end{aligned} \tag{2.3}$$

Let $f(y) = (y_i - x_i)P_y(\tau_G > T - s)$. By Hypothesis 1, $f(y) \in C^2(G)$. From the first term on the right hand side of (2.3), we get

$$\begin{aligned}
\lim_{h\to 0}(1/h)E_x((X_i(h) - x_i)P_{X(h)}(\tau_G > T - s), |X(h) - x| < \varepsilon) \\
&= \lim_{h\to 0}(1/h)E_x(f(X(h)) - f(x), |X(h) - x| < \varepsilon) \\
&= (\tfrac{1}{2}\nabla \cdot a\nabla f + b\nabla f)(x) \\
&= (a(x)\nabla P_x(\tau_G > T - s))_i + \tfrac{1}{2}(\nabla \cdot a)_i(x) \\
&\quad \cdot P_x(\tau_G > T - s) + b_i(x)P_x(\tau_G > T - s).
\end{aligned}$$

Considering (2.2) and (2.3), we can complete the proof of (ii) if we show that the last two terms on the right hand side of (2.3) are $o(h)$. We have by the Schwarz inequality,

$$\begin{aligned}
\limsup_{h\to 0}(1/h)E_x((X_i(h) - x_i)(P_{X(h)}(\tau_G > T - s - h) \\
&\quad - P_{X(h)}(\tau_G > T - s)), |X(h) - x| < \varepsilon) \\
&\le \limsup_{h\to 0}(1/h)E_x((X_i(h) - x_i)^2, |X(h) - x| \le \varepsilon) \\
&\quad \cdot E_x((P_{X(h)}(\tau_G > T - s - h) - P_{X(h)}(\tau_G > T - s))^2) \\
&= 0,
\end{aligned}$$

since $(1/h)E_x((X_i(h) - x_i)^2, |X(h) - x| < \varepsilon)$ remains bounded as $h \to 0$ and $g(y, u) \equiv P_y(\tau_G > T - u) \equiv P_{y,u}(\tau_G > T)$ is jointly continuous in y and u by the strong Feller property. The other term on the right hand side of (2.3), $E_x((X_i(h) - x_i)P_{X(h)}(\tau_G > T - s - h), |X(h) - x| < \varepsilon, \tau_G < h)$, may be treated analogously.

Finally, we must show that (iii) holds. We have

$$\begin{aligned}
E_{x,s}((Y_i^T(s + h) - x_i)(Y_j^T(s + h) - x_j), |Y^T(s + h) - x| < \varepsilon) \\
= \frac{E_x((X_i(h) - x_i)(X_j(h) - x_j), |X(h) - x| < \varepsilon, \tau_G > T - s)}{P_x(\tau_G > T - s)}
\end{aligned}$$

and, using the same manipulations as in (2.3),

$$\begin{aligned}
E_x((X_i(h) - x_i)(X_j(h) - x_j), |X(h) - x| < \varepsilon, \tau_G > T - s) \\
= E_x((X_i(h) - x_i)(X_j(h) - x_j)P_{X(h)}(\tau_G > T - s), |X(h) - x| < \varepsilon) \\
+ E_x((X_i(h) - x_i)(X_j(h) - x_j)(P_{X(h)}(\tau_G > T - s - h) \\
- P_{X(h)}(\tau_G > T - s)), |X(h) - x| < \varepsilon) \\
- E_x((X_i(h) - x_i)(X_j(h) - x_j)P_{x(h)}(\tau_G > T - s - h), \\
|X(h) - x| < \varepsilon, \tau_G < h).
\end{aligned} \tag{2.4}$$

