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SECOND-ORDER APPROXIMATION IN THE CONDITIONAL CENTRAL LIMIT THEOREM

BY DIETER LANDERS AND LOTHAR ROGGE

University of Cologne and University of Duisburg

Let X_n, $n \in \mathbb{N}$ be i.i.d. with mean 0 and variance 1. Let $B \in \sigma(X_n: n \in \mathbb{N})$ be a set such that its distances from the σ-fields $\sigma(X_1, \dots, X_n)$ are of order $O(1/n(\lg n)^{2+\varepsilon})$ for some $\varepsilon > 0$. We prove that for those B the conditional probabilities $P((1/\sqrt{n})\sum_{i=1}^{n} X_i \le t|B)$ can be approximated by a modified Edgeworth expansion up to order $O(1/n)$. An example shows that this is not true any more if the distances of B from $\sigma(X_1, \dots, X_n)$ are only of order $O(1/n(\lg n)^2)$.

1. Introduction and notation. Let X_n, $n \in \mathbb{N}$ be a sequence of i.i.d. real valued random variables with mean 0 and variance 1. Put $S_n = \sum_{i=1}^{n} X_i$ and $S_n^* = (1/\sqrt{n})\sum_{i=1}^{n} X_i$. The conditional central limit theorem of Renyi [5] states that for all $B \in \sigma(X_n: n \in \mathbb{N})$ with $P(B) > 0$ there holds

$$P(S_n^* \le t|B) - \Phi(t) \to_{n \in \mathbb{N}} 0, \qquad (1)$$

where Φ is the distribution function of the standard normal distribution. The conditional central limit theorem plays an important role in the theory of random summation, in random walk problems, in sequential estimation or in the field of Monte Carlo methods. Therefore it seems desirable to have for the conditional central limit theorem a comparable asymptotic theory as there is available for the classical central limit theorem (i.e., $B = \Omega$) by the theorem of Berry–Esseen or—a higher order of approximation—by asymptotic expansions. In [2], Example 1, it was shown that by suitable sets B you can make the convergence order in (1) as bad as you want. Hence "nice" convergence orders and especially asymptotic expansions can only exist for "nice" sets B. In [4] it was shown, that the distances of B from the σ-fields $\mathscr{A}_n := \sigma(X_1, \dots, X_n)$ play an essential role for the convergence order in the conditional central limit theorem. If e.g., $B \in \sigma(X_n: n \in \mathbb{N})$ is such that

$$d(B, \mathscr{A}_n) := \inf_{A \in \mathscr{A}_n} P(A \Delta B) = O\left(\frac{1}{n^{1/2}(\lg n)^{3/2+\varepsilon}}\right)$$

with some $\varepsilon > 0$, then $|P(S_n^* \le t|B) - \Phi(t)| = O(n^{-1/2})$; i.e., the Berry–Esseen order appears (see Corollary 3 of [4]). If, however, $d(B, \mathscr{A}_n) = O(1/n^{1/2}(\lg n)^{3/2})$ one cannot obtain the Berry–Esseen order $O(n^{-1/2})$ any more (see Example 5 in [4]).

In this paper we look for those sets B such that the conditional probabilities $P(S_n^* \le t|B)$ admit an asymptotic expansion up to order $O(1/n)$. If $E(X_1^4) < \infty$

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and if Cramér's condition is fulfilled, i.e., $\limsup_{|t|\to\infty}|E(e^{itX_1})| < 1$, then

$$d(B, \mathscr{A}_n) = O\left(\frac{1}{n(\lg n)^{2+\varepsilon}}\right) \quad \text{for some } \varepsilon > 0 \tag{2}$$

implies that

$$P(S_n^* \le t|B) = \Phi(t) + \frac{\varphi(t)}{\sqrt{n}}(Q_1(t) - a) + O\left(\frac{1}{n}\right) \tag{3}$$

where

$$\varphi(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}, \qquad Q_1(t) = -\frac{E(X_1^3)}{6}(t^2 - 1)$$

and $a = \lim_{n\in\mathbb{N}} E(S_n|B)$ (see Corollary 2). Example 3 shows that this result is not true any more if $\varepsilon > 0$ is weakened in (2) to $\varepsilon = 0$.

Observe that the asymptotic expansion given in (3) differs from the classical Edgeworth expansion by the constant $a = a(B)$; the Edgeworth polynomial $Q_1(t)$ has to be modified—depending on the set B—to the polynomial $Q_1(t) - a(B)$. For all B with $a(B) = 0$ we consequently get the classical Edgeworth expansion.

2. The results. The following theorem is the main result of this paper. If g is a bounded measurable function we use the notation

$$d_1(g, \mathscr{A}_n) = \inf\{E(|g - h|): h\ \mathscr{A}_n\text{-measurable}\}$$

i.e., $d_1(g, \mathscr{A}_n)$ is the $\|\ \|_1$-distance of g from the subspace of all $\mathscr{A}_n = \sigma(X_1, \dots, X_n)$ measurable integrable functions. It is well known that $d_1(g, \mathscr{A}_n) \to 0$ for each $\sigma(X_n: n \in \mathbb{N})$-measurable integrable function g.

We write in the following $E(S_n^* \le t, g)$ instead of $E(g \cdot 1_{\{S_n^* \le t\}})$.

