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THE ROBBINS–SIEGMUND SERIES CRITERION FOR PARTIAL MAXIMA

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Let $X, X_1, X_2, \ldots$ be i.i.d. random variables and let $M_n = \max_{1 \le j \le n} X_j$. For each nondecreasing real sequence $\{b_n\}$ such that $P(X > b_n) \to 0$ and $P(M_n \le b_n) \to 0$ we show that $P(M_n \le b_n \text{ i.o.}) = 1$ if and only if $\sum_n P(X > b_n)\exp\{-nP(X > b_n)\} = \infty$. The restrictions on the b_n's can be removed.

Let $X, X_1, X_2, \ldots$ be a sequence of independent, identically distributed (i.i.d.) random variables and put $M_n = \max_{1 \le j \le n} X_j$. For each real sequence $\{b_n\}$, $P(M_n \le b_n \text{ i.o.})$ assumes a value of either zero or one. In Klass (1984, Remark 2) the following series criterion was presented, identifying which is the case:

THEOREM 1. *Suppose b_n is nondecreasing, $P(X > b_n) \to 0$, and $nP(X > b_n) \to \infty$. Let $n_1 = 1$ and, having constructed $n_1, \ldots, n_k$, let $n_{k+1} = \min\{j > n_k: (j - n_k)P(X > b_{n_k}) \ge 1\}$. Then*

$$(1) \qquad P(M_n \le b_n \text{ i.o.}) = \begin{cases} 1 & \text{if } \sum_{k=1}^{\infty} \exp\{-n_k P(X > b_{n_k})\} = \infty, \\ 0 & \text{if } \sum_{k=1}^{\infty} \exp\{-n_k P(X > b_{n_k})\} < \infty. \end{cases}$$

Moreover, if $P(X > b_n) \to c > 0$ then obviously $P(M_n \le b_n \text{ i.o.}) = 0$; while if $\liminf_{n \to \infty} nP(X > b_n) < \infty$ then $P(M_n \le b_n \text{ i.o.}) = 1$.

The assumption that $\{b_n\}$ be nondecreasing also results in no loss of generality. For a discussion of this detail, see the above paper.

Harry Kesten (private communication) expressed the idea that the complicated series expression in (1) could be simplified. In fact, we obtain a simplification in the form of a series expression which does not involve construction of a monitoring sequence of check points $\{n_k\}$. The series is the same as that used by Robbins and Siegmund (1970). An analogous expression applies to the evaluation of $\liminf_{n \to \infty}(X_1 + \cdots + X_n)/a_n$ whenever $\{X_j\}$ are i.i.d. and nonnegative and $a_n \to \infty$.

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Theorem 2. *Under the assumptions of Theorem* 1,

$$(2)\quad P(M_n \le b_n \text{ i.o.}) = \begin{cases} 1 & \text{if } \sum_{n=1}^{\infty} P(X > b_n)\exp\{-nP(X > b_n)\} = \infty, \\ 0 & \text{if } \sum_{n=1}^{\infty} P(X > b_n)\exp\{-nP(X > b_n)\} < \infty. \end{cases}$$

Proof. Since $nP(X > b_n) \to \infty$ and $(n_{k+1} - n_k)P(X > b_{n_k}) \to 1$, it follows that $n_{k+1}/n_k \to 1$. Hence there exists k_0 such that $n_j P(X > b_{n_{j+1}}) \ge 1$ for $j \ge k_0$. Note also that $y\exp(-jy)$ decreases for $y \ge j^{-1}$. Hence for all $k \ge k_0$,

$$\begin{aligned} \sum_{n_k \le j < n_{k+1}} & P(X > b_j)\exp\{-jP(X > b_j)\} \\ & \ge \sum_{n_k \le j < n_{k+1}} P(X > b_{n_k})\exp\{-jP(X > b_{n_k})\} \\ & \ge e^{-1}(n_{k+1} - n_k)P(X > b_{n_k})\exp\{-n_k P(X > b_{n_k})\} \\ & \ge e^{-1}\exp\{-n_k P(X > b_{n_k})\}. \end{aligned}$$

In the reverse direction,

$$\begin{aligned} \sum_{n_k \le j < n_{k+1}} & P(X > b_j)\exp\{-jP(X > b_j)\} \\ & \le \sum_{n_k \le j < n_{k+1}} P(X > b_{n_{k+1}})\exp\{-jP(X > b_{n_{k+1}})\} \\ & \le e^{(n_{k+1} - n_k)P(X > b_{n_{k+1}})}(n_{k+1} - n_k)P(X > b_{n_{k+1}})\exp\{-n_{k+1}P(X > b_{n_{k+1}})\} \\ & \le 2e^2\exp\{-n_{k+1}P(X > b_{n_{k+1}})\}. \end{aligned}$$

Hence the two series

$$\sum_n P(X > b_n)\exp\{-nP(X > b_n)\}$$

and

$$\sum_k \exp\{-n_k P(X > b_{n_k})\}$$

converge or diverge together. Now Theorem 2 follows from Theorem 1.

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