Let $f(y) = (y_i - x_i)(y_j - x_j)P_y(\tau_G > T - s)$. From the first term on the right hand side of (2.4), we obtain

$$\lim_{h\to 0}(1/h)E_x((X_i(h) - x_i)(X_j(h) - x_j)P_{X(h)}(\tau_G > T - s), \, |X(h) - x| < \varepsilon)$$
$$= \lim_{h\to 0}(1/h)E_x(f(X(h)) - f(x), \, |X(h) - x| < \varepsilon)$$
$$= (\tfrac{1}{2}\nabla \cdot a\nabla f + b\nabla f)(x) = a_{ij}(x)P_x(\tau_G > T - s).$$

To complete the proof, we must show that the last two terms on the right hand side of (2.4) are $o(h)$. We have by the Schwartz inequality,

$$\limsup_{h\to 0}(1/h)E_x((X_i(h) - x_i)(X_j(h) - x_j) \cdot (P_{X(h)}(\tau_G > T - s - h) - P_{X(h)}(\tau_G > T - s)), \, |X(h) - x| < \varepsilon) = 0$$

by the same argument as in the proof of (ii). The other term on the right hand side of (2.4),

$$E_x((X_i(h) - x_i)(X_j(h) - x_j)P_{X(h)}(\tau_G > T - s - h), \, |X(h) - x| < \varepsilon, \, \tau_G < h),$$

is treated analogously. This completes the proof of Lemma 2.1.

For the rest of this section, we assume that Hypotheses 1 and 2 hold. By Hypothesis 2,

$$\lim_{T\to\infty} \frac{\nabla P_x(\tau_G > T - s)}{P_x(\tau_G > T - s)} = \frac{\nabla \varphi_0(x)}{\varphi_0(x)}, \tag{2.5}$$

uniformly for x in compact subsets of G. Hence, formally,

$$\lim_{T\to\infty} L_s^T = \mathscr{L}_G \equiv \frac{1}{2}\nabla \cdot a\nabla + b\nabla + \frac{a\nabla\varphi_0}{\varphi_0}.$$

We want to utilize a theorem of Stroock and Varadhan to show that, in fact, the diffusion $Y^T(\cdot)$ with generator L_s^T converges to a homogeneous diffusion $Y(\cdot)$ which remains in G for all time up to t and has generator $\mathscr{L}_G$. Then we will identify $\mathscr{L}_G$ with L_G to complete the proof of the theorem.

We need to introduce the martingale framework for diffusion processes. Let $L_u = (1/2)\nabla \cdot \hat{a}(u, x)\nabla + \hat{b}(u, x)\nabla$ with $\hat{a}(u, x)$ a $d \times d$ matrix function and $\hat{b}$ a d-vector function on $[0, \infty) \times R^d$. The martingale problem for $\hat{a}$ and $\hat{b}$ is the problem of finding for each $x \in R^d$ and $s \in [0, \infty)$, a probability measure $\hat{P}_{x,s} \in \mathscr{P}(C([s, \infty), R^d))$ which satisfies

(a) $\hat{P}_{x,s}(Z(s) = x) = 1$

(b) $f(Z(t)) - \int_s^t L_u f(Z(u))\, du$

is a $\hat{P}_{x,s}$ martingale for $t \geq s$ and $f \in C_0^\infty(R^d)$.

Consider the following two conditions on $\hat{a}$.

CONDITION 1. *$\hat{a}$ is strictly positive on compact sets.*

CONDITION 2. $\lim_{y \to x} \sup_{0 \le s \le t} \| \hat{a}(s, y) - \hat{a}(s, x) \| = 0$, *for all* $t > 0$ *and* $x \in R^d$.

We will now present three theorems of Stroock and Varadhan which can be found in [9]. We will use the generic $Z(\cdot)$ to denote a sample path of any of the processes considered in connection with these theorems.

SV-I. *If Conditions* 1 *and* 2 *hold and* $\hat{a}$ *and* $\hat{b}$ *are bounded and measurable, then there exists a unique solution to the martingale problem for each* $s \ge 0$ *and* $x \in R^d$. *Furthermore, for any stopping time* τ, $\hat{P}_{Z(\tau),\tau}$ *is a version of the conditional probability of* $\hat{P}_{x,s}$ *given* $\mathscr{F}_\tau$, *the* σ*-field up to the stopping time* τ.