THEOREM 1. *Let X_n, $n \in \mathbb{N}$ be* i.i.d. *with mean* 0, *variance* 1, *and $E(X_1^4) < \infty$. Assume that Cramér's condition is fulfilled. Let g be a bounded measurable function such that*

$$d_1(g, \mathscr{A}_n) = O\left(\frac{1}{n(\lg n)^{2+\varepsilon}}\right) \quad \text{for some } \varepsilon > 0.$$

Then

$$\sup_{t\in\mathbb{R}}\left|E(S_n^* \le t, g) - \Phi(t)E(g) - \varphi(t)\frac{1}{\sqrt{n}}[E(g)Q_1(t) - a]\right| = O\left(\frac{1}{n}\right),$$

where

$$a = \lim_{n\in\mathbb{N}} E(S_n g), \qquad \varphi(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}$$

and

$$Q_1(t) = -\frac{E(X_1^3)}{6}\cdot(t^2 - 1).$$

PROOF. In the proof we use the symbol c for a general constant which may depend on the function g and the distribution of X_1. As $d_1(g^+, \mathscr{A}_n)$, $d_1(g^-, \mathscr{A}_n)$

$\leq d_1(g, \mathscr{A}_n)$, we assume w.l.g. $g \geq 0$. We shall prove that

$$\text{(1)}\qquad \sup_{|t| \leq 2\sqrt{\lg n}} \left| E(S_n^* \leq t, g) - \Phi(t)E(g) - \varphi(t)\frac{1}{\sqrt{n}}[E(g)Q_1(t) - a] \right| = O\left(\frac{1}{n}\right).$$

Let us show first that this implies the assertion. Obviously,

$$\text{(2)}\qquad \sup_{|t| \geq 2\sqrt{\lg n}} \left| \varphi(t)\frac{1}{\sqrt{n}}[E(g)Q_1(t) - a] \right| = O\left(\frac{1}{n}\right).$$

Hence the assertion is shown, if we prove

$$\text{(3)}\qquad \sup_{|t| \geq 2\sqrt{\lg n}} |E(S_n^* \leq t, g) - \Phi(t)E(g)| = O\left(\frac{1}{n}\right).$$

We consider the case $t \leq -2\sqrt{\lg n}$ and $t \geq 2\sqrt{\lg n}$. As $\Phi(-2\sqrt{\lg n}) = O(1/n)$ we obtain by (1) and (2), using $g \geq 0$,

$$\begin{aligned}
&\sup_{t \leq -2\sqrt{\lg n}} |E(S_n^* \leq t, g) - \Phi(t)E(g)| \\
&\quad = \sup_{t \leq -2\sqrt{\lg n}} |E(S_n^* \leq t, g)| + O\left(\frac{1}{n}\right) \\
&\quad = E\left(S_n^* \leq -2\sqrt{\lg n}, g\right) + O\left(\frac{1}{n}\right) \\
&\quad = \left|E\left(S_n^* \leq -2\sqrt{\lg n}, g\right) - \Phi(-2\sqrt{\lg n})E(g)\right| + O\left(\frac{1}{n}\right) \\
&\quad =_{(1),(2)} O\left(\frac{1}{n}\right) + O\left(\frac{1}{n}\right) = O\left(\frac{1}{n}\right).
\end{aligned}$$

As $1 - \Phi(2\sqrt{\lg n}) = O(1/n)$ we furthermore obtain by (1) and (2), using $g \geq 0$,

$$\begin{aligned}
&\sup_{t \geq 2\sqrt{\lg n}} |E(S_n^* \leq t, g) - \Phi(t)E(g)| \\
&\quad = \sup_{t \geq 2\sqrt{\lg n}} |E(S_n^* > t, g) - (1 - \Phi(t))E(g)| \\
&\quad = E\left(S_n^* > 2\sqrt{\lg n}, g\right) + O\left(\frac{1}{n}\right) \\
&\quad = \left|E\left(S_n^* > 2\sqrt{\lg n}, g\right) - \left(1 - \Phi(2\sqrt{\lg n})\right)E(g)\right| + O\left(\frac{1}{n}\right) \\
&\quad = \left|E\left(S_n^* \leq 2\sqrt{\lg n}, g\right) - \Phi(2\sqrt{\lg n})E(g)\right| + O\left(\frac{1}{n}\right) \\
&\quad =_{(1),(2)} O\left(\frac{1}{n}\right) + O\left(\frac{1}{n}\right) = O\left(\frac{1}{n}\right).
\end{aligned}$$

Thus we have shown (3) and hence it suffices to prove (1).

