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ASYMPTOTIC GROWTH OF CONTROLLED GALTON–WATSON PROCESSES[1]

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The almost sure growth behavior of some time-homogeneous Markov chains is studied. They generalize the ordinary Galton–Watson processes with regard to allowing state-dependent offspring distributions and also to controlling the number of reproducing individuals by a random variable that depends on the state of the process. The main assumption is that the mean offspring per individual is nonincreasing while the state increases. These controlled Galton–Watson processes can be included in a general growth model whose divergence rate is determined. In case of processes that differ from the Galton–Watson process only by the state dependence of the offspring distributions, a necessary and sufficient moment condition for divergence with "natural" rate is obtained generalizing the $(x \log x)$ condition of Galton–Watson processes. In addition, some criteria are given when a state-dependent Galton–Watson process behaves like an ordinary supercritical Galton–Watson process.

1. Introduction. The (Bienaymé–) Galton–Watson process $(Z_n, n \in \mathbb{N})$ is a Markov chain on $\mathbb{N}$, the nonnegative integers, where

$$Z_0 \in \mathbb{N} \setminus \{0\}, \qquad Z_{n+1} = \sum_{j=1}^{Z_n} X_{n+1,j}, \qquad n \geqq 0,$$

$\{X_{n,j}, n, j \in \mathbb{N} \setminus \{0\}\}$ are i.i.d. random variables on $\mathbb{N}$. If we interpret Z_n as the number of individuals in the nth generation of a population this means: Each individual generates, independently of all the others with identical distribution, new particles. With $\mu = E(X_{11}) < \infty$ denoting the mean offspring, it is well-known that (besides the trivial case $X_{11} = 1$ a.s.) $\mu \leqq 1 \Leftrightarrow P(Z_n \to 0) = 1$, and $\mu > 1$, $E(X_{11}\log(\max\{X_{11}, 1\})) < \infty \Leftrightarrow Z_n\mu^{-n} \to W$ a.s. with $P(0 < W < \infty) = P(Z_n \to \infty)$. Even if the additional moment condition fails the growth rate is very similar to the exponential μ^n [c.f., Athreya and Ney (1972)].

To obtain more realistic models for population growth two kinds of generalized ("controlled") Galton–Watson processes have been studied. Some authors [Stein (1974), Levy (1975), Roi (1975), Fujimagari (1976), Höpfner (1983, 1985), Klebaner (1983, 1984a, 1984b, 1985)] have admitted state-dependent offspring distributions,

$$Z_0 \in \mathbb{N} \setminus \{0\}, \quad Z_{n+1} = \sum_{j=1}^{Z_n} X_{n+1,j}(Z_n), \qquad n \geqq 0,$$

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$\{X_{n,j}(k), n, j, k \in \mathbb{N} \setminus \{0\}\}$ are independent, $\{X_{n,j}(k), n, j \in \mathbb{N} \setminus \{0\}\}$ are identically distributed on $\mathbb{N}$, i.e., the offspring distributions of individuals of one generation remain independently distributed but now they depend on the number of individuals of the generation. Sevast'yanov and Zubkov (1974) and Zubkov (1974) have retained i.i.d. offspring but have controlled the number of reproducing particles by a function $\varphi: \mathbb{N} \to \mathbb{N}$,

$$Z_0 \in \mathbb{N}, \qquad Z_{n+1} = \sum_{j=1}^{\varphi(Z_n)} X_{n+1,j}, \qquad n \geqq 0.$$

Yanev (1975) has replaced the function φ by identically distributed random functions φ_n that are essentially independent of $\{X_{n,j}\}$. Roi (1975), who has dealt with state-dependent generalizations of multi-type Galton–Watson processes, has considered a branching process with state-dependent emigration. Before producing offspring each individual emigrates, independently of the others, with a probability $p(Z_n)$. This process can also be regarded as a φ-branching process with $\varphi_n(k)$ independently $b_{k,1-p(k)}$-binomial distributed.

Most of the results obtained for these processes deal with criteria for a.s. extinction (resp. boundedness) or divergence with positive probability. If the $(2 + \delta)$ moments of the offspring distributions are uniformly bounded Levy (1975) has shown that $\mu(i) = E(X_{11}(i)) = 1 + A/2i$, where $A < \liminf \operatorname{var}(X_{11}(n))$ or $A > \limsup \operatorname{var}(X_{11}(n))$ describes the critical boundary between these two types of asymptotic behavior. Roi (1975) has obtained similar results using only second moment restrictions. For the φ-controlled branching process Zubkov (1974) has obtained the boundary $\mu\varphi(i)/i = 1 + A/2i\mu$, where $A < \operatorname{var}(X_{11})$ or $A > \operatorname{var}(X_{11})$, $\mu = E(X_{11})$.

There are only a few statements on the growth rate of these processes. Fujimagari (1976), Roi (1975), and Klebaner (1984a, b) have shown that the state-dependent Galton–Watson processes diverge exponentially with rate μ^n if $\mu(i) \to \mu > 1$ sufficiently fast and the second moments of the offspring distributions satisfy some boundary conditions. In Klebaner (1985) these second moment conditions are relaxed. Klebaner (1983, 1984a) and Höpfner (1983, 1985) have described some classes of offspring distributions, where Z_n/n does not converge almost surely but in distribution to a Γ distribution. These cases belong to the class of processes with $\mu(i) = 1 + A/i$ and A sufficiently large. Therefore, $\mu(i)$ is close to a function that does not allow divergence. Concerning the φ-controlled processes Zubkov (1974) has proved that they grow exponentially if $\mu\varphi(i)/i$ converges to a constant greater than 1 sufficiently fast.

In the following we want to study state-dependent and φ-controlled Galton–Watson processes:

$$Z_0 \in \mathbb{N}, \qquad Z_{n+1} = \sum_{j=1}^{\varphi_{n+1}(Z_n)} X_{n+1,j}(Z_n), \qquad n \geqq 0,$$

where $\{X_{n,j}(k), \varphi_m(l), n, j, k, m, l \in \mathbb{N}\}$ are independent and $\{X_{n,j}(k), n, j \in \mathbb{N}\}$ respectively $\{\varphi_m(l), m \in \mathbb{N}\}$ are identically distributed on $\mathbb{N}$. Using the methods of Stein (1974) and Levy (1975) it is not difficult to obtain criteria for

$P(\limsup Z_n < \infty) = 1$ or $P(Z_n \to \infty) > 0$ in case of deterministic φ_n. We will confine ourselves to studying the divergence rate of $(Z_n, n \in \mathbb{N})$. We are especially interested in the cases with $\mu(i)E(\varphi_1(i))/i \to 1$ sufficiently slowly where we expect divergence rates which are slower than exponential. The results are collected in Section 2. For the proofs, given in Section 4, we cannot apply any method using the special properties of generating functions of the Galton-Watson processes. Our proofs base on discrete Martingale theory as can be found, e.g., in the book of Hall and Heyde (1980). The main ideas are independent of the special model of controlled Galton–Watson processes. We include them—after some modification—in a more general growth model. Its divergence behavior is determined in Section 3.

2. Results on controlled Galton–Watson Processes. $(Z_n, n \in \mathbb{N})$ is a time-homogeneous Markov chain on $\mathbb{N}$ as described at the end of Section 1:

$$\mu(k) = E(X_{11}(k)), \qquad \nu(k) = E(\varphi_1(k)), \qquad \tau^2(k) = \operatorname{var}(\varphi_1(k))$$

exist. Note that

$$E(Z_{n+1}|Z_n) = \nu(Z_n)\mu(Z_n).$$

To obtain divergence of (Z_n) with positive probability we assume:

(2.1) $$\nu(k)\mu(k) \quad \text{is increasing for } k \geqq x_0 > 0.$$

(2.2) $$\nu(k)\mu(k) \geqq k(1 + Ak^{-\alpha}) \quad \text{for some } A > 0, \alpha \in [0,1), \text{ and all } k \geqq x_0.$$

The lower bound in condition (2.2) is not far away from $\nu(k)\mu(k)/k - 1 \sim k^{-1}$, where as well as a.s. boundedness, divergence can take place.

The essential assumption we need for most of our results is

(2.3) $$\nu(k)\mu(k)/k \quad \text{is nonincreasing for } k \geqq x_0.$$

It means that the mean increment per individual decreases while the number of individuals of the same generation increases. Such a condition has also been assumed for state-dependent Markov branching processes [Küster (1983)]. As already remarked there, condition (2.3) is—although very restrictive—natural in many real processes. On account of limited food or other limited resources the offspring will decrease while the population increases. Moreover, we assume that the variances of the controlling random variables $\varphi_n(k)$ are not too large compared with their expectation,

(2.4) $$\tau^2(k)/\nu(k)^2 = O(k^{-\alpha-\delta}), \quad \text{for some } \delta > 0, \alpha \text{ is given by condition (2.2)}.$$

For technical reasons we extend $\nu(k)\mu(k)$ to a continuous differentiable function on $\mathbb{R}^+$. We can do it in such a way that (2.1)–(2.3) is now satisfied for all $x \geqq x_0$ (x_0 chosen sufficiently large).