Now consider the case in which Conditions 1 and 2 still hold, but the coefficients are only locally bounded and measurable. That is, for each $R > 0$, there exists a constant M_R with $\| \hat{a} \| \le M_R$, $| \hat{b} | \le M_R$ for $|x| \le R$, $0 \le t \le R$. Let $\{G_m\}_{m=1}^\infty$ be an increasing sequence of bounded open sets with $[0, \infty) \times R^d = \bigcup_{m=1}^\infty G_m$. Pick a bounded a_m satisfying Conditions 1 and 2 and a bounded b_m with $a_m = \hat{a}$ and $b_m = \hat{b}$ on G_m. Let $L_u^m = \frac{1}{2} \nabla \cdot a_m \nabla + b_m \nabla$. By SV-I, the martingale problem for L_u^m has a unique solution, $P_{x,s}^m$, for each $(s, x) \in [0, \infty) \times R^d$. Let $\tau_m = \inf\{t \ge s: Z(t) \notin G_m\}$. Stroock and Varadhan prove

SV-II. *If* $\hat{a}$ *and* $\hat{b}$ *are locally bounded and measurable and* $\hat{a}$ *satisfies Conditions* 1 *and* 2, *then there exists at most one solution to the martingale problem starting from any* $(s, x) \in [0, \infty) \times R^d$. *Moreover, if* $P_{x,s}^n$ *is the unique solution for* a_n *and* b_n, *then for each* $(s, x) \in [0, \infty) \times R^d$, *a solution exists for* $\hat{a}$ *and* $\hat{b}$ *if and only if* $\lim_{n \to \infty} P_{x,s}^n(\tau_n \le t) = 0$ *for each* $(s, x) \in [0, \infty) \times R^d$ *and* $t > s$. *If a solution* $\hat{P}_{x,s}$ *exists, then* $\hat{P}_{x,s} = P_{x,s}^n$ *on* $\mathscr{F}_{\tau_n}$.

Finally, with $\{G_m\}_{m=1}^\infty$ as above we have the following key theorem of Stroock and Varadhan.

SV-III. *Let* $\hat{a}(t, x)$ *and* $\hat{b}(t, x)$ *be locally bounded measurable functions which are continuous in* x *for each* $t \ge 0$. *Assume that for* $(s, x) \in [0, \infty) \times R^d$, *the martingale problem for* $\hat{a}$ *and* $\hat{b}$ *has a unique solution* $\hat{P}_{x,s}$. *Suppose that for each* $n \ge 1$, $a_n(t, x)$ *and* $b_n(t, x)$ *are measurable functions on* $[0, \infty) \times R^d$ *and assume that for each* $T > 0$ *and* m,

$$\sup_{n > 1} \sup_{0 \le s \le T} \sup_{x \in G_m} (\| a_n(s, x) \| + | b_n(s, x) |) < \infty$$

and

$$\lim_{n \to \infty} \int_0^T \sup_{x \in G_m} (\| \hat{a}(s, x) - a_n(s, x) \| + | \hat{b}(s, x) - b_n(s, x) |) \, ds = 0$$

If $P_{x,s}^n$ *is a solution to the martingale problem starting from* $(s, x) \in [0, \infty) \times R^d$ *for* a_n *and* b_n, *then* $P_{x,s}^n \Rightarrow \hat{P}_{x,s}$ *as* $n \to \infty$.

(We should mention that the topology on $C([0, \infty), R^d)$ is the topology of uniform convergence on bounded t-intervals.)