There exist $\mathscr{A}_\nu$-measurable g_ν (see [6]) such that

$$E(|g - g_\nu|) = d_1(g, \mathscr{A}_\nu) =: \varepsilon_\nu. \tag{4}$$

Let $\mathbb{N}_1 := \{2^k\colon k \in \mathbb{N}\}$ and put

$$h_2 = g_2 \quad \text{and} \quad h_\nu = g_\nu - g_{2k} \quad \text{for } \nu \text{ with } 2^k < \nu \leq 2^{k+1}. \tag{5}$$

Then h_ν is $\mathscr{A}_\nu$-measurable and we obtain by assumption

$$E(|h_\nu|) \leq 2\varepsilon_{2^k} \leq \frac{c}{\nu(\lg \nu)^{2+\varepsilon}}, \qquad 2^k < \nu \leq 2^{k+1}. \tag{6}$$

Let $j(n) = [n/\lg n]$ and put $N_n = \{\nu \in \mathbb{N}_1\colon \nu \leq j(n)\} \cup \{j(n)\}$. Since $g = \sum_{\nu \in N_n} h_\nu + g - g_{j(n)}$ and since by assumption $E(|g - g_{j(n)}|) = \varepsilon_{j(n)} = O(1/n)$, we obtain

$$E(S_n^* \leq t, g) = \sum_{\nu \in N_n} E(S_n^* \leq t, h_\nu) + O\left(\frac{1}{n}\right) \quad \text{uniformly in } t \in \mathbb{R}. \tag{7}$$

If $\nu < n$ then $\omega \to F_{n-\nu}(\sqrt{n/(n-\nu)} \cdot t - (1/\sqrt{n-\nu})S_\nu(\omega))$ is a version of $P(S_n^* \leq t|\mathscr{A}_\nu)$, where F_n is the distribution function S_n^*. Since h_ν is $\mathscr{A}_\nu$-measurable we obtain that

$$E(S_n^* \leq t, h_\nu) = \int h_\nu(\omega) F_{n-\nu}\left(\sqrt{\frac{n}{n-\nu}}\, t - \frac{1}{\sqrt{n-\nu}} S_\nu(\omega)\right) P(d\omega). \tag{8}$$

Let $K_n(t) = \Phi(t) + (\varphi(t)/\sqrt{n})Q_1(t)$. Since Cramér's condition is fulfilled, we have by the classical asymptotic expansion that

$$\sup_{y \in \mathbb{R}} |F_{n-\nu}(y) - K_{n-\nu}(y)| = O\left(\frac{1}{n-\nu}\right) \quad \text{for } n - \nu \to \infty$$

and hence

$$\sup_{\nu \in N_n,\, y \in \mathbb{R}} |F_{n-\nu}(y) - K_{n-\nu}(y)| = O\left(\frac{1}{n}\right). \tag{9}$$

From (7) and (8) we obtain with $D_n = F_n - K_n$

$$\begin{aligned} E(S_n^* \leq t, g) = {} & \sum_{\nu \in N_n} \int h_\nu(\omega) K_{n-\nu}\left(\sqrt{\frac{n}{n-\nu}}\, t - \frac{1}{\sqrt{n-\nu}} S_\nu(\omega)\right) P(d\omega) \\ & + \sum_{\nu \in N_n} \int h_\nu(\omega) D_{n-\nu}\left(\sqrt{\frac{n}{n-\nu}}\, t - \frac{1}{\sqrt{n-\nu}} S_\nu(\omega)\right) P(d\omega) + O\left(\frac{1}{n}\right) \end{aligned} \tag{10}$$

uniformly in $t \in \mathbb{R}$.

By (9) we have for all $t \in \mathbb{R}$

$$\begin{aligned} & \left| \sum_{\nu \in N_n} \int h_\nu(\omega) D_{n-\nu}\left(\sqrt{\frac{n}{n-\nu}}\, t - \frac{1}{\sqrt{n-\nu}} S_\nu(\omega)\right) P(d\omega) \right| \\ & \leq \frac{c}{n} \sum_{\nu \in N_n} E(|h_\nu|). \end{aligned} \tag{11}$$

By (6) we have

$$\sum_{\nu \in N_n} E(|h_\nu|) = O(1). \tag{12}$$

Then (10), (11), and (12) imply

$$\begin{aligned} E(S_n^* \le t, g) = \sum_{\nu \in N_n} \int h_\nu(\omega) K_{n-\nu}\left(\sqrt{\frac{n}{n-\nu}}\, t - \frac{1}{\sqrt{n-\nu}} S_\nu(\omega)\right) P(d\omega) \\ + O\left(\frac{1}{n}\right) \end{aligned} \tag{13}$$

uniformly in $t \in \mathbb{R}$. Consequently, it suffices to prove uniformly in $|t| \le 2\sqrt{\lg n}$

$$\sum_{\nu \in N_n} \int h_\nu(\omega) K_{n-\nu}(t) P(d\omega) = \Phi(t)E(g) + \frac{\varphi(t)}{\sqrt{n}} E(g) Q_1(t) + O\left(\frac{1}{n}\right), \tag{14}$$

$$\begin{aligned} \sum_{\nu \in N_n} \int h_\nu(\omega) \left[K_{n-\nu}\left(\sqrt{\frac{n}{n-\nu}}\, t - \frac{1}{\sqrt{n-\nu}} S_\nu(\omega)\right) - K_{n-\nu}(t) \right] P(d\omega) \\ = -\frac{a}{\sqrt{n}} \varphi(t) + O\left(\frac{1}{n}\right). \end{aligned} \tag{15}$$

PROOF OF (14). We have

$$\begin{aligned} K_{n-\nu}(t) &= \Phi(t) + \frac{\varphi(t)}{\sqrt{n-\nu}} Q_1(t) \\ &= \Phi(t) + \frac{\varphi(t)}{\sqrt{n}} Q_1(t) \left(1 - \frac{\nu}{n}\right)^{-1/2} \\ &= \Phi(t) + \frac{\varphi(t)}{\sqrt{n}} Q_1(t) \left[1 + \sum_{k=1}^{\infty} \binom{-1/2}{k} \left(-\frac{\nu}{n}\right)^k \right]. \end{aligned}$$