As $E(Z_{n+1}|Z_n) = \nu(Z_n)\mu(Z_n)$ we expect that the real sequence $(z_n, n \in \mathbb{N})$,

$$z_0 = x_0, \qquad z_{n+1} = \nu(z_n)\mu(z_n), \qquad n \geqq 0,$$

will describe, besides a random factor, the divergence behavior of the process

$(Z_n, n \in \mathbb{N})$. In some cases it is possible to use instead of (z_n) the solution $h(x)$ of the corresponding differential equation,

$$h(0) = x_0, \qquad h'(x) = h(x)\log(\nu(h(x))\mu(h(x))/h(x)), \qquad x \geqq 0,$$

which is the inverse function of

$$h^{-1}(x) = \int_{x_0}^{x} (\tau \log(\nu(\tau)\mu(\tau)/\tau))^{-1} d\tau, \qquad x \geqq x_0.$$

THEOREM 1. *Assume conditions* (2.1)–(2.4), *and there is* $\beta \in (1 + \alpha, 2]$ *such that for all* $k \geqq x_0$, *the* β *moment* $M(k)$ *of* $X_{11}(k)$ *exists and*

$$\nu(k)M(k) = O(k^{\beta-\alpha-\delta}) \quad \textit{for some } \delta > 0. \tag{2.5}$$

Then,

$$Z_n/z_n \to W \quad a.s., \quad \textit{where } P(0 < W < \infty) = P(Z_n \to \infty) = P\Big(\limsup_{n\to\infty} Z_n = \infty\Big).$$

If $\nu(k)\mu(k)/k \to 1$ *then* $P(W = 1) = P(Z_n \to \infty)$ *and we can replace* z_n *by* $h(n)$. *In case of* $P(Z_n \geqq x_0$ *for some* $n \in \mathbb{N} | Z_0) > 0$ *we have* $P(W > 0) > 0$.

Considering the lower bound of $\nu(k)\mu(k)/k$ we see that the growth rate can be less than exponential. The smallest rate is of order $n^{1/\alpha}$. Except for the case of exponential growth the limit of Z_n/z_n degenerates on $\{Z_n \to \infty\}$. Regarding the special case of the ordinary Galton–Watson process Theorem 1 does not contain the divergence result under the weakest moment condition, the $(x \log x)$ condition. We need the finiteness of the $(1 + \delta)$ moment of the offspring distribution. This is remedied in the following result on processes without φ controls, i.e., $\varphi_n(k) = k$. One of the additional assumptions is that the offspring distributions are being dominated by a probability distribution:

(2.6) There is a random variable $Y \geqq x_0$ such that for all $x \geqq x_0$

$$\sup_{k \geqq x_0} P(X_{11}(k) - \mu(k) > x) \leqq P(Y > x).$$

THEOREM 2. *Assume* $\varphi_n(k) = k$ *for all* $n \in \mathbb{N}$, $k \geqq x_0$, *and in addition to* (2.1)–(2.3):

(2.7) *There is* $\alpha^* \in (0,1)$ *such that* $(\mu(x) - 1)x^{\alpha^*}$ *is nondecreasing for* $x \geqq x_0$.

Let (2.6) *be satisfied and*

$$E(Yh^{-1}(Y)) < \infty. \tag{2.8}$$

Then,

$$Z_n/z_n \to W \quad a.s., \qquad \textit{with } P(0 < W < \infty) = P(Z_n \to \infty) = P\Big(\limsup_{n\to\infty} Z_n = \infty\Big).$$

If $\mu(k) \to 1$ *then* $P(W = 1) = P(Z_n \to \infty)$ *and we can replace* z_n *by* $h(n)$. *In case of* $P(Z_n \geqq x_0$ *for some* $n \in \mathbb{N} | Z_0) > 0$ *we obtain* $P(W > 0) > 0$.

The inverse of the growth rate of the process is important for the moment condition we need for the a.s. convergence of $Z_n/z_n \to W$, $P(W > 0) > 0$. This result is similar to those of branching processes in varying environment [Goettge (1976)] and Markov branching processes with state-dependent offspring distributions [Küster (1983)], where it was also possible to show necessity of the moment condition needed for divergence with "natural" rate. In some of the present cases we can prove the necessity of $E(Yh^{-1}(Y)) < \infty$ for the convergence of $Z_n/z_n \to W$ with $P(0 < W < \infty) > 0$.

THEOREM 3. *Assume the conditions* (2.1)–(2.3), (2.7) *and, instead of* (2.6),

(2.9) $$\inf_{k \geqq x_0} P(X_{11}(k) - \mu(k) > x) \geqq P(Y > x) \quad \text{for all } x \geqq x_0.$$

Let $Z_n/z_n \to W$ a.s. with $P(0 < W < \infty) > 0$ be satisfied. Then, $\mu(x) \to \mu > 1$ implies

$$E(Yh^{-1}(Y)) < \infty$$

and $\mu(x) \to 1$ implies

$$E(Y/(\mu(Y) - 1)) < \infty.$$

If in addition

(2.10) *There is $\alpha^{**} > 0$ such that for all $k \geqq x_0$ $(\mu(k) - 1)k^{\alpha^{**}}$ is nonincreasing.*

is satisfied, then $E(Yh^{-1}(Y)) < \infty$ is valid also in case of $\mu(x) \to 1$.

These results show that, essentially, $E(Yh^{-1}(Y)) < \infty$ is a necessary and sufficient condition for the divergence of (Z_n) with "natural" rate (z_n).

REMARK 1. We obtain similar results for inhomogeneous controlled Galton–Watson processes. The conditions have to be valid for all functions $\mu_n(k)$, $\nu_n(k)$, $\tau_n^2(k)$, where $\mu_n(k) = E(X_{n,1}(k))$, etc. In addition we need an assumption like $\inf\{\nu_n(k)\mu_n(k) | n \in \mathbb{N}\} > k$, for all $k \geq x_0$, as we cannot consider a slowdown of the growth rate on account of inhomogeneity. Therefore we cannot include the results of Goettge (1976) on inhomogeneous branching processes.

Let $(Z_n, n \in \mathbb{N})$ be a state-dependent Galton–Watson process without φ controls and $\mu(k) \to \mu > 1$, not necessarily in a monotone way. An ordinary Galton–Watson process with $\mu(k) = \mu$ diverges like μ^n. Now, we can ask, how stable is this result when the basic assumption of i.i.d. offsprings is relaxed, i.e., when is the growth rate of the state-dependent branching process μ^n? Klebaner (1984b, 1985) has studied this problem in detail for L^r convergence, $1 \leqq r \leqq 2$. A necessary and sufficient condition for a.s. convergence is given in our next theorem.

THEOREM 4. *Let $(Z_n, n \in \mathbb{N})$ be a state-dependent Galton–Watson process, $\varphi_n(k) = k$ and $\mu(k) \to \mu > 1$, where the convergence is not necessarily monotone.*

Assume that, for $k \geqq x_0$ sufficiently large and some $\beta \in (1,2]$, the β moment $M(k)$ of $X_{11}(k)$ exists and satisfies

(2.11) $\quad M(k)k^{1-\beta+\delta}$ *is uniformly bounded for some* $\delta > 0$.

Then, for $\varepsilon > 0$ arbitrary,

$$Z_n(\mu-\varepsilon)^{-n} \to \infty \quad a.s., \qquad Z_n(\mu+\varepsilon)^{-n} \to 0 \quad a.s. \text{ on } \{\limsup Z_n \geqq x_0\}$$

and

$$\{\omega | Z_n(\omega)\mu^{-n} \to W, 0 < W < \infty\} = \left\{\omega \Big| \sum_{n=1}^{\infty} \log \mu(Z_n(\omega))/\mu \text{ is convergent}\right\} \quad a.s.$$

In case of $\mu(k) \to \mu$ monotone (from above or below) for k sufficiently large,

$$Z_n\mu^{-n} \to W \quad a.s. \quad \text{and} \quad P(0 < W < \infty) = P(Z_n \to \infty)$$

if and only if

$$\sum_{n=1}^{\infty} |\mu - \mu(n)|/n < \infty.$$

3. An auxiliary growth model. If the second moments of the offspring distributions exist we can regard the controlled Galton–Watson process satisfying conditions (2.1)–(2.5), $\beta = 2$, as a special case of a more general growth model: $(X_n, n \in \mathbb{N})$ is a sequence of nonnegative random variables that are related recursively by

$$X_{n+1} = g(X_n) + \xi_{n+1} + R_{n+1}.$$

The function g: $\mathbb{R}^+ \to \mathbb{R}^+$ satisfies

(3.1) $\quad g(x) > x \quad \text{for } x \geqq x_0 > 0.$

(3.2) $\quad g(x)$ is increasing, $\quad g(x)/x$ is nonincreasing for $x \geqq x_0$.