We will use the above theorem to show that $Q_{x,0}^{T,t} \Rightarrow Q_x^t \in \mathscr{P}(C([0, t], R^d))$

where Q_x^t solves the martingale problem for $\mathscr{L}_G$, but first we must rephrase the above theory for R^d to meet our needs on G. We replace R^d by G and now let $\{G_m\}_{m=1}^\infty$ be an increasing sequence of open sets with $\cup_{m=1}^\infty G_m = G$. Let $L_u = \frac{1}{2}\nabla \cdot \hat{a}(u, x)\nabla + \hat{b}(u, x)\nabla$ with $\hat{a}$ and $\hat{b}$ locally bounded on G, that is bounded on compact subsets of G. We will define the martingale problem on G up to time t to be the problem of finding for each $0 \le s \le t$ and $x \in G$, a probability measure $\hat{P}_{x,s} \in C([s, \infty), R^d)$ satisfying,

(a) $\hat{P}_{x,s}(Z(s) = x) = 1$

(b) $f(Z(v)) - \int_s^v L_u f(Z(u))\, du$ is a $\hat{P}_{x,s}$ martingale for all $f \in C_0^\infty(G)$, $s \le v \le t$

(c) $\hat{P}_{x,s}(Z(u) \in G, s \le u \le t) = 1$

If one looks at the proof of SV-III [9, Theorem 11.1.4] and of the key lemma [9, Lemma 11.1.1] upon which it is based, it will be clear that SV-III holds in our context. In fact, we have stated SV-III in such a manner that one need only replace R^d by G to obtain the appropriate theorem for our context. Thus, since (2.5) holds, in order to prove that $Q_{x,0}^{T,t} \Rightarrow Q_x^t$, and that Q_x^t has generator $\mathscr{L}_G$, we need only verify the following three statements.

(1) $\{Q_x^{T,t}\}$, $x \in G$, solves the martingale problem on G up to t for L_s^T.

(2) There exists a unique solution to the martingale problem on G up to t for $\mathscr{L}_G$.

(3) The generator for the process corresponding to Q_x^t is $\mathscr{L}_G$.

(1) and (3) simply express the equivalence of the martingale problem for a given operator to the problem of finding a process with the given operator as the generator. We prove these first and then prove (2). For $f \in C_0^\infty(G)$, we have,

$$(d/du)E_{x,r}(f(Y^T(u))) = \lim_{h\to 0}(1/h)E_{x,r}(E_{Y^T(u),u}(f(Y^T(u+h)) - f(Y^T(u))))$$
$$= E_{x,r}(L_u^T f(Y^T(u))).$$

Thus,

$$E_{x,r}(f(Y^T(u))) = f(x) + \int_r^u E_{x,r}(L_s^T f(Y^T(s)))\, ds, \tag{2.6}$$

for $0 \le r \le u \le t$ and $x \in G$. Hence for $0 \le r \le v \le u \le t$,

$$E_{x,r}\left(f(Y^T(u)) - \int_r^u L_s^T f(Y^T(s))\, ds \mid \mathscr{F}_v\right)$$
$$= f(Y^T(v)) - \int_r^v L_s^T f(Y^T(s))\, ds$$
$$+ E_{Y^T(v),v}(f(Y^T(u)) - f(Y^T(v))) - \int_v^u E_{Y^T(v),v}(L_s^T f(Y^T(s)))\, ds$$
$$= f(Y^T(v)) - \int_r^v L_s^T f(Y^T(s))\, ds, \quad \text{by (2.6).}$$

This proves (1). We now prove (3). Since $f(Y(u)) - \int_0^u \mathscr{L}_G f(Y(s))\,ds$ is a Q_x^t martingale for $0 \le u \le t$, $x \in G$ and all $f \in C_0^\infty(G)$, we have,

$$E_{x,u}(f(Y(u+h)) - f(x)) = \int_u^{u+h} E_{x,u}(\mathscr{L}_G f(Y(s)))\,ds$$

and hence,

$$\lim_{h\to 0}(1/h)E_{x,u}(f(Y(u+h)) - f(x)) = \mathscr{L}_G f(x)$$

for $0 \le u < t$, $x \in G$ and all $f \in C_0^\infty(G)$.