Let $r(x) = (1-x)^{-1/2} - 1 = \sum_{k=1}^{\infty} \binom{-1/2}{k} (-x)^k$ for $|x| < 1$. Then $|r(x)| \le c|x|$ for $|x| \le \frac{1}{2}$. We obtain uniformly in $t \in \mathbb{R}$

$$\begin{aligned} \sum_{\nu \in N_n} K_{n-\nu}(t) E(h_\nu) &= \Phi(t) E(g_{j(n)}) + \sum_{\nu \in N_n} \frac{\varphi(t)}{\sqrt{n}} Q_1(t) \left(1 + r\left(\frac{\nu}{n}\right)\right) E(h_\nu) \\ &= \Phi(t) E(g) + \frac{\varphi(t)}{\sqrt{n}} Q_1(t) E(g) \\ &\quad + \frac{\varphi(t)}{\sqrt{n}} Q_1(t) \sum_{\nu \in N_n} r\left(\frac{\nu}{n}\right) E(h_\nu) + O\left(\frac{1}{n}\right). \end{aligned}$$

Therefore (14) is shown—even uniformly in $t \in \mathbb{R}$—if we prove

$$\frac{1}{\sqrt{n}} \sum_{\nu \in N_n} r\left(\frac{\nu}{n}\right) E(h_\nu) = O\left(\frac{1}{n}\right). \tag{16}$$

As $|r(\nu/n)| \leq c(\nu/n)$ if $(\nu/n) \leq \frac{1}{2}$, (16) follows from formula (F1) (see end of the proof).

PROOF OF (15). Let

$$u = u_{t,n,\nu}(\omega) = \sqrt{\frac{n}{n-\nu}}\,t - \frac{1}{\sqrt{n-\nu}} S_\nu(\omega).$$

By the Taylor expansion

$$K_{n-\nu}(u) - K_{n-\nu}(t) = (u-t)K'_{n-\nu}(t) + \tfrac{1}{2}(u-t)^2 K''_{n-\nu}(\xi)$$

with $\xi = \xi_{t,n,\nu}(\omega) \in [u, t]$. We prove that uniformly in $|t| \leq 2\sqrt{\lg n}$

$$(15)_1 \qquad \sum_{\nu \in N_n} K'_{n-\nu}(t) \int h_\nu(\omega)(u_{t,n,\nu}(\omega) - t) P(d\omega) = -a\frac{\varphi(t)}{\sqrt{n}} + O\left(\frac{1}{n}\right)$$

and

$$(15)_2 \qquad \sum_{\nu \in N_n} \int h_\nu(\omega)(u_{t,n,\nu}(\omega) - t)^2 K''_{n-\nu}(\xi(\omega)) P(d\omega) = O\left(\frac{1}{n}\right).$$

Obviously, $(15)_1$ and $(15)_2$ imply (15).

PROOF OF $(15)_1$. We have

$$K'_{n-\nu}(t) = \varphi(t) + \frac{1}{\sqrt{n-\nu}}(\varphi Q_1)'(t)$$

and

$$u_{t,n,\nu}(\omega) - t = t\left(\sqrt{\frac{n}{n-\nu}} - 1\right) - \frac{1}{\sqrt{n-\nu}} S_\nu(\omega)$$
$$= tr\left(\frac{\nu}{n}\right) - \frac{1}{\sqrt{n-\nu}} S_\nu(\omega).$$

To prove $(15)_1$ it suffices to prove uniformly in $|t| \leq 2\sqrt{\lg n}$ that

$$(17) \qquad t\varphi(t) \sum_{\nu \in N_n} r\left(\frac{\nu}{n}\right) E(h_\nu) = O\left(\frac{1}{n}\right),$$

$$(18) \qquad -\varphi(t) \cdot \sum_{\nu \in N_n} \frac{1}{\sqrt{n-\nu}} E(h_\nu S_\nu) = -a\frac{\varphi(t)}{\sqrt{n}} + O\left(\frac{1}{n}\right),$$

$$(19) \qquad t(\varphi Q_1)'(t) \sum_{\nu \in N_n} \frac{1}{\sqrt{n-\nu}} r\left(\frac{\nu}{n}\right) E(h_\nu) = O\left(\frac{1}{n}\right),$$

$$(20) \qquad -(\varphi Q_1)'(t) \sum_{\nu \in N_n} \frac{1}{n-\nu} E(h_\nu S_\nu) = O\left(\frac{1}{n}\right).$$

As $|r(\nu/n)| \le c(\nu/n)$ if $(\nu/n) \le \frac{1}{2}$, (17) and (19) follow from (F1) (uniformly in $t \in \mathbb{R}$).