In case of controlled Galton–Watson processes (3.1) and (3.2) are satisfied for $g(x) = \nu(x)\mu(x)$ on account of (2.1)–(2.3). The random variable ξ_n is mesurable with respect to $\mathscr{F}_n \supset \mathscr{F}(X_1, \dots, X_n)$ and

(3.3) $\quad E(\xi_n | \mathscr{F}_{n-1}) = 0.$

The second moments of the $\{\xi_n\}$ exist satisfying:

(3.4)
There is a function σ^2: $\mathbb{R}^+ \to \mathbb{R}^+$ such that
$E(\xi_{n+1}^2 I(X_n \geqq x_0) | \mathscr{F}_n) \leqq \sigma^2(X_n) I(X_n \geqq x_0)$ a.s.,
$\sigma^2(x) = o(1)g(x)^2 \log(g(x)/x) <$
$(2b^{-1} + 2(1-3b)^{-2})^{-1} g(x)^2 \log(g(x)/x)$ for some
$b \in (0, 1/3)$ and all $x \geqq x_0$.

$I(A)$ denotes the indicator function of the set A. The sequence which seems to be suitable to describe the asymptotic behavior of $(X_n, n \in \mathbb{N})$ is $(x_n, n \in \mathbb{N})$, where

$$x_{n+1} = g(x_n), \qquad n \in \mathbb{N}.$$

Note, that we have $x_n \to \infty$ for $n \to \infty$. Suppose $x_n \to x' < \infty$; then $g(x') = x'$, as g is continuous due to (3.2). But this is a contradiction to assumption (3.1).

The variances of the $\{\xi_n\}$ are related with (x_n) by:

$$\text{If } (y_n, n \in \mathbb{N}) \text{ is a sequence with } \liminf_{n\to\infty} (\log y_n)/(\log x_n) \geqq 1 \text{ then } \sum_{n=1}^{\infty} \sigma^2(y_n)/g(y_n)^2 < \infty. \tag{3.5}$$

In case of controlled Galton–Watson processes with second moments (3.3)–(3.5) are satisfied due to the assumptions (2.2), (2.4), and (2.5), $R_n = 0$. The same is true for Galton–Watson processes with weaker moment conditions, considering proper truncation. Remainder terms are collected in R_n.

PROPOSITION 1. *For the process* $(X_n, n \in \mathbb{N})$ *as given above assume* (3.1)–(3.5). *The random variables* $(R_n, n \in \mathbb{N})$ *satisfy*

$$\sum_{n=0}^{\infty} |R_n| g(X_n)^{-1} I(X_n \geqq x_0) < \infty \quad a.s. \tag{3.6}$$

Then,

$$X_n/x_n \to W \quad a.s. \qquad \text{with } P(0 < W < \infty) = P\Big(\limsup_{n\to\infty} X_n > x_0\Big).$$

If $g(x)/x \to 1$ *then* $P(W = 1) = P(\limsup_{n\to\infty} X_n > x_0)$ *and we can replace* x_n *by* $h(n)$, *where*

$$h(0) = x_0, \qquad h'(x) = h(x)\log(g(h(x))/h(x)), \qquad \text{for } x \geqq 0.$$

The main idea of the proof is to examine $\log X_n$ instead of X_n. On $\{X_n \geqq x_0\}$ we can write

$$\log X_{n+1} = \log X_n + \log(g(X_n)/X_n) + \log(1 + (\xi_{n+1} + R_{n+1})/g(X_n))$$

and

$$\log x_{n+1} = \log x_n + \log(g(x_n)/x_n).$$

As $\log(g(x)/x)$ is nonincreasing $\log X_{n+1}$ is expected to grow faster than $\log x_{n+1}$ if $X_n < x_n$ and slower if $X_n > x_n$. The following theorem shows that, with some additional assumptions, such property leads to convergence of $\log X_n - \log x_n$.

PROPOSITION 2. *Let* $(X_n, n \in \mathbb{N})$ *be a stochastic process on* $\mathbb{R}^+$,

$$X_{n+1} = X_n + m(X_n) + \xi_{n+1} + R_{n+1}, \qquad n \in \mathbb{N},$$

where $m\colon \mathbb{R}^+ \to \mathbb{R}^+ \setminus \{0\}$ *is a function with*

$$m(x) \text{ is nonincreasing}, \qquad x + m(x) \text{ is increasing}. \tag{3.7}$$

ξ_{n+1} *is measurable with respect to* $\mathscr{F}_{n+1} \supset \mathscr{F}(X_0, \dots, X_{n+1})$ *and*

$$E(\xi_{n+1} | \mathscr{F}_n) = 0, \qquad E(|\xi_{n+1}|^p | \mathscr{F}_n) = O(1) m(X_n)^d, \qquad \text{for some } 1 \leqq d < p \leqq 2. \tag{3.8}$$

The random variables $\{R_n, n \in \mathbb{N}\}$ *satisfy*

$$\lim_{n\to\infty} x_n^{-1} \sum_{j=1}^{n} |R_j| = 0 \quad a.s., \tag{3.9}$$

where $(x_n, n \in \mathbb{N})$ *is the real sequence defined by the recursive relation*

$$x_{n+1} = x_n + m(x_n), \qquad n \in \mathbb{N}.$$

Then,

$$\lim_{n\to\infty} X_n/x_n = 1 \quad a.s.$$

If in addition

$$\sum_{n=0}^{\infty} E(|\xi_{n+1}|^p|\mathscr{F}_n) < \infty \quad a.s., \qquad \sum_{n=0}^{\infty} |R_n| < \infty \quad a.s., \tag{3.10}$$

is valid, then

$$\lim_{n\to\infty} X_n - x_n = W \quad a.s.$$

W is a finite random variable. If $m(x) \to 0$ *then* $W = 0$ *and we can replace* x_n *by* $f(n)$, *where* $f(x)$ *is the function given by*

$$f(0) = x_0, \qquad f'(x) = m(f(x)), \qquad x \geqq 0.$$

PROOF OF PROPOSITION 2. Using the recursive relations of (X_n) we can write

$$X_{n+1} = X_0 + \sum_{j=0}^{n} m(X_j) + \sum_{j=0}^{n} (\xi_{j+1} + R_{j+1}).$$

By (3.8) and Jensen's inequality we obtain

$$\sum_{j=0}^{n} E(|\xi_{j+1}|^p|\mathscr{F}_j)\left(\sum_{k=0}^{j} m(X_k)\right)^{-p} = \sum_{j=0}^{n} \mathrm{O}(1)m(X_j)^d\left(\sum_{k=0}^{j} m(X_k)^d\right)^{-p/d}.$$

The convergence of these series is a result of the theorem of Abel and Dini [cf., Knopp (1928)] that says: If $(a_n, n \geqq 0)$ is a sequence in $\mathbb{R}^+\backslash\{0\}$, then

$$\sum_{j=0}^{\infty} a_j\left(\sum_{k=0}^{j} a_k\right)^{-c} \begin{cases} < \infty, & \text{if } c > 1 \\ = \infty, & \text{if } c \leqq 1 \text{ and } \sum_{j=0}^{\infty} a_j = \infty. \end{cases}$$

But,

$$\sum_{j=0}^{\infty} E(|\xi_{j+1}|^p|\mathscr{F}_j)\left(\sum_{k=0}^{j} m(X_k)\right)^{-p} < \infty,$$

and the strong law of large numbers (SLLN) for martingales [Hall and Heyde (1980), p. 35] leads to

(3.11)

$$X_{n+1} = \begin{cases} \left(\sum_{j=0}^{n} m(X_j)\right)(1 + \mathrm{o}(1)) + \sum_{j=0}^{n} R_{j+1}, & \text{on } \sum_{j=0}^{\infty} m(X_j) = \infty, \\ X_0 + \sum_{j=0}^{\infty} (m(X_j) + \xi_{j+1}) + \sum_{j=0}^{n} R_{j+1} + \mathrm{o}(1), & \text{on } \sum_{j=0}^{\infty} m(X_j) < \infty. \end{cases}$$

Now we compare X_n with x_n. Note that $x_n \to \infty$, as $m(x)$ is continuous and strictly positive due to (3.8).