Finally, we must show that (2) holds. Let $\{G_n\}_{n=1}^\infty$ be an increasing sequence of open sets with $G = \cup_{n=1}^\infty G_n$. Let $b_n(x)$ be bounded on R^d with $b_n = b + (a\nabla\varphi_0)/\varphi_0$ on G_n and consider the martingale problem (on R^d) for a and b_n. Let $L^n = \frac{1}{2}\nabla \cdot a\nabla + b_n\nabla$. Note that $L^n = \mathscr{L}_G$ on G_n. From SV-I, there exists a unique solution to the martingale problem for L^n starting from time 0. Call the solution $\{P_x^n\}$, $x \in R^d$, and denote the sample paths by $X_n(\cdot)$. In order to show that there exists a unique solution to the martingale problem on G up to time t for $\mathscr{L}_G$, we use SV-II, which carries over to our framework. Let $\tau_n = \inf\{t \ge 0: X_n(t) \notin G_n\}$. We need to show that

$$\lim_{n\to\infty} P_x^n(\tau_n < s) = 0 \quad \text{for all} \quad 0 \le s \le t \quad \text{and} \quad x \in G. \tag{2.7}$$

Using $L\varphi_0 = -\lambda_0\varphi_0$, one can check that $\mathscr{L}_G(1/\varphi_0) = \lambda_0(1/\varphi_0)$. Thus we have for $0 \le s \le t$, $x \in G$ and $n \ge 1$,

$$\begin{aligned}
E_x\varphi_0^{-1}(X_n(s \wedge \tau_n)) &= \varphi_0^{-1}(x) + E_x \int_0^{s\wedge\tau_n} L^n\varphi_0^{-1}(X_n(u))\,du \\
&= \varphi_0^{-1}(x) + E_x \int_0^{s\wedge\tau_n} \mathscr{L}_G\varphi_0^{-1}(X_n(u))\,du \\
&= \varphi_0^{-1}(x) + \lambda_0 E_x \int_0^{s\wedge\tau_n} \varphi_0^{-1}(X_n(u))\,du \\
&= \varphi_0^{-1}(x) + \lambda_0 E_x \int_0^{s\wedge\tau_n} \varphi_0^{-1}(X_n(u\wedge\tau_n))\,du \\
&\le \varphi_0^{-1}(x) + \lambda_0 E_x \int_0^{s} \varphi_0^{-1}(X_n(u\wedge\tau_n))\,du \\
&= \varphi_0^{-1}(x) + \lambda_0 \int_0^{s} E_x\varphi_0^{-1}(X_n(u\wedge\tau_n))\,du
\end{aligned}$$

By Gronwall's inequality, this gives us

$$E_x\varphi_0^{-1}(X_n(s\wedge\tau_n)) \le \varphi_0^{-1}(x)\exp(\lambda_0 s)$$

and

$$\lim_{n\to\infty} E_x\varphi_0^{-1}(X_n(s\wedge\tau_n)) \le \varphi_0^{-1}(x)\exp(\lambda_0 s). \tag{2.8}$$

Since $\varphi_0 > 0$ on G and $\varphi_0 = 0$ on ∂G, (2.8) implies that $\lim_{n\to\infty} P_x^n(\tau_n < s) = 0$. This completes the proof of (2).

To complete the proof of our theorem, we must identify $\mathscr{L}_G$. That is, we must show that $(\nabla\varphi_0)/\varphi_0 = (\nabla g_0)/g_0 - \nabla h_{g_0}$, or equivalently, $\varphi_0 = g_0\exp(-h_{g_0})$, where g_0 is the square root of the density of μ_0 and μ_0 is a certain probability measure at which $\inf_{\{\mu\in\mathscr{P}(R^d):\mathrm{supp}\mu\subset\bar{G}\}} I(\mu)$ is attained (h_{g_0} was defined in (1.9)). By Proposition 3,

