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ROTATIONAL REPRESENTATIONS OF STOCHASTIC MATRICES

BY JOHN HAIGH[1]

University of Guelph

Let (p_{ij}) be the matrix of a recurrent Markov chain with stationary vector $\pi > 0$, and let $\{J_i\}$ be a partition of the unit circle into sets $J_1, \cdots, J_n$, $m(J_i) = \pi_i$, where m is Lebesgue measure. Suppose f_t defines rotation through distance t. The conditions under which p_{ij} can be written as $m(f_t(J_i) \cap J_j)/m(J_i)$ for all i and j, where each J_i is the union of at most $b(n)$ arcs, have recently been examined by Steve Alpern and Joel Cohen. Cohen conjectured that $b(n) = n - 1$, and proved $b(2) = 1$. Alpern proved that Cohen's conjecture was false for n sufficiently large, and gave bounds for $b(n)$. We give a construction that shows that $b(3) = 2$, and prove that $b(n)$ is nondecreasing.

1. Introduction. Let m denote Lebesgue measure on $X = [0, 1)$, and let $f_t(x) = x + t (\text{mod } 1)$ be the shift transformation on X. Given any partition $J = \{J_i: i = 1, 2, \cdots, n, m(J_i) > 0\}$ of X, Cohen (1981) pointed out that if

$$p_{ij} = m(f_t(J_i) \cap J_j)/m(J_i) \tag{1}$$

then $P = (p_{ij})$ is the matrix of a recurrent Markov chain on $\{1, 2, \cdots, n\}$, with stationary vector $\pi = (\pi_1, \cdots, \pi_n)$, $\pi_i = m(J_i)$. Conversely, given a stochastic matrix P, Cohen considered whether a representation of P in the form (1) exists, when each J_i is the union of at most $b(n)$ intervals. He conjectured that, if P is irreducible, such a representation always exists, and that $b(n) = n - 1$; proved this conjecture when $n = 2$; and showed that, for any n, there was some irreducible matrix P for which $b(n) \geq n - 1$.

Alpern (1983) showed that it was possible to extend the class of matrices from irreducible to recurrent, and used an elegant construction to show that, given any recurrent P, such a representation is possible, with $t = 1/t_0$, $t_0 =$ l.c.m. $(1, 2, \cdots, n)$, and $b(n) \leq c(n)$, where

$$c(n) = t_0(n^2 - n + 1) \leq \exp(\beta n) \tag{2}$$

for some constant β. In itself, this does not disprove Cohen's conjecture, that $b(n) = n - 1$, for some more parsimonious construction might exist. But Alpern also constructed particular stochastic matrices that imply that the conjecture is false, at least for $n = 6$ and $n \geq 8$.

We shall prove that, when P is recurrent, then $b(3) = 2$. We shall also show that, in contrast to the 2×2 case, essentially different representations of P

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might exist when $n \geq 3$. Finally, we show that $b(n) \geq b(n-1)$, and use Alpern's construction to make an explicit conjecture about $b(n)$.

2. The case $n = 3$. Given any 3×3 recurrent matrix P with stationary vector π, we have

$$\pi_i = \pi_1 p_{1i} + \pi_2 p_{2i} + \pi_3 p_{3i} \quad i = 1, 2, 3 \tag{3}$$

$$\pi_i = \pi_i p_{i1} + \pi_i p_{i2} + \pi_i p_{i3} \quad i = 1, 2, 3. \tag{4}$$

Equating the two expressions for π_i in (3) and (4), we have

$$\pi_2 p_{21} - \pi_1 p_{12} = \pi_1 p_{13} - \pi_3 p_{31} = \pi_3 p_{32} - \pi_2 p_{23} = v \quad \text{(say)}. \tag{5}$$

It turns out to be more convenient to define $f_t(x) = x + t \pmod 2$, and work on the interval $[0, 2)$, so that $m(J_i) = 2\pi_i$. We now specify the division points $x_1, x_2, \cdots, x_5$, that partition $[0, 2)$ into the six intervals in the order shown, by fixing the lengths of these intervals.

```
              I11        I22       I31       I12       I21       I32
         |------------|-------|-------|-------|-------|-------)
      x0 = 0          x1      x2      x3      x4      x5      x6 = 2 ≡ 0
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We choose

$$m(I_{i1}) = 2\pi_i - \pi_i p_{ii}, \quad m(I_{i2}) = \pi_i p_{ii} \tag{6}$$

and we define $t = 1 - v$ (v as in (5)). Then, writing $f_t(x_i) = y_i$, we can easily calculate the values of $y_0, \cdots, y_5$, and verify that these points are juxtaposed with $x_0, \cdots, x_6$ as shown