$$\begin{aligned} X_{n+1} - x_{n+1} &= (X_n - x_n)\big(1 + I(X_n \neq x_n)(m(X_n) - m(x_n))/(X_n - x_n)\big) \\ &\quad + \xi_{n+1} + R_{n+1} \\ &= U_{n+1} \sum_{j=0}^{n+1} (\xi_j + R_j)/U_j \end{aligned}$$

with

$$U_{n+1} = \prod_{j=0}^{n} \big(1 + I(X_j \neq x_j)(m(X_j) - m(x_j))/(X_j - x_j) - I(X_j = x_j)2^{-1}\big),$$

$R_0 = X_0 - x_0$, $\xi_0 = 0$. U_n is a positive random variable, measurable with respect to F_{n-1}, $(U_n, n \in \mathbb{N})$ is nonincreasing. Hence, we obtain similarly to (3.11)

$$U_{n+1} \sum_{j=0}^{n+1} \xi_j/U_j = \begin{cases} \mathrm{o}\left(\sum_{j=0}^{n} m(X_j) \right), & \text{on } \sum_{j=0}^{\infty} m(X_j) = \infty, \\ \mathrm{o}(1) + \lim_{n \to \infty} U_n \sum_{j=0}^{n} \xi_j/U_j, & \text{on } \sum_{j=0}^{\infty} m(X_j) < \infty. \end{cases}$$

Using (3.9) and (3.11) this results in the first statements of the theorem. If we assume condition (3.10) we obtain $U_{n+1}\Sigma_{j=0}^{n+1} R_j/U_j$ is a.s. convergent and the SLLN for martingales yields also the convergence of $U_{n+1}\Sigma_{j=0}^{n+1}\xi_j/U_j$. Therefore, $X_n - x_n$ has to converge a.s. In case of $m(x) \to 0$ the limit is 0. We obtain this result by the relations

$$\{\omega | X_n(\omega) - x_n \to 0\} \supset \{U_n \to 0\} \supset \left\{ \omega \middle| \sum_{n=0}^{\infty} |m(X_n(\omega)) - m(x_n)| = \infty \right\}$$

and the following lemma.

Lemma 1. *Let $m(x)$ be a function on $\mathbb{R}^+ \backslash \{0\}$ satisfying (3.7). $(x_n, n \in \mathbb{N})$ is the sequence with $x_{n+1} = x_n + m(x_n)$, $n \geqq 0$. If $m(x) \to 0$ then*

$$\sum_{n=0}^{\infty} |m(x_n) - m(x_n + c)| = \infty \quad \textit{for all } c \neq 0.$$

Proof of Lemma 1. Suppose, $\Sigma_n |m(x_n) - m(x_n + c)| < \infty$ for some $c \neq 0$. We can assume $c > 0$. We can write

$$\begin{aligned} \infty &> \sum_{n=0}^{\infty} m(x_n) - m(x_n + c) \\ &\geqq \sum_{j=1}^{\infty} (m(\log j + 1) - m(c + \log j)) \mathrm{card}\{n | \log j \leqq x_n < \log j + 1\} \\ &\geqq \mathrm{O}(1) + \sum_{j=1}^{\infty} (m(\log j + 1) - m(c + \log j))/jm(\log j). \end{aligned}$$

As the last sum is finite we obtain for $n > k$,

$$\sum_{j=k}^{n} m(c+\log j)/jm(\log j) \geqq \gamma_k + \sum_{j=k}^{n} m(\log j+1)/jm(\log j)$$
$$\geqq \gamma_k + \sum_{j=k}^{n} (\log j)/j\log j + 1,$$

where (γ_k) is a sequence which is independent of n and $\gamma_k \to 0$ for $k \to \infty$. Hence,

$$\sum_{j=k}^{n} m(c+\log j)/jm(\log j) \geqq (\log n) - (\log k) + \gamma_k + \gamma k^{-1}\log n, \tag{3.12}$$

for some constant $\gamma > -\infty$. Now we can write

$$\infty > \sum_{j=1}^{\infty} (jm(\log j))^{-1} \sum_{k=j+1}^{[je^c-1]} (m(\log k) - m(\log k+1))$$
$$\geqq \sum_{k=2}^{\infty} (m(\log k) - m(\log k+1))(m(\log k))^{-1}$$
$$\cdot \sum_{j=[(k+1)e^{-c}]}^{k-1} m(c+\log j)/jm(\log j).$$

Considering (3.12) we obtain

$$\infty > \sum_{k=2}^{\infty} (c+o(1))(m(\log k) - m(\log k+1))/m(\log k),$$

which is, in case of $c > 0$, equivalent to

$$0 < \prod_{k=2}^{\infty} m(\log k+1)/m(\log k) = \lim_{n\to\infty} m(\log n)/m(\log 2).$$

But the limit is 0, if $m(x) \to 0$. Hence, $c = 0$, which proves our lemma.

To finish with the proof of Proposition 2 we only have to show that $x_n - f(n) \to 0$, if $m(x) \to 0$. Considering the definition of $f(x)$ and the monotonicity of $m(x)$ we obtain

$$f(n) + m(f(n) + m(f(n))) \leqq f(n+1) \leqq f(n) + m(f(n)).$$

As $x + m(x)$ is increasing and $x_0 = f(0)$ this implies

$$x_n \geqq f(n) \quad \text{for all } n \geqq 0.$$

If $x_n \geqq f(n) + m(f(n))$, then

$$0 \leqq x_{n+1} - f(n+1) \leqq x_n + m(f(n) + m(f(n))) - f(n+1) \leqq x_n - f(n).$$

If $x_n < f(n) + m(f(n))$, then

$$0 \leqq x_{n+1} - f(n+1) \leqq f(n) + m(f(n)) + m(f(n) + m(f(n))) - f(n+1)$$
$$\leqq m(f(n)).$$

As $m(x) \to 0$, these inequalities imply the convergence of $x_n - f(n)$ to a constant $c \geqq 0$. But $c > 0$ is impossible, because in this case

$$x_{n+1} - f(n+1) \leqq x_0 - f(0) + \sum_{j=0}^{n} m(x_j) - m(f(j) + m(f(j)))$$

$$\leqq O(1) + \sum_{j=0}^{n} m(x_j) - m(x_j - c/2)$$

converges to $-\infty$, according to Lemma 1. Therefore,

$$x_n - f(n) \to 0. \tag{3.13}$$

REMARK 2. If in Proposition 2 only an inequality

$$X_{n+1} \geqq X_n + m(X_n) + \xi_{n+1} + R_{n+1}, \qquad n \geqq 0$$

is valid, then

$$\liminf_{n\to\infty} X_n/x_n \geqq 1 \quad \text{a.s.}$$

Instead of $(X_n, n \in \mathbb{N})$ we have to regard $(Y_n, n \in \mathbb{N})$ with

$$Y_0 = X_0, \qquad Y_{n+1} = Y_n + m(Y_n) + \xi_{n+1} + R_{n+1}, \qquad n \geqq 0.$$

As $E(|\xi_{n+1}|^p|\mathscr{F}_n) = O(1)m(Y_n)^d$ and $Y_n \leqq X_n$ due to the monotonicity of $m(x)$ and $x + m(x)$, we obtain $Y_n/x_n \to 1$ a.s. and $\liminf X_n/x_n \geqq 1$ a.s..

REMARK 3. We can generalize Proposition 2 by replacing the function $m(x)$ by random functions $m_n(x)$ each satisfying (3.8), $m_n(x)$ has to be measurable with respect to $\mathscr{F}_{n-1}$ and has to satisfy (3.9). In the results the real sequence (x_n) will be replaced by a random sequence $Y_{n+1} = Y_n + m_{n+1}(Y_n)$. It is clear that we obtain statements on inhomogeneous processes (X_n) in particular.

Now we can start with the proof of Proposition 1.