$$\lim_{t\to\infty}(1/t)\log P_x(\tau_G > t) = -\inf_{\{\mu\in\mathscr{P}(R^d):\,\mathrm{supp}\,\mu\subset\bar{G}\}} I\mu)$$

and by Hypothesis 2, $\lim_{t\to\infty}(1/t)\log P_x(\tau_G > t) = -\lambda_0$. Hence $\lambda_0 = \inf_{\{\mu\in\mathscr{P}(R^d):\mathrm{supp}\mu\subset\bar{G}\}} \mathrm{I}(\mu)$. Define $g_0 \equiv (\varphi_0\tilde{\varphi}_0)^{1/2}$, and let μ_0 be the probability measure with density g_0^2. Define $W \equiv ½ \log(\tilde{\varphi}_0/\varphi_0)$. One can check that W satisfies (1.9) for $g = g_0$; hence in fact, $h_{g_0} = W = ½ \log(\tilde{\varphi}_0/\varphi_0)$. Plugging $g_0 = (\varphi_0\tilde{\varphi}_0)^{1/2}$, $h_{g_0} = ½ \log(\hat{\varphi}_0/\varphi_0)$ into (1.8), one can check that $I(\mu_0) = \lambda_0$. Hence $\inf_{\{\mu\in\mathscr{P}(R^d):\mathrm{supp}\mu\in\bar{G}\}} I(\mu)$ is attained at μ_0. This completes the proof of the theorem.

3. Examples.

EXAMPLE 1. Consider Brownian motion with a constant drift, b, in one dimension. The generator is $L = ½(d^2/dx^2) + b(d/dx)$. Let $G = (-c, c)$. The operator $-L$ with Dirichlet conditions at $(-c, c)$ has as its smallest eigenvalue $\lambda_0 = (\pi^2/8c^2) + (b^2/2)$. The corresponding nonnegative eigenfunction is $\varphi_0 = e^{-bx}\cos(\pi/2c)x$. The adjoint operator $-\tilde{L} = -((d^2/dx^2) - b(d/dx))$ also has $\lambda_0 = (\pi^2/8c^2) + (b^2/2)$ as its smallest eigenvalue. The corresponding eigenfunction (normalized so that

$$\int_{-c}^{c} \varphi_0\tilde{\varphi}_0(X)\,dx = 1) \quad \text{is} \quad \tilde{\varphi}_0 = \frac{e^{bx}}{c}\cos\frac{\pi}{2c}x.$$

Thus, one-dimensional Brownian motion with a constant drift, conditioned to remain in $(-c, c)$ up to time T, converges as $T\to\infty$ to the diffusion which never leaves $(-c, c)$ with generator

$$\begin{aligned} L_c = L + \frac{\varphi'}{\varphi}\frac{d}{dx} &= \frac{1}{2}\frac{d^2}{dx^2} + b\frac{d}{dx} + \left(-\frac{\pi}{2c}\tan\frac{\pi}{2c}x - b\right)\frac{d}{dx} \\ &= \frac{1}{2}\frac{d^2}{dx^2} - \frac{\pi}{2c}\tan\frac{\pi}{2c}\frac{d}{dx}. \end{aligned}$$

The invariant measure for the process has density $\varphi_0\tilde{\varphi}_0(X) = (1/c)\cos^2(\pi/2c)x$, $-c < x < c$. In particular, note that the limiting process does not depend on the original constant drift b.