PROOF OF (18). As $(1/\sqrt{n-\nu}) = (1/\sqrt{n})\,(1 + r(\nu/n))$ we have

$$\sum_{\nu \in N_n} \frac{1}{\sqrt{n-\nu}} E(h_\nu S_\nu) = \frac{1}{\sqrt{n}} \sum_{\nu \in N_n} E(h_\nu S_\nu) + \frac{1}{\sqrt{n}} \sum_{\nu \in N_n} r\left(\frac{\nu}{n}\right) E(h_\nu S_\nu).$$

Hence (18) is shown if we prove

$$\frac{1}{\sqrt{n}} \sum_{\nu \in N_n} E(h_\nu S_\nu) = \frac{a}{\sqrt{n}} + O\left(\frac{1}{n}\right), \tag{21}$$

$$\frac{1}{\sqrt{n}} \sum_{\nu \in N_n} r\left(\frac{\nu}{n}\right) E(h_\nu S_\nu) = O\left(\frac{1}{n}\right). \tag{22}$$

As h_ν is $\mathscr{A}_\nu$-measurable and $E(X_1) = 0$ we have

$$\sum_{\nu \in N_n} E(h_\nu S_\nu) = \sum_{\nu \in N_n} E(h_\nu S_{j(n)}) = E(g_{j(n)} \cdot S_{j(n)}).$$

By Lemma 4 we have

$$E(g_{j(n)} \cdot S_{j(n)}) - a = O\left(\frac{1}{\sqrt{j(n)} \lg j(n)}\right) = O\left(\frac{1}{\sqrt{n}}\right).$$

Hence we obtain (21). Since h_ν are uniformly bounded we have

$$\begin{aligned}
\left|\sum_{\nu \in N_n} r\left(\frac{\nu}{n}\right) E(h_\nu S_\nu)\right| &\le c\frac{1}{n} \sum_{\nu \in N_n} \nu E(|h_\nu S_\nu|) \\
&\le c\frac{1}{\sqrt{n}} \sum_{\nu \in N_n} \sqrt{\nu}\, E(|h_\nu S_\nu|) \\
&\le c\frac{1}{\sqrt{n}} \sum_{\nu \in N_n} \left(\nu E(|h_\nu|) - \sqrt{\nu}\, E\left(|h_\nu S_\nu| 1_{\{|S_\nu| > \sqrt{\nu}\}}\right)\right) \\
&\le c\frac{1}{\sqrt{n}} \sum_{\nu \in N_n} \left(\nu E(|h_\nu|) + E\left(|h_\nu S_\nu^2|\right)\right) \\
&= O\left(\frac{1}{\sqrt{n}}\right) \quad \text{according to (F1) and (F2).}
\end{aligned}$$

This implies (22) and hence (18) is shown.

PROOF OF (20). We have

$$\sum_{\nu \in N_n} E(|h_\nu S_\nu|) \le \sum_{\nu \in N_n} E(|h_\nu|) + \sum_{\nu \in N_n} E\left(|h_\nu S_\nu^2|\right) = O(1) \quad \text{by F(1) and (F2).}$$

This proves (20). Consequently, $(15)_1$ is shown.

PROOF OF $(15)_2$. Since $\sup\{|K''_{n-\nu}(\xi)|: \xi \in \mathbb{R}, n \in \mathbb{N}, \nu < n\} < \infty$ is suffices to show

$$\sum_{\nu \in N_n} \int |h_\nu(\omega)||u_{t,n,\nu}(\omega) - t|^2 P(d\omega) = O\left(\frac{1}{n}\right) \tag{23}$$

uniformly in $|t| \le 2\sqrt{\lg n}$. We have

$$(u_{t,n,\nu}(\omega) - t)^2 \le 2\left(\frac{\nu}{n-\nu}\right)^2 t^2 + 2\frac{S_\nu^2(\omega)}{n-\nu}. \tag{24}$$

According to (F1) there holds for $|t| \le 2\sqrt{\lg n}$

$$\begin{aligned} \sum_{\nu \in N_n} \left(\frac{\nu}{n-\nu}\right)^2 t^2 E(|h_\nu|) &\le c\frac{1}{n^2}\lg n \cdot \sum_{\nu \in N_n} \nu^2 E(|h_\nu|) \\ &\le c\frac{1}{n^2}(\lg n)\frac{n}{\lg n} \sum_{\nu \in N_n} \nu E(|h_\nu|) = O\left(\frac{1}{n}\right). \end{aligned} \tag{25}$$

Furthermore by (F2)

$$\sum_{\nu \in N_n} \frac{1}{n-\nu} E\left(|h_\nu| S_\nu^2\right) = O\left(\frac{1}{n}\right). \tag{26}$$

Now (24)–(26) imply (23), i.e., $(15)_2$.

It remains to prove the following two formulas which we have used in the preceding proof.

$$\sum_{\nu \in N_n} \nu(\lg \nu)^{1+\varepsilon/2} E(|h_\nu|) = O(1), \tag{F1}$$

$$\sum_{\nu \in N_n} E\left(|h_\nu| S_\nu^2\right) = O(1). \tag{F2}$$

PROOF OF (F1). Using (6) we obtain

$$\begin{aligned} \sum_{\nu \in N_n} \nu(\lg \nu)^{1+\varepsilon/2} E(|h_\nu|) &\le_{(6)} c \sum_{\nu \in N_n} \nu(\lg \nu)^{1+\varepsilon/2} \frac{1}{\nu(\lg \nu)^{2+\varepsilon}} \\ &= c \sum_{\nu \in N_n} \frac{1}{(\lg \nu)^{1+\varepsilon/2}} = O(1). \end{aligned}$$