PROOF OF PROPOSITION 1. We want to apply Proposition 2 to the process $(\log X_n, n \in \mathbb{N})$. $\log(g(e^x)e^{-x})$ satisfies condition (3.7), but it is not clear how to split $\log(1 + (\xi_{n+1} + R_{n+1})/g(X_n))$ in terms satisfying (3.8) or (3.9). The main difficulty is that $1 + (\xi_{n+1} + R_{n+1})/g(X_n)$ does not have to be bounded away from 0. We enforce this by regarding special partial sequences. Let $(T(n), n \in \mathbb{N})$ be a sequence of stopping times on $\mathbb{N} \cup \{\infty\}$,

$$T(0) = 0, \qquad T(n) = \inf\{k > T(n-1) | X_k \geqq (1-b)g(X_{T(n-1)}), X_k \geqq x_0\}, \qquad n \geqq 0,$$

where $b \in (0, \frac{1}{3})$ is fixed by assumption (3.4), $\inf \varnothing = \infty$. We define

$$F_n^T = F_{T(n)}, \qquad X_n^T = X_{T(\max\{k \leqq n | T(k) < \infty\})},$$

$$g_n^T(x) = \begin{cases} g(x), & \text{if } T(n) < \infty, \\ x, & \text{if } T(n) = \infty. \end{cases}$$

We divide the proof of Proposition 1 into four steps. We will show:

Step 1. $\liminf_{n\to\infty}(\log X_n^T)/(\log x_n) \geqq I(T(n) < \infty \text{ for all } n)$ a.s.
Step 2. $P(T(n+1) \neq T(n)+1 \text{ i.o.}) = 0$.
Step 3. $\{T(n) = \infty \text{ for some } n\} \subset \{\limsup X_n \leqq x_0\}$ a.s.
Step 4. Conclusion.

STEP 1. We define

$$\xi_{n+1}^T = I\big(\xi_{T(n)+1} \leqq bg\big(X_n^T\big)\big)\max\big\{-b, \xi_{T(n)+1}/g\big(X_n^T\big)\big\},$$

$$R_{n+1}^T = I\big(\xi_{n+1}^T > -b\big)\max\big\{-2b, R_{T(n)+1}/g\big(X_n^T\big)\big\},$$

where $\xi_\infty = R_\infty = 0$. As $(\xi_{n+1}^T + R_{n+1}^T)$ is bounded from below by $-3b$ Taylor's formula leads to

$$\log X_{n+1}^T \geqq \log X_n^T + \log\big(g_n^T\big(X_n^T\big)/X_n^T\big) + \xi_{n+1}^T + R_{n+1}^T$$
$$-\big(\xi_{n+1}^T\big)^2(1-3b)^{-2} - \big(R_{n+1}^T\big)^2(1-3b)^{-2}.$$

Using (3.3), which implies $E(\xi_{T(n)+1}|\mathscr{F}_n^T) = 0$, and (3.4) we can write

$$\big|E\big(\xi_{n+1}^T|\mathscr{F}_n^T\big)\big| = bP\big(\xi_{T(n)+1} < -bg\big(X_n^T\big)|\mathscr{F}_n^T\big)$$
$$+\Big|E\Big(I\big(|\xi_{T(n)+1}| > bg\big(X_n^T\big)\big)\xi_{T(n)+1}/g\big(X_n^T\big)|\mathscr{F}_n^T\Big)\Big|$$
$$\leqq 2b^{-1}\sigma^2\big(X_n^T\big)g\big(X_n^T\big)^{-2}I(T(n) < \infty),$$

Similarly,

$$E\Big(\big(\xi_{n+1}^T\big)^2|\mathscr{F}_n^T\Big) \leqq 2\sigma^2\big(X_n^T\big)g\big(X_n^T\big)^{-2}I(T(n) < \infty),$$

and by (3.4) there is $c \in (0,1)$ with

$$\Big|E\big(\xi_{n+1}^T|\mathscr{F}_n^T\big) + (1-3b)^{-2}E\Big(\big(\xi_{n+1}^T\big)^2|\mathscr{F}_n^T\Big)\Big| \leqq c\log\big(g_n^T\big(X_n^T\big)/X_n^T\big). \tag{3.14}$$

Now, we define a martingale difference sequence by

$$\eta_{n+1} = \xi_{n+1}^T - E\big(\xi_{n+1}^T|\mathscr{F}_n^T\big) + \Big(\big(\xi_{n+1}^T\big)^2 - E\Big(\big(\xi_{n+1}^T\big)^2|\mathscr{F}_n^T\Big)\Big)(1-3b)^{-2}$$

and we know that

$$E\big(\eta_{n+1}^2|\mathscr{F}_n^T\big) = O(1)\log\big(g_n^T\big(X_n^T\big)/X_n^T\big).$$

Therefore,

$$\log X_{n+1}^T \geqq \log X_n^T + (1-c)\log\big(g_n^T\big(X_n^T\big)/X_n^T\big) + \eta_{n+1}$$
$$+R_{n+1}^T - \big(R_{n+1}^T\big)^2(1-3b)^{-2}$$

is an inequality which satisfies the assumption of Proposition 2 modified by Remarks 2 and 3, $(R_n^T + (R_n^T)^2(1-3b)^{-2})$ satisfies (3.9) on account of (3.5).

Hence, we obtain for the sequence $(y_n, n \in \mathbb{N})$, where

$$y_0 = x_0, \qquad y_{n+1} = y_n\big(g(y_n)/y_n\big)^{1-c}, \qquad n \geqq 0 \tag{3.15}$$
$$\liminf_{n\to\infty}\big(\log X_n^T\big)/\log y_n \geqq I(T(n) < \infty \text{ for all } n) \text{ a.s.}$$

As $y_n \to \infty$ (X_n^T) is divergent on $\{T(n) < \infty \text{ for all } n\}$.

Next, we compare (y_n) with (x_n). Iteration of the recursive relations yields

$$\log x_{n+1} - \log y_{n+1} = u_{n+1} \sum_{j=1}^{n} c\big(\log g(y_j)/y_j\big)/u_j,$$

where (u) is positive and nonincreasing. Hence,

$$0 \leqq \log x_n - \log y_n \leqq c(1-c)^{-1}\log y_n.$$

As (X_n^T) diverges on $\{T(n) < \infty \text{ for all } n\}$ we can choose $c > 0$ in (3.14) arbitrary small, for n sufficiently large. Therefore, we can replace $\log y_n$ in (3.15) by $\log x_n$,

$$\liminf_{n\to\infty}\big(\log X_n^T\big)/\log x_n \geqq I(T(n) < \infty \text{ for all } n) \quad \text{a.s.} \tag{3.16}$$

STEP 2. Using (3.16) and (3.5) we obtain

$$\sum_{n=1}^{\infty} \sigma^2(X_n^T)/g(X_n^T)^2 I(T(n) < \infty) < \infty \quad \text{a.s.}$$

This implies

$$\sum_{n=1}^{\infty} P\big(\xi_{T(n)+1} < -b2^{-1}g(X_n^T)|\mathscr{F}_n^T\big) < \infty \quad \text{a.s.}$$

and the conditioned version of the Borel–Cantelli lemma [cf., Hall and Heyde (1980), p. 32] yields

$$P\big(\xi_{T(n)+1} < -b2^{-1}g(X_n^T) \text{ i.o.}\big) = 0.$$

By (3.6),

$$\sum_{n=1}^{\infty} I\big(R_{T(n)+1} < -2bg(X_n^T)\big) < \infty \quad \text{a.s.},$$

therefore, considering the definition of $T(n)$,

$$P(T(n+1) \neq T(n) + 1 \text{ i.o.}) = 0. \tag{3.17}$$

STEP 3. Obviously,

$$\{T(n) = \infty \text{ for some } n\} \subset \{\limsup X_n < \infty\}.$$

On $\{\limsup X_n > x_0\}$ we can define stopping times $\{T^0(n), n \geqq -1\}$,

$$T^0(-1) = 0, \qquad T^0(n+1) = \inf\{k > T^0(n) | X_k \geqq x_0\}, \qquad n \geqq -1.$$

Then we can write on $\{\limsup X_n > x_0\} \cap \{\sup\{X_n | n \geqq 0\} \leqq M\}$

$$\begin{aligned} X_{T^0(n+1)} &\geqq X_{T^0(n)} + X_{T^0(n)}\big(-1 + g(X_{T^0(n)})/X_{T^0(n)}\big) + \xi_{T^0(n)+1} \\ &\geqq X_{T^0(n)} + x_0(-1 + g(M)/M) + \xi_{T^0(n)+1} \\ &\geqq n\left(x_0(-1 + g(M)/M) + n^{-1}\sum_{j=0}^{n} \xi_{T^0(j)+1}\right). \end{aligned}$$

But $n^{-1}\sum_{j=0}^{n}\xi_{T^0(j)+1} \to 0$ a.s. by the SLLN for martingales as the conditioned variances of $\xi_{T^0(j)+1}$ are bounded by $(2b^{-1} + 2(1 - 3b)^{-2})^{-1}g(M)^2 \cdot \log(g(x_0)/x_0)$. Therefore, $X_{T^0(n)}$ diverges on $\{\limsup X_n > x_0\} \cap \{\sup\{X_n | n \geqq 0\} \leqq M\}$, which is a contradiction. As M has been chosen arbitrary this means

$$P(x_0 < \limsup X_n < \infty) = 0.$$

STEP 4. On $\{T(n) < \infty$ for all $n\}$ which includes $\{\limsup X_n > x_0\}$, according to Step 3, we write

$$\log X_{n+1} = \log X_n + \log(g(X_n)/X_n) + \log(1 + (\xi_{n+1} + R_{n+1})/g(X_n)).$$

As (3.16) holds true for $\log X_n$ instead of $\log X_n^T$, on account of (3.17), (3.5) and (3.6) imply that $\xi_{n+1}/g(X_n)$ and $(\log((1 + \xi_{n+1} + R_{n+1})/g(X_n)) - \xi_{n+1}/g(X_n))$ satisfy (3.9) or (3.10) on $\{T(n) < \infty$ for all $n\}$. The arguments are the same as in Step 1. Applying Proposition 1 proves Proposition 2.