EXAMPLE 2. Consider three-dimensional Brownian motion conditioned to remain in the disc, $G = \{x \in R^3\colon |x| < c\}$. The generator, $L = ½\Delta$, with the Dirichlet condition on $|x| = c$ is self adjoint. Hence we may pick $\varphi_0 = \tilde{\varphi}_0$ with $\int_G \varphi_0^2(X)\,dx = 1$. The smallest eigenvalue is $\lambda_0 = (\pi^2/2c^2)$ and $\varphi_0 = (\sqrt{2\pi c}\, r)^{-1}\sin(\pi r/c)$, $r = (x^2 + y^2 + z^2)^{1/2}$. Thus the process conditioned to remain in G up to time T converges as $T \to \infty$ to a limiting diffusion which never leaves $|x| < c$, with generator

$$L_c = \frac{1}{2}\Delta + \frac{\nabla\varphi_0}{\varphi_0}\nabla = \frac{1}{2}\frac{d^2}{dr^2} + \frac{1}{r^2}\frac{d^2}{d\theta^2} + \frac{\pi}{c}\cot\frac{\pi}{c}r\frac{d}{dr}.$$

The invariant measure has density $\varphi_0^2 = (1/2\pi c r^2)\sin^2(\pi r/c)$. Note that the radial process, $r(t) = (X(t) + Y(t) + Z(t))^{1/2}$, corresponding to the limiting process, has the same behavior near $r = 0$ as near $r = c$. More precisely, its behavior is symmetric about $r = c/2$. The original radial process has generator $L = \frac{1}{2}(d^2/dr^2) + (1/r)(d/dr)$. Thus the repulsion from the origin for the original process is given by $(1/r)(d/dr)$. For the conditioned process, the repulsion from the origin is given by

$$\frac{1}{r}\left(1 - \frac{\pi^2}{3c^2}r^2 + O(r^4)\right)\frac{d}{dr}$$

which is slightly smaller. This is because the conditioned process must "think twice" before repelling from $r = 0$ since the process is not allowed to reach $r = c$.

EXAMPLE 3. Consider standard Brownian motion, $X(t)$, with generator $L = \frac{1}{2}(d^2/dx^2)$. Let $G = (0, \infty)$. This case is not covered by our theorem since G is not compact, and indeed, the spectrum comes all the way down to zero and there exists no minimum eigenvalue and corresponding eigenfunction. However, this example shows that a limiting process can exist nonetheless, although it will not be positive recurrent. Lemma 2.1 is still valid in the noncompact case and we have

$$P_x(\tau_G > t) = P_0(\sup_{0\le s\le t} X(s) < x) = 2\int_0^x \frac{\exp(-v^2/2t)}{\sqrt{2\pi t}}\,dv.$$

Thus the conditioned process, $Y^{T,t}(s)$, $0 \le s \le t$, has generator

$$L_s^T = \frac{1}{2}\frac{d^2}{dx^2} + \left(\exp\left(\frac{-x^2}{2(T-s)}\right) \Big/ \int_0^x \exp\left(\frac{-v^2}{2(T-s)}\right)dv\right)\frac{d}{dx}.$$

Formally,

$$L_s^T \to \frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} \equiv \mathscr{L}_G, \quad \text{as} \quad T \to \infty.$$

The Stroock-Varadhan theory goes thru here. For the theory did not require compactness, and the only place we used the fact that

$$\lim_{T\to\infty}(\nabla P_x(\tau_G > T - s)/P_x(\tau_G > T - s))$$

has the specific form $(\nabla\varphi_0/\varphi_0)$, where $-L\varphi_0 = \lambda_0\varphi_0$, was in verifying statement 2 (page 373)—that a (necessarily unique) solution to the martingale problem existed for the limiting generator $\mathscr{L}_G$. That causes no problem in the present case. The generator, $\mathscr{L}_G = \frac{1}{2}(d^2/dx^2) + (1/x)(d/dx)$, is a familiar Bessel process—the radial process of a standard three-dimensional Brownian motion. Thus, one-dimensional Brownian motion starting from $x > 0$, conditioned to remain positive up to time T, converges as $T \to \infty$ to the radial process of three-dimensional Brownian motion. In particular, this limiting process is transient—one-dimensional Brownian motion conditioned to remain positive for all time runs off to infinity.

