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ON COMPLETE CONVERGENCE IN THE LAW OF LARGE NUMBERS FOR SUBSEQUENCES

BY ALLAN GUT

Uppsala University

Shorter and more elementary proofs of some results of Asmussen and Kurtz are given. We determine first those subsequences for which mean zero is the necessary and sufficient requirement for complete convergence and then give integrability conditions in terms of the growth of the subsequences in the case when a moment of order greater than one exists.

1. Introduction. A sequence $\{U_n\}_{n=1}^{\infty}$ of random variables is said to *converge completely* to the constant c if

$$\sum_{n=1}^{\infty} P(|U_n - c| > \varepsilon) < \infty \quad \text{for all } \varepsilon > 0.$$

This definition was introduced in Hsu and Robbins (1947), where it was shown that the sequence of arithmetic means of i.i.d. random variables converges completely to the expected value of the summands provided their variance is finite. The converse was proved by Erdös (1949, 1950).

This result has been generalized in various ways; in particular, motivated by the study of supercritical branching processes, Asmussen and Kurtz (1980) solved the Hsu–Robbins–Erdös problem for *subsequences*, i.e. complete convergence for subsequences of the sequence of arithmetic means of i.i.d. random variables is considered. In this context we also refer to Athreya and Kaplan (1976), Proposition 1; Athreya and Kaplan (1978), Lemmas 4.2 and 4.3; and Nerman (1981), Proposition 4.1.

The purpose of this note is to present shorter and simpler proofs of the results of Asmussen and Kurtz (1980). Their proofs are very technical and some of the steps depend in turn upon technical results from Kurtz (1972). Further, their assumptions are expressed in terms of a "smoothing function," whereas our assumptions will be more directly related to the subsequence.

As mentioned above, finite variance is required for complete convergence. However, since the law of large numbers really is a first-moment problem we characterize those subsequences for which finite expectation is necessary and sufficient for complete convergence. This is done in Section 3. In Section 4 such sequences are considered for which the summands have a finite moment of some order > 1 (and ≤ 2).

Since we do not cover all possible subsequences with this approach, Section 5 contains some remarks on "missing" sequences as well as some other comments.

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2. Preliminaries. Throughout, X and $\{X_n\}_{n=1}^{\infty}$ are i.i.d. random variables and $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$. Further, $\{n_k\}_{k=1}^{\infty}$ is a strictly increasing subsequence of the positive integers, whose inverse, ψ, is defined by

$$\psi(x) = \text{Card}\{k; n_k \leq x\}, \qquad x > 0 \quad \text{and} \quad \psi(0) = 0.$$

Set $M(x) = \sum_{k=1}^{[x]} n_k$, $x > 0$, $\beta_k = n_{k+1}/M(k)$ and $\gamma_k = n_k/n_{k+1}$, $k = 1, 2, \dots$.

LEMMA 2.1. *For any random variable* X,

(*i*)
$$\sum_{k=1}^{\infty} n_k P(|X| \geq n_k) = EM(\psi(|X|)).$$

(*ii*) *If*

(2.1)
$$0 < \liminf_{k \to \infty} \beta_k \leq \limsup_{k \to \infty} \beta_k < \infty$$

then

$$EM(\psi(|X|)) < \infty \Leftrightarrow E|X| < \infty.$$

PROOF. Since $\{|X| \geq n_k\} = \{\psi(|X|) \geq k\}$ (i) follows by partial summation. Further, (2.1) implies that there exist $C_1 > 0$ and $C_2 < \infty$ such that $C_1 x \leq M(\psi(x)) \leq C_2 x$, from which (ii) follows. □

LEMMA 2.2.

(*i*) *If* $\sum_{k=1}^{\infty} P(|S_{n_k}| \geq n_k \varepsilon) < \infty$ *for all* $\varepsilon > 0$, *then*

$$EM(\psi(|X|)) < \infty.$$

If, in addition,

(2.2)
$$\limsup_{k \to \infty} \beta_k < \infty,$$

then $E|X| < \infty$ *and* $EX = 0$.

(*ii*) *If* $\sum_{k=1}^{\infty} P(|S_{n_k}| \geq M(k)\varepsilon) < \infty$ *for all* $\varepsilon > 0$ *and* (2.2) *holds, then*

$$E|X| < \infty \quad \textit{and} \quad EX = 0.$$

PROOF OF (i). To prove that $EM(\psi(|X|)) < \infty$ one can use an argument due to Erdös (1949) (see also Katz (1963), p. 317; Baum and Katz (1965), p. 114; and Gut (1983), Section 7).

If (2.2) also holds, then there exists $C < \infty$, such that $x \leq CM(\psi(x))$, from which it follows that $E|X| < \infty$. Finally, the law of large numbers yields $EX = 0$.