```
            ↓     ↓                 ↓     ↓                  ↓     ↓
  |---------------------|------|---------------|------|---------------|-------)
 0=x0      y3    y4   x1     x2  y5      y0  x3     x4  y1      y2  x5      x6 = 2
```

(i.e., that $x_0 \leq y_3$, $y_4 \leq x_1$ etc.) and that

$$\begin{aligned} x_3 - y_0 &= 2\pi_1 p_{13} & y_1 - x_4 &= 2\pi_1 p_{12} \\ x_1 - y_4 &= 2\pi_2 p_{21} & y_5 - x_2 &= 2\pi_2 p_{23} \\ x_5 - y_2 &= 2\pi_3 p_{32} & y_3 - x_0 &= 2\pi_3 p_{31}. \end{aligned} \tag{7}$$

Thus, if $J_i = I_{i1} \cup I_{i2}$, clearly $m(J_i) = 2\pi_i$, and relations (7) show that (1) holds for $\{p_{ij}: i \neq j\}$; but, since the sets $\{J_i\}$ partition $[0, 2)$, (1) also holds for $\{p_{ii}\}$. Hence any 3×3 recurrent matrix has a representation of the form (1), with each J_i being the union of at most 2 intervals.

Cohen also asked whether such a representation, using the minimum number of intervals, was unique up to cyclic permutations of the sets $\{I_{ij}\}$. The answer is that it may not be unique. Suppose $\pi_1 p_{11} \geq |v|$ and $\pi_2 p_{22} \geq |v|$ (this is clearly possible, since it can happen that $v = 0$), and choose

$$m(I_{11}) = \pi_1 p_{11} + \pi_1 p_{12} + \pi_2 p_{21}$$
$$m(I_{21}) = \pi_2 p_{22} + \pi_2 p_{23} + \pi_3 p_{32}$$
$$m(I_{31}) = \pi_3 p_{33}$$

and

$$m(I_{i2}) = 2\pi_i - m(I_{i1}) \quad \text{for} \quad i = 1, 2, 3.$$

Then, placing the intervals on $[0, 2)$ in the order I_{11}, I_{21}, I_{31}, I_{12}, I_{22}, I_{32}, and choosing $t = 1 - v$, it is again straightforward to verify that the division points in these intervals are mapped in the order

```
          ↓      ↓          ↓      ↓      ↓   ↓
|--------------|---------|------|---------|---------|---------|----------------)
x0 = 0    y4  x1  y5  x2    x3  y0  x4  y1  x5  y2    y3  x6 = 2
```

and that results similar to (7), that guarantee (1), hold. If, of course, $\pi_i p_{ii} \geq |v|$ for each of $i = 1, 2, 3$, there are two more, different, decompositions of J_1, J_2, J_3 into at most two intervals, that will yield (1).

Notice that, in all the above examples, we had $t = 1 - v$ or $t = 1 + v$. These cases are not essentially different, as, even when $v \neq 0$, the decompositions can be paired off, with each pair corresponding to an ordering of six intervals with a "forward" shift $1 - v$, and a reverse ordering of the same intervals, with a "backward" shift $1 + v$.

Cohen (1981), who worked chiefly with the 2×2 case, was not much concerned with the one case in which P is recurrent, but reducible, namely the identity matrix, since this is trivial. However, both Alpern's (1983) construction, and those given above, work equally well with reducible or irreducible matrices, provided only that they are recurrent.

3. The cases $n \geq 4$. The proof that $b(3) = 2$ relies on being able to write π_i in the two different ways (3) and (4) which, via (5), lead to the value of t and the decomposition (6). There is no obvious parallel of (5) when $n \geq 4$.

Alpern's construction leads, via (2), to be upper bound $c(3) = 42$. An examination of the details shows that the upper bound in the 3×3 case can actually be reduced from 42 to about 10, but it also becomes apparent that this construction is rather extravagant in the number of intervals it can generate, since we now know that $b(3) = 2$.

The basis of Cohen's conjecture, that $b(n) = n - 1$, was to use a matrix that is very close to the identity matrix. In his Lemma 2, which disproves this conjecture, Alpern used a matrix corresponding to a reducible Markov chain with cyclic classes of sizes $1, c_1, c_2, \cdots, c_r$. Thus, if we partition any positive integer m as $m = c_1 + c_2 + \cdots + c_r$, and define

(8) $$H(m) = \text{Max}\{\text{l.c.m.}\ (c_1, c_2, \cdots, c_r)\text{: all partitions}\},$$

Alpern's result implies that $b(n) \geq H(n-1)$. We offer the conjecture that $b(n) = H(n-1)$. Some values are

n	2	3	4	5	6	7	8	9	10	11	12	13
$H(n-1)$	1	2	3	4	6	6	12	15	20	30	30	60

The permutation matrix (p_{ij}) used to prove $b(n) \geq H(n-1)$ is recurrent, but reducible. From it, define $Q = (q_{ij})$, where $q_{ij} = \varepsilon$ when $p_{ij} = 0$, and $q_{ij} = 1 - \varepsilon(n-1)$ when $p_{ij} = 1$, where ε is strictly positive, but very small. Only minor changes in the argument used for (p_{ij}) are needed to show that $b(n) \geq H(n-1)$ for the irreducible matrix Q.

We now prove that $b(n) \geq b(n-1)$. Note that, if our conjecture that $b(n) = H(n-1)$ is true, the table of values of $H(n-1)$ shows that we do not have the strict inequality $b(n) > b(n-1)$, for all values of n.

PROOF. Let P_0 be some $(n-1) \times (n-1)$ recurrent matrix in which any rotational representation requires some J_i to contain $b(n-1)$ intervals. Let P_1 be the $n \times n$ matrix whose principal submatrix is P_0, and $p_{nn} = 1$. Suppose $m(J_n) = a$, and that P_1 has a representation in which every J_i is the union of at most r intervals. Since $f_t(J_n) = J_n$, we see that the intervals in J_n can be split into families, each family consisting of equally sized intervals whose left endpoints are a multiple of $t = p/k$ apart.

Remove this J_n from $[0, 1)$, coalesce the remaining intervals, define $t = p(1-a)/k$; this gives a representation of P_0 on $[0, 1-a)$, using at most r intervals, so $r \geq b(n-1)$; but $b(n) \geq r$, so $b(n) \geq b(n-1)$. □

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SCHOOL OF MATHEMATICAL AND PHYSICAL SCIENCES
THE UNIVERSITY OF SUSSEX
MATHEMATICS AND PHYSICS BUILDING
FALMER, BRIGHTON BN1 9QH
UNITED KINGDOM