Appendix. A sufficient condition for Hypothesis 1 to hold is that there exists a $C^2(G)$ solution to the parabolic equation with discontinuous boundary data,

$$(*)\qquad u_t = Lu,\quad u(x,0) = 1 \quad \text{for } x \in G,\quad u(y,t) = 0 \quad \text{for } y \in \partial G \text{ and } t > 0.$$

For then an application of Ito's formula gives us $P_x(\tau_G > t) = u(x, t)$.

Hypotheses 2 and 3 are similar and, in fact, may be equivalent under sufficient smoothness assumptions. Our operator, $-L$, with the Dirichlet boundary condition on ∂G, is positive and has a compact resolvent. Since the resolvent is compact, the spectrum will consist only of complex eigenvalues clustering at infinity, [1].

In the case that $a^{-1}b(= \nabla Q)$ is a gradient function, then L and $\tilde{L}$ are self adjoint with respect to the densities e^{2Q} and e^{-2Q} respectively. Hence, there exists complete orthonormal sequences (with respect to the densities e^{2Q} and e^{-2Q}) of eigenfunctions, $\{\varphi_n\}$ and $\{\tilde{\varphi}_n\}$, for $-L$ and $-\tilde{L}$, with corresponding eigenvalues $\{\lambda_n\}$ and $[\tilde{\lambda}_n\}$. A $C^2(G)$ solution to $(*)$ is given by $u(x, t) = \sum_{n=0}^{\infty} c_n \varphi_n(x) e^{-\lambda_n t}$, *with* $c_n = \int_G \varphi_n e^{2Q}\, dx$. Since $\varphi_0 > 0$, we have $c_0 > 0$. Thus $P_x(\tau_G > t) = u(x, t) = \sum_{n=0}^{\infty} c_n \varphi_n(x) e^{-\lambda_n t}$ satisfies Hypotheses 1 and 2.

Now assume $Q \in C^2(\bar{G})$. Then one can check that $\varphi_n = \Psi_n e^{-Q}$, $\tilde{\varphi}_n = \tilde{c}\, \Psi_n e^{Q}$ and $\lambda_n = \tilde{\lambda}_n = \gamma_n$ where $\tilde{c}$ is a normalizing constant and Ψ_n with $\int_G \Psi_n^2\, dx = 1$ satisfies

$$-\tfrac{1}{2}\nabla \cdot a\nabla\Psi_n + \tfrac{1}{2}(\nabla Q a \nabla Q + \nabla \cdot (a\nabla Q))\Psi_n = \gamma_n \Psi_n, \quad \Psi_n = 0 \quad \text{on} \quad \partial G.$$

The solution to

$$(**)\qquad u_t = Lu,\quad u(x,0) = f(x) \quad \text{for } x \in G,\quad u(y,t) = 0 \quad \text{for } y \in \partial G \text{ and } t > 0$$

is $u_f(x, t) = \sum_{n=0}^{\infty} d_n \exp(-\lambda_n t)\varphi_n(x)$, with $d_n = \int_G f \varphi_n e^{2Q}\, dx$. Since $E_x(f(X(t)), \tau_G > t) = u_f(x, t)$, we see that $P_x(X(t) \in dy, \tau_G > t)$ has density

$$\sum_{n=0}^{\infty} \exp(-\lambda_n t)\varphi_n(x)\varphi_n(y)\exp(2Q(y)) = \sum_{n=0}^{\infty} \exp(-\lambda_n t)\varphi_n(x)\, \frac{\tilde{\varphi}_n(y)}{\tilde{c}}.$$

Thus Hypothesis 3 is satisfied.

If $a^{-1}b$ is not a gradient, the operator $-L$ with the Dirichlet boundary condition on ∂G is not self adjoint and a complete set of eigenvalues may not exist. Even if a complete set does exist, it may not be possible to expand functions in convergent eigenfunction expansions since the eigenfunctions are not orthogonal. Such a convergent eigenfunction expansion is a sufficient condition for the 3 hypotheses to hold. In [7, page 543], it is claimed that such an expansion does exist in $d = 2$ dimensions.

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