PROOF OF F2. We have

$$\begin{aligned} E\left(|h_\nu S_\nu^2|\right) &\le cE\left(S_\nu^2 1_{\{|S_\nu| > \sqrt{3}\sqrt{\nu}(\lg \nu)^{1/2+\varepsilon/4}\}}\right) + 3\nu(\lg \nu)^{1+\varepsilon/2} E(|h_\nu|) \\ &=: cA_\nu + 3B_\nu. \end{aligned} \tag{27}$$

Furthermore—using $E(|Y|) \leq \sum_{k=0}^{\infty} P\{|Y| > k\}$—

$$A_\nu = 3\nu(\lg \nu)^{1+\varepsilon/2} E\left(\frac{S_\nu^2}{3\nu(\lg \nu)^{1+\varepsilon/2}} 1_{\{S_\nu^2 > 3\nu(\lg \nu)^{1+\varepsilon/2}\}} \right)$$

$$\leq c\nu(\lg \nu)^{1+\varepsilon/2} \sum_{k \in \mathbb{N}} P\left\{S_\nu^2 > k3\nu(\lg \nu)^{1+\varepsilon/2}\right\}$$

$$= c\nu(\lg \nu)^{1+\varepsilon/2} \sum_{k \in \mathbb{N}} P\left\{|S_\nu^*| > \sqrt{3}\sqrt{k}\,(\lg \nu)^{1/2+\varepsilon/4}\right\}$$

and hence by Remark 5

$$\leq c\nu(\lg \nu)^{1+\varepsilon/2} \frac{1}{\nu} \sum_{k \in \mathbb{N}} \frac{1}{k^2(\lg \nu)^{2+\varepsilon}}.$$

Therefore

$$\sum_{\nu \in N_n} A_\nu \leq c \sum_{\nu \in N_n} \frac{1}{(\lg \nu)^{1+\varepsilon/2}} = O(1). \tag{28}$$

Since $\sum_{\nu \in N_n} B_\nu = O(1)$ by (F1), we obtain (F2) by (27) and (28).

REMARK. By similar methods one obtains an $o(n^{-1/2})$ approximation order in Theorem 1 for nonlattice distributions with finite third moment under the weaker condition

$$d(g, \mathscr{A}_n) = O\left(\frac{1}{n^{1/2}(\lg n)^{3/2+\varepsilon}} \right) \quad \text{for some } \varepsilon > 0.$$

COROLLARY 2. *Let X_n, $n \in \mathbb{N}$, be* i.i.d. *with mean* 0, *variance* 1, *and* $E(X_1^4) < \infty$. *Assume that Cramér's condition is fulfilled. Let B be a measurable set with $P(B) > 0$ such that*

$$d(B, \mathscr{A}_n) = O\left(\frac{1}{n(\lg n)^{2+\varepsilon}} \right) \quad \text{for some } \varepsilon > 0.$$

Then

$$\sup_{t \in \mathbb{R}} \left| P\{S_n^* \leq t|B\} - \Phi(t) - \varphi(t)\frac{1}{\sqrt{n}}[Q_1(t) - \hat{a}] \right| = O\left(\frac{1}{n}\right),$$

where $\hat{a} = \lim_{n \in \mathbb{N}} E(S_n|B)$.

PROOF. Apply Theorem 1 with $g = 1_B$ and observe that $d_1(1_B, \mathscr{A}_n) \leq d(B, \mathscr{A}_n)$.

The following example shows that the assertion of the preceding corollary is not true if the assumptions is weakened to $d(B, \mathscr{A}_n) = O(1/n(\lg n)^2)$. The random variables in this example are even standard normally distributed.

EXAMPLE 3. Let X_n, $n \in \mathbb{N}$ be standard normally distributed. Since $P\{S_\nu^* \geq \sqrt{\lg \nu}\} \geq c(1/\sqrt{\nu \lg \nu})$, $\nu \geq 2$, it is easy to see that there exist Borel sets $C_\nu \subset \mathbb{R}^\nu$ with $C_\nu = -C_\nu$ such that for some suitable $\nu_0 \in \mathbb{N}$

$$(1) \qquad B_\nu = (X_1, \dots, X_\nu)^{-1}(C_\nu), \qquad \nu \geq \nu_0, \quad \text{are disjoint,}$$

$$(2) \qquad B_\nu \subset \{|S_\nu^*| \geq \sqrt{\lg \nu}\}, \qquad \nu \geq \nu_0,$$

$$(3) \qquad P(B_\nu) = \frac{1}{\nu^2 (\lg \nu)^2}, \qquad \nu \geq \nu_0.$$

We shall prove that for some $\nu_1 \geq \nu_0$

$$(4) \qquad \Phi(1)P(B_\nu) - P(S_n^* \leq 1, B_\nu) \geq c\frac{1}{n}\frac{1}{\nu \lg \nu}$$

if $\nu_1 \leq \nu \leq n/(\lg n)^2$. Let us show first that (1)–(4) lead to an example of the desired kind. Put $B = \sum_{\nu \geq \nu_1} B_\nu$. Then according to (3) for $n \geq \nu_1$

$$d(B, \mathscr{A}_n) \leq P\left(B \Delta \sum_{\nu=\nu_1}^{n} B_\nu\right) = \sum_{\nu=n+1}^{\infty} P(B_\nu) = O\left(\frac{1}{n(\lg n)^2}\right).$$