Instead of $\log X_n$ one can also study the transformation $G(X_n)$,

$$G(x) = \int_{x_0}^{x} (g(y) - y)^{-1}\,dy, \qquad x \geqq x_0,$$

to get results as in Proposition 1. Keller et al. (1984) have used this approach. If we do not consider rest terms Taylor's formula leads to

$$G(X_{n+1}) \approx G(X_n) + 1 + \xi_{n+1}(g(g(X_n)) - g(X_n))^{-1},$$

an expression which is easy to handle compared with

$$\log X_{n+1} \approx \log X_n + \log(g(X_n)/X_n) + \xi_{n+1}/g(X_n).$$

But when we use the transformation $G(X_n)$ we need more restrictions on $g(x)$ as G depends on g.

REMARK 4. Remarks 2 and 3 in connection with Proposition 2 are also valid for Proposition 1. For example a recursion inequality leads to

$$\liminf_{n\to\infty} X_n/x_n > 0 \quad \text{a.s. on } \{\limsup X_n > x_0\}.$$

4. Proofs of the theorems 1–4. PROOF OF THEOREM 1. We want to apply Proposition 1. As the variances of $\{X_{n,j}(k)\}$ might not exist we have to use truncated random variables. Define

$$\overline{X}_{n,j}(k) = X_{n,j}(k) I(X_{n,j}(k) \leqq k), \qquad \bar{\mu}(k) = E(\overline{X}_{11}(k)).$$

Then,

$$Z_{n+1} \geqq \sum_{j=1}^{\varphi_{n+1}(Z_n)} \overline{X}_{n+1,j}(Z_n)$$

and by (2.3)–(2.5)

$$\text{(4.1)}\quad \operatorname{var}\left(\sum_{j=1}^{\varphi_{n+1}(k)} \overline{X}_{n+1,j}(k)\right) = \nu(k)\operatorname{var}\overline{X}_{11}(k) + \tau^2(k)\bar{\mu}(k)^2$$
$$\leqq \nu(k)M(k)\mathrm{O}(k^{2-\beta}) + \tau^2(k)\mu(k)^2 = \mathrm{O}(k^{2-\alpha-\delta}).$$

We can assume $\alpha > 0$. If $k \geqq y_0$, $y_0 \geqq x_0$ sufficiently large, (2.5) and (2.2) imply

$$E(Z_{n+1}|Z_n = k) \geqq \nu(k)\bar{\mu}(k) \geqq k(1 + k^{-\alpha}A/2).$$

Then (3.4) and (3.5) are satisfied with $g(x) = x(1 + x^{-\alpha}A/2)$ and $x_{n+1} = g(x_n)$. Hence, we obtain by Proposition 1, considering Remark 4,

$$\liminf_{n\to\infty} Z_n/x_n > 0 \quad \text{a.s. on } \left\{\limsup_{n\to\infty} Z_n > y_0\right\}.$$

As Z_n is a homogeneous Markov chain on $\mathbb{N}$ with $E(Z_{n+1} - Z_n|Z_n) > 0$ on $\{Z_n \geqq x_0\}$

$$\{\limsup Z_n > y_0\} = \{\limsup Z_n \geqq x_0\} = \{\limsup Z_n = \infty\} \quad \text{a.s.}$$

Thus, considering the growth rate of (x_n),

$$\text{(4.2)}\qquad Z_n^{-1}I(Z_n \geqq x_0) = \mathrm{O}(n^{-1/\alpha}).$$

Now, we write

$$\text{(4.3)}\quad Z_{n+1} = \nu(Z_n)\mu(Z_n) + \left(-\nu(Z_n)\bar{\mu}(Z_n) + \sum_{j=1}^{\varphi_{n+1}(Z_n)} \overline{X}_{n+1,j}(Z_n)\right)$$
$$-\nu(Z_n)(\mu(Z_n) - \bar{\mu}(Z_n)) + \sum_{j=1}^{\varphi_{n+1}(Z_n)} X_{n+1,j}(Z_n)I\big(X_{n+1,j}(Z_n) > Z_n\big).$$

Define the second term as ξ_{n+1}; then (4.1) and (4.2) imply (3.4) and (3.5), $g(x) = \nu(x)\mu(x)$. Now, we only have to show that the last two terms in (4.3) can be collected in a random variable R_n satisfying (3.6):

$$\sum_{n=1}^{\infty} P\left(\sum_{j=1}^{\varphi_{n+1}(Z_n)} X_{n+1,j}(Z_n)I\big(X_{n+1,j}(Z_n) > Z_n\big) > 0 \Big| Z_n\right) I(Z_n \geqq x_0)$$
$$\leqq \sum_{n=1}^{\infty} \nu(Z_n)P(X_{11}(Z_n) > Z_n|Z_n)I(Z_n \geqq x_0) \leqq \sum_{n=1}^{\infty} \nu(Z_n)M(Z_n)Z_n^{-\beta}I(Z_n \geqq x_0),$$

which is finite by (2.5) and (4.2). Therefore, the last term of (4.3) becomes 0 eventually. Similarly we can show, using (2.5), that

$$\sum_{n=1}^{\infty} \nu(Z_n)(\mu(Z_n) - \bar{\mu}(Z_n))(\nu(Z_n)\mu(Z_n))^{-1}I(Z_n \geqq x_0)$$
$$= \sum_{n=1}^{\infty} \mathrm{O}\big(Z_n^{-\alpha-\delta}\big)I(Z_n \geqq x_0) < \infty.$$

Hence, we can apply Proposition 1.

As $(Z_n, n \in \mathbb{N})$ is a time-homogeneous Markov chain and $Z_n \to \infty$ on $\{\limsup Z_n \geqq x_0\}$ we obtain by standard argumentation that $P(Z_n \geqq x_0$ for some $n|Z_0) > 0$ implies $P(Z_n \to \infty) > 0$.

Before starting with the proof of Theorem 2 we collect some properties of z_n and $h(n)$ in a lemma.

LEMMA 2. *Let* $\nu(x) = x$ *and assume* (2.1)–(2.3) *and* (2.7). *Then*

$$(\mu(x) - 1)^{-1} = O(h^{-1}(x)). \tag{4.4}$$

$$M_1 \leqq \operatorname{card}\{n | z_n \leqq k\}/h^{-1}(k) \leqq M_2 \quad \text{for some } 0 < M_1 \leqq M_2 < \infty \text{ and all } k \geqq 1. \tag{4.5}$$

$$\text{If } (y_n, n \geqq 0) \text{ is a real sequence with } y_n \geqq c z_n,\ c > 0, \text{ then } \sum y_n^{-1} I(y_n \geqq k) = O(h^{-1}(k)/k). \tag{4.6}$$

PROOF. In case of $\mu(x) \to \mu > 1$ it is easy to prove (4.4) and (4.5). Let $\mu(x) \to 1$. Then, by (2.7)

$$\begin{aligned}(\mu(x) - 1)\int_{x_0}^{x}(t \log \mu(t))^{-1} dt &= \int_{x_0}^{x}(\mu(x) - 1)((\mu(t) - 1)t)^{-1}(1 + o(1))\,dt \\ &\geqq \int_{x_0}^{x} x^{-\alpha^*} t^{-1+\alpha^*}(1 + o(1))\,dt = (1 + o(1))/\alpha^*,\end{aligned}$$

which proves (4.4).