The proof of (ii) is similar. □

3. The case $0 < \liminf(n_{k+1}/M(k)) \leq \limsup(n_{k+1}/M(k)) < \infty$. It follows from Lemma 2.2 (i) that $EM(\psi(|X|)) < \infty$ is always a necessary condition for complete convergence. For $E|X| < \infty$ to be the correct integrability condition one would need something like $M(\psi(x)) < Cx$ for some finite constant C. The following theorem emerges.

THEOREM 3.1.

(*i*) *Suppose that*

$$\liminf_{k \to \infty} \beta_k > 0. \tag{3.1}$$

(a) *If* $E|X| < \infty$ *and* $EX = 0$, *then*

$$\sum_{k=1}^{\infty} P(|S_{n_k}| \geq n_k \varepsilon) < \infty \quad \textit{for all } \varepsilon > 0. \tag{3.2}$$

(b) *If* (3.2) *holds, then* $EM(\psi(|X|)) < \infty$.

(*ii*) *If, moreover,*

$$\limsup_{k \to \infty} \beta_k < \infty, \tag{3.3}$$

then $E|X| < \infty$ *and* $EX = 0$ *are necessary and sufficient for* (3.2) *to hold.*

REMARK 3.1. The following example, given to me by an associate editor, shows that (3.2) may hold even though the expected value does not exist. Let X have a symmetric distribution. A necessary and sufficient condition for the weak law of large numbers to hold is that $xP(|X| > x) \to 0$ as $x \to \infty$. Now, choose $\{n_k\}$ such that $P(|S_{n_k}| \geq n_k 2^{-k}) \leq 2^{-k}$.

REMARK 3.2. A particular case for which (3.1) holds is when $\limsup_{k \to \infty} \gamma_k < 1$, i.e. the case when there is at least geometric increase.

REMARK 3.3. The sufficiency has, for $\limsup \gamma_k < 1$, been proved in Athreya and Kaplan (1978), Lemma 4.2 (in a more general setup), by essentially using the method which is used to prove the classical strong law of large numbers (see e.g. Chung (1974), pp. 126–7).

PROOF OF THEOREM 3.1

Sufficiency. Thus, suppose that $EX = 0$ and set $T_k = S_{M(k)} - S_{M(k-1)}$. The strong law of large numbers in particular implies that $(M(k))^{-1} \cdot S_{M(k)} \to 0$ a.s. as $k \to \infty$ and hence also that

$$(M(k))^{-1} \cdot T_k \to 0 \text{ a.s.} \quad \text{as } k \to \infty, \tag{3.4}$$

which in view of (3.1) yields

$$n_k^{-1} \cdot T_k \to 0 \text{ a.s.} \quad \text{as } k \to \infty. \tag{3.5}$$

Since $\{T_k\}_{k=1}^{\infty}$ are independent random variables it follows from the Borel–Cantelli lemma and the fact that $T_k \overset{d}{=} S_{n_k}$ for all k that

$$\sum_{k=1}^{\infty} P(|S_{n_k}| \geq \varepsilon n_k) = \sum_{k=1}^{\infty} P(|T_k| \geq \varepsilon n_k) < \infty \quad \text{for all } \varepsilon > 0, \tag{3.6}$$

i.e. (3.2) holds.

Necessities. Immediate from Lemma 2.2 (i).

EXAMPLES. Sequences like $n_k = 2^k$, $n_k = k!$, $n_k = 2^{2^k}$ are covered by Theorem 3.1 (cf. also Gut (1979), p. 1060 and Gut (1983), Section 2).

4.

THEOREM 4.1. *Suppose that*

$$x^{-r} \cdot M(\psi(x)) \to \infty \quad \text{as } x \to \infty \text{ for some } r > 1. \tag{4.1}$$

The following are equivalent:

$$EM(\psi(|X|)) < \infty \quad \text{and} \quad EX = 0 \tag{4.2}$$

$$\sum_{k=1}^{\infty} P(|S_{n_k}| \geq n_k \varepsilon) < \infty \quad \text{for all } \varepsilon > 0. \tag{4.3}$$

REMARK 4.1. If $n_k = k$, then $M(k) = k(k+1)/2$, (4.1) holds and (4.2) amounts to $EX^2 < \infty$ and $EX = 0$, i.e. the classical result of Hsu–Robbins–Erdös is obtained.

REMARK 4.2. The typical case covered by Theorem 4.1 is $n_k = k^d$, where d is a fixed, positive integer.

REMARK 4.3. The restriction (4.1) is due to the fact that we have been unable to produce a proof covering *all* possible subsequences for which (3.1) fails to hold. For some further comments, see Section 5C.

The proof of Theorem 4.1 consists of a modification of the proof used in Gut (1978), p. 474, in which an iteration of an inequality of Hoffmann-Jørgensen (1974), p. 164, is fundamental.

PROOF. Suppose that (4.2) holds and that the random variables have a symmetric distribution. From (4.1) we know that $E|X|^r < \infty$ for some $r > 1$. We can, and do, choose an $r < 2$. Choose j so large that $2^j(r-1) > 1$.