Hence the assumption of Corollary 2 is fulfilled with $\varepsilon = 0$. As $C_\nu = -C_\nu$ and $P \circ X_1 = P \circ (-X_1)$ we have $E(X_j 1_{B_\nu}) = 0$ for all j, $\nu \in \mathbb{N}$. Hence $E(S_n 1_{B_\nu}) = 0$ for all n, $\nu \in \mathbb{N}$ whence $\hat{a} = \lim_{n \in \mathbb{N}} E(S_n | B) = 0$. Using (1), (3), and (4) we obtain

$$n(\Phi(1)P(B) - P(S_n^* \leq 1, B)) \to_{n \in \mathbb{N}} \infty.$$

Since $Q_1(1) = 0$ this contradicts the assertion of Corollary 2 (with $t = 1$). Therefore it remains to prove (4).

PROOF OF (4). Since Φ is the distribution function of S_n^*, we have that $\Phi(\sqrt{n/(n-\nu)}\, t - (1/\sqrt{n-\nu}) S_\nu(\omega))$ is a version of $P(S_n^* \leq t | \mathscr{A}_\nu)$. Hence for $\nu < n$

$$A_{n,\nu} := \Phi(1)P(B_\nu) - P(S_n^* \leq 1, B_\nu)$$
$$= \int_{B_\nu} \left[\Phi(1) - \Phi\left(\sqrt{\frac{n}{n-\nu}} - \frac{1}{\sqrt{n-\nu}} S_\nu(\omega)\right)\right] P(d\omega).$$

Put

$$t_{n,\nu}(\omega) = \sqrt{\frac{n}{n-\nu}} - \frac{1}{\sqrt{n-\nu}} S_\nu(\omega);$$

then

$$1 - t_{n,\nu}(\omega) = 1 - \left(1 - \frac{\nu}{n}\right)^{-1/2} + \frac{1}{\sqrt{n-\nu}} S_\nu(\omega)$$
$$= -r\left(\frac{\nu}{n}\right) - \frac{1}{\sqrt{n-\nu}} S_\nu(\omega).$$

Hence we have by the Taylor expansion

$$\Phi(1) - \Phi(t_{n,\nu}(\omega)) = \left[-r\left(\frac{\nu}{n}\right) + \frac{1}{\sqrt{n-\nu}} S_\nu(\omega)\right]\Phi'(1)$$
$$- \frac{1}{2}\left[-r\left(\frac{\nu}{n}\right) + \frac{1}{\sqrt{n-\nu}} S_\nu(\omega)\right]^2 \Phi''(1)$$
$$+ \frac{1}{6}\left[-r\left(\frac{\nu}{n}\right) + \frac{1}{\sqrt{n-\nu}} S_\nu(\omega)\right]^3 \Phi'''(\xi_{n,\nu}(\omega))$$

with $\xi_{n,\nu}(\omega) \in [1, t_{n,\nu}(\omega)]$. As $\Phi''(1) = -\Phi'(1)$ and $\int_{B_\nu} S_\nu(\omega)P(d\omega) = 0$, we obtain for $\nu \le n/2$

$$A_{n,\nu} = -r\left(\frac{\nu}{n}\right)\Phi'(1)P(B_\nu) + \frac{1}{2}\Phi'(1)\int_{B_\nu}\left[-r\left(\frac{\nu}{n}\right) - \frac{1}{\sqrt{n-\nu}} S_\nu(\omega)\right]^2 P(d\omega)$$
$$+ \frac{1}{6}\int_{B_\nu}\left[-r\left(\frac{\nu}{n}\right) + \frac{1}{\sqrt{\nu-\nu}} S_\nu(\omega)\right]^3 \Phi'''(\xi_{n,\nu}(\omega))P(d\omega)$$
$$\ge -r\left(\frac{\nu}{n}\right)\Phi'(1)P(B_\nu) + \frac{1}{2}\Phi'(1)\frac{1}{n}\int_{B_\nu} S_\nu^2(\omega)P(d\omega)$$
$$- \frac{c}{6}\int_{B_\nu}\left[\frac{\nu}{n} + \sqrt{\frac{\nu}{n}}\,|S_\nu^*(\omega)|\right]^3 P(d\omega).$$

Since S_ν^* is standard normally distributed, we obtain using (2) by a rather direct computation

$$\int_{B_\nu} |S_\nu^*(\omega)|^3 P(d\omega) \le c(\lg \nu)^{3/2} P(B_\nu).$$

Hence by (2)

$$A_{n,\nu} \ge -c\frac{\nu}{n}P(B_\nu) + \frac{1}{2}\Phi'(1)\frac{\nu \lg \nu}{n}P(B_\nu) - c\left(\frac{\nu}{n}\right)^{3/2}(\lg \nu)^{3/2}P(B_\nu)$$
$$\ge c\frac{\nu \lg \nu}{n}P(B_\nu)$$

if $\nu_1 \le \nu \le n/(\lg n)^2$. According to (3) this implies (4).