Note, that $z_n/h(n) \to 1$ in case of $\mu(x) \to 1$, this result corresponds to (3.13), where $x_n = \log z_n$. Then, we obtain (4.5) as

$$|h^{-1}(k(1 + o(1))) - h^{-1}(k)| \leqq o(1)/(\mu(k) - 1) = o(h^{-1}(k)).$$

To prove (4.6) we split the sum into two terms,

$$\sum_n y_n^{-1} I(y_n \geqq k) \leqq \sum_n (c z_n)^{-1} I(z_n \geqq k/c) + \sum_n k^{-1} I(z_n < k/c).$$

Now, using (4.5),

$$\sum_n k^{-1} I(z_n < k/c) = k^{-1}\operatorname{card}\{n | z_n < k/c\} = O(h^{-1}(k/c)/k).$$

If $\mu(x) \to \mu > 1$, define $k_0 = \inf\{n | z_n \geqq k/c\}$, and we can write

$$\sum_n z_n^{-1} I(z_n \geqq k/c) = \sum_{n=k_0}^{\infty} z_n^{-1} \leqq c k^{-1} \sum_{n=k_0}^{\infty} \mu^{k_0 - n} = O(k^{-1}).$$

If $\mu(x) \to 1$, then $z_n/h(n) \to 1$ and we only have to show

$$\int_{h^{-1}(k/c)}^{\infty} h(t)^{-1} dt = O(h^{-1}(k)/k).$$

By substitution and partial integration we can write

$$\begin{aligned}\int_{h^{-1}(k)}^{\infty} h(t)^{-1} dt &= \int_k^{\infty} t^{-2}(\log \mu(t))^{-1} dt \\ &= (k \log \mu(k))^{1} - \int_k^{\infty} t^{-1}\mu'(t)(\log \mu(t))^{-2} dt.\end{aligned}$$

As

$$\mu'(t) \geqq -\alpha^*(\mu(t)-1)/t = (-\alpha^* + o(1))(\log\mu(t))/t$$

and $\alpha^* < 1$ we obtain

$$\int_{h^{-1}(k)}^{\infty} h(t)^{-1}\,dt = O\big((k\log\mu(k))^{-1}\big) = O\big(h^{-1}(k)/k\big).$$

Now, statement (4.6) follows from $h^{-1}(k/c) = O(h^{-1}(k))$.

Proof of Theorem 2. Let $\overline{X}_{n,j}(k)$, $\bar{\mu}(k)$ be defined as in the proof of Theorem 1. First we examine the case $\mu(x) \to 1$. For fixed $c \in (0,1)$, $cz_1 \geqq x_0$, we define stopping times $\{t(n), n \in \mathbb{N}\}$ on $\mathbb{N} \cup \{\infty\}$ by

$$t(0) = 0, \qquad t(n+1) = \inf\{k > t(n) | Z_k \geqq cz_{n+1}\}, \qquad n \geqq 0.$$

Define

$$Z_n^t = Z_{t(\sup\{k \leqq n | t(k) < \infty\})}, \qquad X_{n+1,j}^t(k) = X_{t(n)+1,j}(k) I(t(n) < \infty),$$
$$\overline{X}_{n+1,j}^t(k) = \overline{X}_{t(n)+1,j}(k) I(t(n) < \infty), \qquad n \geqq 0.$$

We can write

$$\begin{aligned}(4.7)\qquad Z_{n+1}^t \geqq I(t(n) < \infty)\Bigg(& Z_n^t\mu(Z_n^t) + \sum_{k=1}^{Z_n^t}\big(\overline{X}_{n+1,k}^t(Z_n^t) - \bar{\mu}(Z_n^t)\big) \\ & - Z_n^t\big(\mu(Z_n^t) - \bar{\mu}(Z_n^t)\big) \\ & + \sum_{k=1}^{Z_n^t} X_{n+1,k}^t(Z_n^t) I\big(X_{n+1,k}^t(Z_n^t) > Z_n^t\big)\Bigg),\end{aligned}$$

with equality if $t(n+1) = t(n) + 1 < \infty$. We want to collect the last two terms in a random variable R_n satisfying (3.6).

$$\begin{aligned}&\sum_{n=1}^{\infty} P\left(\sum_{j=1}^{Z_n^t} X_{n+1,k}^t(Z_n^t) I\big(X_{n+1,k}^t(Z_n^t) > Z_n^t\big) > 0 | Z_n^t\right) \\ &\quad \leqq \sum_{n=1}^{\infty} I(t(n) < \infty) Z_n^t P\big(X_{n+1,1}(Z_n^t) > Z_n^t | Z_n^t\big) \leqq O(1)\sum_{n=1}^{\infty} E(YI(Y > cz_n)) \\ &\quad = O(1)\sum_{k=1}^{\infty} E(YI(k < Y \leqq k+1))\operatorname{card}\{n | z_n \leqq k/c\} \\ &\quad = O(1)E\big(Yh^{-1}(Y)\big) < \infty\end{aligned}$$

by (4.5) and assumption (2.8). This means that the last term on the r.h.s. of (4.6) becomes 0 eventually. Similarly we can show that

$$\sum_{n=1}^{\infty}\big(\mu(Z_n^t) - \bar{\mu}(Z_n^t)\big) I(t(n) < \infty) < \infty.$$

Thus, the last two terms in (4.6) satisfy (3.6). As

$$\operatorname{var}\left(\sum_{j=1}^{k} \bar{X}_{1,j}(k)\right) \leqq kE\big(Y^2 I(Y \leqq k)\big) + k^2 P(Y > k) = \mathrm{o}(1) k^2 \log \mu(k),$$

(3.4) is satisfied for k sufficiently large.

To show (3.5) note that, by (4.6), for (y_n) with $\liminf(\log y_n)/(\log z_n) \geqq 1$

$$\sum_{n=1}^{\infty} y_n^{-1} E\big(Y^2 I(Y \leqq y_n)\big) \leqq \sum_{j=1}^{\infty} E\big(Y^2 I(j-1 < Y \leqq j)\big) \sum_n y_n^{-1} I(y_n \geqq j)$$
$$= \mathrm{O}(1) E\big(Y h^{-1}(Y)\big) < \infty.$$

Therefore, we obtain by Proposition 1 together with Remark 4:

$$\liminf_{n \to \infty} Z_n^t / z_n \geqq I(t(n) < \infty \text{ for all } n) \quad \text{a.s.} \tag{4.8}$$

It is clear that $\{t(n) = \infty \text{ for some } n\} = \{\limsup Z_n < \infty\}$. We will show that $t(n+1) = t(n) + 1 < \infty$ eventually on $\{\liminf Z_n^t / z_n \geqq 1\}$. It is sufficient to show that for some $1 > c' > c$

$$\sum_{n=1}^{\infty} P\left(\sum_{j=1}^{Z_n^t} X_{n+1,k}^t(Z_n^t) \leqq c z_n | Z_n^t\right) I\big(Z_n^t \geqq c' z_n\big) < \infty \quad \text{a.s.}$$

But the nth term of this sum is bounded by

$$P\left(\big(Z_n^t\big)^{-1} \left| \sum_{j=1}^{Z_n^t} \big(\bar{X}_{n+1,j}^t(Z_n^t) - \bar{\mu}(Z_n^t)\big) \right| \geqq 1 - c/c' | Z_n^t \right) I\big(Z_n^t \geqq c' z_n\big)$$
$$= \mathrm{O}(1) \big(Z_n^t\big)^{-1} E\big(Y^2 I(Y \geqq Z_n^t) | Z_n^t\big) I\big(Z_n^t \geqq c' z_n\big),$$

and by (4.6) we obtain

$$\sum_{n=1}^{\infty} \big(Z_n^t\big)^{-1} E\big(Y^2 I(Y \leqq Z_n^t) | Z_n^t\big) I\big(Z_n^t \geqq c' z_n\big)$$
$$\leqq \sum_{j=1}^{\infty} E\big(Y^2 I(j-1 < Y \leqq j)\big) \sum_n \big(Z_n^t\big)^{-1} I\big(Z_n^t \geqq \max\{j, c' z_n\}\big)$$
$$= \mathrm{O}(1) E\big(Y h^{-1}(Y)\big) < \infty.$$

Therefore $P(\{t(n+1) \neq t(n) + 1 < \infty\} \text{ i.o.}) = 0$ a.s. and, on $\{t(n) < \infty$ for all $n\}$, (4.7) is an equality almost every time. Thus, we can replace (4.8) by

$$\lim_{n \to \infty} Z_n^t / z_n = I(t(n) < \infty \text{ for all } n) = I\Big(\limsup_{n \to \infty} Z_n = \infty\Big) \quad \text{a.s.,}$$

where we can replace Z_n^t by Z_n as $\lim_{n \to \infty} z_{n+n'}/z_n = 1$ for all $n' \in \mathbb{N}$. This proves Theorem 2 for $\mu(x) \to 1$.