By applying the above mentioned inequality, Markov's inequality and the moment inequalities of Marcinkiewicz and Zygmund (1937) (see also Gut (1978), Lemmas 2.3 and 2.4), we obtain

$$\begin{aligned} P\big(|S_{n_k}| \geq 3^j n_k \varepsilon\big) &\leq C_j n_k P(|X| \geq n_k \varepsilon) + D_j\big(P(|S_{n_k}| \geq n_k \varepsilon)\big)^{2^j} \\ &\leq C_j n_k P(|X| \geq n_k \varepsilon) + D_j\big((n_k \varepsilon)^{-r} B_r n_k E|X|^r\big)^{2^j}, \end{aligned}$$

where C_j and D_j are numerical constants depending on j only, and B_r depends on r only. Thus

$$\sum_{k=1}^{\infty} P\big(|S_{n_k}| \geq 3^j n_k \varepsilon\big) \leq C_j \sum_{k=1}^{\infty} n_k P(|X| \geq n_k \varepsilon) + D_j\big(\varepsilon^{-r} B_r E|X|^r\big)^{2^j} \sum_{k=1}^{\infty} n_k^{-2^j(r-1)}.$$

Now, the first sum on the RHS is finite by Lemma 2.1 (i) and the second sum is dominated by $\sum_{n=1}^{\infty} n^{-2^j(r-1)}$, which converges since the exponent was chosen to be > 1. In view of the arbitrariness of ε this proves (4.3) in the symmetric case. The desymmetrization is standard.

The implication (4.3) $\Rightarrow$ (4.2) follows from Lemma 2.2 (i) and the law of large numbers.

5. Some remarks

A. The integrability condition (1.4) used in Asmussen and Kurtz (1980) is $\int_0^\infty t\gamma(t)P(|X| > t)\,dt < \infty$, where $\gamma(t)$ is the derivative of the inverse of the smoothing function mentioned in Section 1. In our notation this condition can be compared with $\sum n(\psi(n) - \psi(n-1))P(|X| \geq n) = \sum n_k \cdot 1 \cdot P(|X| \geq n_k) < \infty$, which in view of Lemma 2.1 is the same as $EM(\psi(|X|)) < \infty$ in general, and $E|X| < \infty$ if (2.1) holds.

B. In the proof of the sufficiency in Theorem 3.1 the assumption (3.1) was only used in the transition from (3.4) to (3.5). If, instead, one uses independence and Borel–Cantelli on (3.4) (as was done on (3.5) in the proof) the sufficiency of the following result emerges. The necessity follows from Lemma 2.2 (ii).

THEOREM 5.1. *Suppose that*

$$E|X| < \infty \quad \text{and} \quad EX = 0. \tag{5.3}$$

Then

$$\sum_{k=1}^{\infty} P\big(|S_{n_k}| \geq \varepsilon M(k)\big) < \infty \quad \text{for all } \varepsilon > 0. \tag{5.4}$$

If (5.4) *holds and* $\limsup_{k\to\infty}\beta_k < \infty$, *then* (5.3) *holds.*

C. As mentioned in the introduction, our results do not cover *all* subsequences. In Sections 3 and 4, alternative proofs have been presented for those cases where short and simple proofs are available.

A sequence like $n_k = [e^{\sqrt{k}}]$ is, for example, not covered by the above results. For this case, one notes that $\beta_k \to 0$ as $k \to \infty$, $\psi(x) \sim (\log x)^2$, $M(x) \sim 2\sqrt{x} \cdot e^{\sqrt{x}}$ and hence that $M(\psi(x)) \sim 2x\log x$ as $x \to \infty$. Thus $E|X|\log^+|X| < \infty$ and $EX = 0$ are necessary conditions for $\sum P(|S_{n_k}| \geq n_k\varepsilon) < \infty$ for every $\varepsilon > 0$ in view of Lemma 2.2 (i). However, by following, essentially, the proof for the classical strong law, see e.g. Chung (1974), pp. 126–7, together with Hoffmann-Jørgensen's inequality (cf. Section 4) applied to $S'_{n_k} = \sum_{i=1}^{n_k} X_i I\{|X_i| < \varepsilon n_k\}$, one can also prove sufficiency for this case. (Corollary I of Asmussen and Kurtz (1980) applies to this example with (in their notation) $\psi(t) = \exp(\sqrt{t})$ and $t_k = k$).

More generally, under certain assumptions on the regularity of how the subsequence increases, together with the assumption that

$$n_i \sum_{k=i}^{\infty} (M(k))^{-1} \leq C < \infty \quad \text{for all } i, \tag{5.5}$$

a positive result can be proved by this procedure. Since such proofs do not introduce an important novelty we do not give any details.

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UPPSALA UNIVERSITY
DEPARTMENT OF MATHEMATICS
THUNBERGSVÄGEN 3
752 38 UPPSALA, SWEDEN