LEMMA 4. *Let X_n, $n \in \mathbb{N}$ be* i.i.d. *with mean* 0, *variance* 1, *and* $E(X_1^4) < \infty$. *Let g be a bounded measurable function and g_n be $\mathscr{A}_n$-measurable with* $E(|g - g_n|) = d_1(g, \mathscr{A}_n)$. *Assume that*

$$d_1(g, \mathscr{A}_n) = O\left(\frac{1}{n \lg n}\right).$$

Then $\sqrt{n \lg n}\,(E(S_n g_n) - a) = O(1)$ *where* $a = \lim_{n \in \mathbb{N}} E(S_n g)$.

PROOF. We show firs. that

$$(1)\qquad \sup_{k>n} \sqrt{n \lg n}\, |E(S_k g_k) - E(S_n g_n)| = O(1).$$

Let $a_{kn} := E(S_k g_k) - E(S_n g_n)$. As g_n is $\mathscr{A}_n$-measurable and hence independent of $S_k - S_n$ for $k \geq n$ we have $E(S_k g_n) = E(S_n g_n)$. Therefore

$$(2)\qquad a_{kn} = E(S_k(g_k - g_n)), \qquad k \geq n.$$

For $k > n$ there exist $j \in \mathbb{N}_0$ and $0 \leq \nu < n2^j$ such that $k = n2^j + \nu$. Hence by (2)

$$(3)\qquad a_{kn} = E(S_{n2^j+\nu}(g_{n2^j+\nu} - g_n))$$
$$= E(S_{n2^j+\nu}(g_{n2^j+\nu} - g_{n2^j})) + \sum_{\mu=1}^{j} E(S_{n2^\mu}(g_{n2^\mu} - g_{n2^{\mu-1}})).$$

Let $m \leq r \leq 2m$. Then

$$(4)\qquad |E(S_r(g_r - g_m))| \leq E(|S_r|\,|g_r - g_m|)$$
$$\leq cE\left(|S_r| 1_{\{|S_r^*| > \sqrt{3 \lg r}\}}\right) + \sqrt{3r \lg r}\, E(|g_r - g_m|).$$

We have using our assumption

$$(5)\qquad \sqrt{r \lg r}\, E(|g_r - g_m|) \leq 2\sqrt{r \lg r}\, d_1(g, \mathscr{A}_m) \leq c \frac{1}{\sqrt{m \lg m}}.$$

Furthermore—using Remark 5—

$$(6)\qquad E\left(|S_r| 1_{\{|S_r^*| > \sqrt{3 \lg r}\}}\right) \leq c\sqrt{r \lg r} \sum_{k \in \mathrm{N}} P\{|S_r^*| > \sqrt{3}\, k \sqrt{\lg r}\}$$
$$\leq c\sqrt{r \lg r}\, \frac{1}{r} \sum_{k \in \mathbb{N}} \frac{1}{k^4 (\lg r)^2} \leq c \frac{1}{\sqrt{m \lg m}}.$$

Together with (4) and (5) this implies for all $m \in \mathbb{N}$

$$(7)\qquad \max_{m \leq r \leq 2m} \sqrt{m \lg m}\, |E(S_r(g_r - g_m))| \leq c.$$

From (7) and (3) we obtain for all k, n with $k > n$

$$\sqrt{n \lg n}\, |a_{kn}| \leq_{(3),(7)} c + \sum_{\mu=1}^{j} \sqrt{n \lg n}\, |E(S_{n2^\mu}(g_{n2^\mu} - g_{n2^{\mu-1}}))|$$
$$\leq_{(7)} c + \sum_{\mu=1}^{j} \sqrt{n \lg n} \cdot \frac{c}{\sqrt{n 2^{\mu-1} \lg(n 2^{\mu-1})}}$$
$$\leq c + c \sum_{\mu=1}^{j} \frac{1}{\sqrt{2^{\mu-1}}} \leq c.$$

By (2) this implies (1).

From (1) we obtain $\sqrt{n \lg n}\,(E(S_n g_n) - a) = O(1)$ with $a = \lim_{k \in \mathbb{N}} E(S_k g_k)$ [this limit exists by (1)]. Therefore it remains to show that

$$|E(S_n g_n) - E(S_n g)| \to 0. \tag{8}$$

We have by (6)

$$|E(S_n g_n) - E(S_n g)| \le E\Big(|g_n - g|\, |S_n| 1_{\{|S_n| \le \sqrt{3n \lg n}\}}\Big) + cE\Big(|S_n| 1_{\{|S_n^*| > \sqrt{3 \lg n}\}}\Big)$$
$$\le \sqrt{3n \lg n}\, E(|g_n - g|) + c\frac{1}{\sqrt{n \lg n}}.$$

Consequently, (8) follows from our assumption.

The last inequality of the preceding proof shows that also

$$\sqrt{n \lg n}\,(E(S_n g) - a) = O(1).$$

For the sake of completeness we cite the following result (see e.g., [1], Theorem 17.11).

REMARK 5. Let X_n, $n \in \mathbb{N}$ be i.i.d. with mean 0, variance 1, and $E(X_1^4) < \infty$. Then for all $t \ge \sqrt{3 \lg n}$

$$P\{|S_n^*| \ge t\} \le c\frac{1}{nt^4}.$$

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MATHEMATICAL INSTITUTE
UNIVERSITY OF COLOGNE
WEYERTAL 86-90
D-5000 COLOGNE
WEST GERMANY

FACHBEREICH 11
UNIVERSITY-GH DUISBURG
LOTHARSTRAßE 65
D = 4100 DUISBURG
WEST GERMANY