In case of $\mu(x) \to \mu > 1$ we can show, applying Proposition 1 again, that for all $c \in (0,1)$

$$\liminf_{n \to \infty} Z_n / z_n^c > 0 \quad \text{on} \quad \limsup_{n \to \infty} Z_n \geqq x_0.$$

This follows from the inequality

$$Z_{n+1} \geqq Z_n \bar{\mu}(Z_n) + \sum_{k=1}^{Z_n} (\bar{X}_{n+1,k}(Z_n) - \bar{\mu}(Z_n)),$$

as $\bar{\mu}(x) \geq \mu(x)^c$ for all x sufficiently large. Now with similar arguments as can be found in Asmussen and Hering [(1983), p. 43], in the case of ordinary Galton–Watson process, we can prove that

$$\sum_n Z_n I(Z_n \leqq k) = \mathrm{O}(k) \quad \text{and} \quad \sum_n (Z_n)^{-1} I(Z_n \geqq k) = \mathrm{O}(k^{-1}).$$

Thus, using only $E(Y) < \infty$, we can replace (4.7) by

$$(4.9) \qquad \begin{aligned} Z_{n+1} = Z_n \mu(Z_n) &+ \left(\sum_{k=1}^{Z_n} (\bar{X}_{n+1,k}(Z_n) - \bar{\mu}(Z_n)) \right) \\ &+ R_{n+1} - Z_n(\mu(Z_n) - \bar{\mu}(Z_n)), \end{aligned}$$

where the second term on the r.h.s. satisfies (3.4) and the third (3.6). Considering the stronger moment condition $E(Yh^{-1}(Y)) < \infty$ [$\Leftrightarrow E(Y \log Y) < \infty$] the last term can be included in R_{n+1} and, applying Proposition 1, we have proved Theorem 2 also in case of $\mu(x) \to \mu > 1$.

Proof of Theorem 3. Let $\mu(x) \to \mu > 1$. We obtain from (4.9), similarly as in the proof of Proposition 1, the equation

$$\begin{aligned} \log Z_{n+1} - \log z_{n+1} = \mathrm{O}(1) &+ \sum_{j=0}^{n} \big(\log \mu(Z_j) - \log \mu(z_j)\big) I(Z_j \geqq x_0) \\ &- \sum_{j=0}^{n} \big(\mu(Z_j) - \bar{\mu}(Z_j)\big)(1 + \mathrm{o}(1)). \end{aligned}$$

By (2.9) we can write

$$-\sum_{j=0}^{n} \mu(Z_j) - \bar{\mu}(Z_j) \geqq -\sum_{j=0}^{n} E\big(YI(Y > Z_j) | Z_j\big),$$

and as

$$\sum_{j=1}^{n} E\big(YI(Y > wz_j)\big) = E\big(Yh^{-1}(Y)\big)/\mathrm{O}(1),$$

for all $w > 0$, using (4.5), it is sufficient for the proof of $E(Yh^{-1}(Y)) < \infty$ to show that

$$\sum_{j=0}^{\infty} \big(\log \mu(Z_j) - \log \mu(z_j)\big) I\big(Z_j \geqq x_0\big) < \infty \quad \text{a.s.}$$

on $\{\liminf Z_n/z_n > w > 0\}$. We can assume that $w \leqq 1$, then

$$I(\liminf Z_n/z_n > w\} \sum_{j=0}^{n} \left(\log\mu(Z_j) - \log\mu(z_j)\right)I(Z_j \geqq x_0)$$
$$\leqq O(1) + \sum_{j=0}^{n} \left(\log\mu(wz_j) - \log\mu(z_j)\right).$$

As $z_{j+1} \geqq \mu z_j$ for all $j \geqq 0$, there is $k_0 \in \mathbb{N}$ with $wz_j \geqq z_{j-k_0}$ for all $j \geqq k_0$ (choose k_0 with $w\mu^{k_0} \geq 1$). Therefore,

$$\sum_{j=0}^{n} \left(\log\mu(wz_j) - \log\mu(z_j)\right) \leqq \sum_{j=0}^{k_0-1} \log\mu(wz_j) - \sum_{j=n-k_0+1}^{n} \log\mu(z_j) < \infty.$$

Now, let $\mu(x) \to 1$. For $M > 0$

$$Z_n P(X_{n+1,1}(Z_n) > Mz_{n+1}|Z_n)I(Z_n/z_n \to W)$$
$$\leqq Z_n z_{n+1}^{-1} M^{-1}\mu(Z_n)I(Z_n/z_n \to W) \to WM^{-1}I(Z_n/z_n \to W)$$

and

$$P\left(\sum_{j=1}^{Z_n} X_{n+1,j}(Z_n) > Mz_{n+1}|Z_n\right)I(Z_n/z_n \to W)$$
$$\geqq \left(W - W^2M^{-1} + o(1)\right)z_n P(Y > Mz_{n+1})I(Z_n/z_n \to W).$$

Therefore,

$$\infty > I(Z_n/z_n \to W < M) \sum_{k=1}^{\infty} P\left(\sum_{j=1}^{Z_k} X_{k+1,j} > Mz_{k+1}|Z_k\right)$$
$$\geqq I(Z_n/z_n \to W < M) \sum_{j=1}^{\infty} P(j < Y \leqq j+1)$$
$$\cdot \sum_k z_k I(z_k \leqq j/M)\left(W - W^2M^{-1} + o(1)\right).$$

Now, by (2.7),

$$\int_0^{h^{-1}(j)} h(t)\,dt = \int_{x_0}^{j} (\log\mu(t))^{-1}\,dt \geqq (\log\mu(j))^{-1}(1 - \alpha^* + o(1))^{-1},$$

and we obtain

$$\infty > I(Z_n/z_n \to W)I(0 < W < M)E(Y/(\mu(Y) - 1))/O(1).$$

Hence, as M is arbitrary, $P(Z_n/z_n \to W > 0) > 0$ implies $E(Y/(\mu(Y) - 1)) < \infty$. If in addition (2.10) is valid we can show similar to (4.4) that

$$h^{-1}(x) = O(1)/(\mu(x) - 1).$$

Thus, $E(Y/(\mu(Y) - 1)) < \infty$ just implies $E(Yh^{-1}(Y)) < \infty$.

PROOF OF THEOREM 4. Choose $0 < \varepsilon < \mu - 1$. As $\mu + \varepsilon \geqq \mu(n) \geqq \mu - \varepsilon$, for all n sufficiently large, we obtain by Theorem 1, considering Remark 4,

$$\liminf Z_n(\mu - \varepsilon)^{-n} > 0, \qquad \limsup Z_n(\mu + \varepsilon)^{-n} < \infty \quad \text{a.s. on } \{Z_n \to \infty\}. \tag{4.10}$$

$\varepsilon > 0$ arbitrary proves our first statement. Now, write

$$Z_{n+1} = I(Z_n > 0) Z_n \mu(Z_n) \left(1 + (Z_n \mu(Z_n))^{-1} \sum_{j=1}^{Z_n} X_{n+1,j}(Z_n) - \mu(Z_n) \right). \tag{4.11}$$

We can prove similarly as in the proof of Theorem 1 that

$$\sum_{n=0}^{\infty} \log Z_{n+1} - \log Z_n \mu(Z_n) < \infty \quad \text{a.s. on } \{Z_n \to \infty\},$$

where we use (2.11) and (4.10). Hence, regarding (4.11),

$$Z_n(\omega)\mu^{-n} \to W, \qquad 0 < W < \infty, \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \log \mu(Z_n(\omega))/\mu \text{ converges.}$$

This proves the second statement of Theorem 4.

For the last result assume $\mu(k) \uparrow \mu$, the other case is similar. Due to the monotonicity we obtain

$$\sum_{n=1}^{\infty} \log \mu - \log \mu(Z_n) < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \mu - \mu(Z_n) < \infty.$$

Assume $Z_n \mu^{-n}$ converges to a positive limit on a set A with positive probability. This is equivalent to $\sum_{n=1}^{\infty} \mu - \mu(Z_n) < \infty$ on A. Hence, using the monotonicity of $\mu(x)$

$$\sum_{n=1}^{\infty} \mu - \mu(\mu^n w) < \infty \quad \text{for some } w > 0.$$

As $\operatorname{card}\{n | \mu^j w \leqq n \leqq \mu^{j+1} w\} \leqq \mu^j w(\mu - 1)$ we obtain, using again the monotonicity of $\mu(x)$,

$$\infty > \sum_{j=1}^{\infty} \mu^{-j-1} w^{-1} \sum_n (\mu - \mu(n)) I(\mu^j w \leqq n < \mu^{j+1} w) \geqq \sum_{n=1}^{\infty} (\mu - \mu(n))/n.$$

Similarly,

$$\infty > \sum_{n=1}^{\infty} (\mu - \mu(n))/n \geqq \sum_{n=1}^{\infty} \mu - \mu((\mu - \varepsilon)^n),$$

and, using (4.10), this implies

$$\sum_{j=1}^{\infty} \mu - \mu(Z_j) < \infty \quad \text{on } \{Z_n \to \infty\}.$$

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