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ON THE DISTRIBUTIONS OF SUMS OF SYMMETRIC RANDOM VARIABLES AND VECTORS

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Let F be a probability distribution on $\mathbb{R}$. Then there exist symmetric (about zero) random variables X and Y whose sum has distribution F if and only if F has mean zero or no mean (finite or infinite). Now suppose F is a probability distribution on $\mathbb{R}^n$. There exist spherically symmetric (about the origin) random vectors X and Y whose sum $X + Y$ has distribution F if and only if all the one-dimensional distributions obtained by projecting F onto lines through the origin have either mean zero or no mean.

1. Introduction and summary. Call a probability distribution F on $\mathbb{R}$ *balanced* if it either has mean zero or no mean (finite or infinite). Simons (1976) showed that a necessary condition for F to be the distribution of a sum $X + Y$, where X and Y are (possibly dependent) random variables symmetric about zero, is that F be balanced. Section 2 will show that this condition is also sufficient. An easy corollary is that for *any* distribution F, there exist three symmetric (about zero) random variables X, Y, and Z whose sum has distribution F.

Section 3 considers the distributions of sums of spherically symmetric (about the origin) random vectors in $\mathbb{R}^n$. Let F be a probability distribution on $\mathbb{R}^n$. We will say that F is *balanced* if it is balanced in the one-dimensional sense in every direction, that is, if all the one-dimensional distributions obtained by projecting F onto lines through the origin are balanced. Theorem 3.1 says that there exist spherically symmetric random vectors $\mathbf{X}$ and $\mathbf{Y}$ whose sum $\mathbf{X} + \mathbf{Y}$ has distribution F if and only if F is balanced in the multidimensional sense. Again, an easy corollary is that for any distribution F on $\mathbb{R}^n$, there exist three spherically symmetric random vectors $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ whose sum $\mathbf{X} + \mathbf{Y} + \mathbf{Z}$ has distribution F.

An obvious corollary of the one-dimensional result is that a sum $X + Y$ of symmetric (about 0) random variables X and Y can be symmetrically distributed about $C \neq 0$. An example of such behavior has previously been exhibited by Chen and Shepp (1983). In their example the summands and the sum all have Cauchy distributions. See also Ferguson (1962), who showed how to obtain an n-dimensional random vector $\mathbf{X}$ for which the scalar product $\mathbf{t}'\mathbf{X}$ is a Cauchy random variable for every $\mathbf{t} \in \mathbb{R}^n$, but for which there does not exist a $\mathbf{b} \in \mathbb{R}^n$, such that $\mathbf{t}'\mathbf{b}$ is the center of symmetry of the $\mathbf{t}'\mathbf{X}$ distribution for every $\mathbf{t}$. In Section 4 the

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Ferguson–Chen–Shepp construction is generalized to show that, for every positive integer n, there exist n-dimensional Cauchy random vectors $\mathbf{X}$ and $\mathbf{Y}$, spherically symmetric about the origin, such that the sum $\mathbf{X}+\mathbf{Y}$ has an n-dimensional Cauchy distribution which is spherically symmetric about a point other than the origin.

In the remainder of this paper, "symmetric" will mean "symmetric about zero" or "symmetric about the origin" if the center of symmetry is not otherwise specified.

2. Sums of symmetric random variables.

We begin with some definitions.

DEFINITION 2.1. Let $\mathscr{D}(\mathbb{R}^n)$ be the set of all probability distributions on $\mathbb{R}^n$.

DEFINITION 2.2. $F \in \mathscr{D}(\mathbb{R})$ is *balanced* if

$$\int_0^{\infty} x\, dF(x) = \int_{-\infty}^{0} (-x)\, dF(x).$$

If $F \in \mathscr{D}(\mathbb{R}^n)$ and $\mathbf{X} \sim F$, then $\mathbf{X}$ (or F) is *balanced* if the distribution of the scalar product $\mathbf{t}'\mathbf{X}$ is balanced for all $\mathbf{t} \in \mathbb{R}^n$.

DEFINITION 2.3. Let $\mathbf{X} \sim F \in \mathscr{D}(\mathbb{R}^n)$. Then $\mathbf{X}$ (or F) is *spherically symmetric* if $\mathscr{L}(\mathbf{X}) = \mathscr{L}(M\mathbf{X})$ for all $n \times n$ orthogonal matrices M and *centrally symmetric* if $\mathscr{L}(\mathbf{X}) = \mathscr{L}(-\mathbf{X})$.

DEFINITION 2.4. $\mathscr{S}_2^{(n)} =: \{F \in \mathscr{D}(\mathbb{R}^n) | \exists$ spherically symmetric $\mathbf{X}$ and $\mathbf{Y}$ on $\mathbb{R}^n$ with $\mathbf{X}+\mathbf{Y} \sim F\}$.

$\mathscr{C}_2^{(n)} =: \{F \in \mathscr{D}(\mathbb{R}^n) | \exists$ centrally symmetric $\mathbf{X}$ and $\mathbf{Y}$ on $\mathbb{R}^n$ with $\mathbf{X}+\mathbf{Y} \sim F\}$.

$\mathscr{S}_3^{(n)}$ and $\mathscr{C}_3^{(n)}$ are defined analogously.

This section will characterize $\mathscr{S}_2 =: \mathscr{S}_2^{(1)} = \mathscr{C}_2^{(1)}$ and $\mathscr{S}_3 =: \mathscr{S}_3^{(1)} = \mathscr{C}_3^{(1)}$.

THEOREM 2.5. *If $F \in \mathscr{D}(\mathbb{R})$, then $F \in \mathscr{S}_2$ if and only if F is balanced.*

DEFINITION 2.6. For $u > 0$ and $v > 0$, let $G(u, v)$ be the mean 0 distribution putting all mass on $\{-u, v\}$. Let $G(0,0)$ be the point mass at 0.

LEMMA 2.7. *For any $G(u, v)$ distribution, there exist $\mathscr{U}[-1,1]$ (i.e., uniform on $[-1,1]$) random variables U_1 and U_2 and a constant b for which $b(U_1 + U_2) \sim G(u, v)$.*

PROOF. Let $U_1 \sim \mathscr{U}[-1,1]$, and let $\theta =: (v-u)/(v+u)$. (Set $\theta = 1$ if $uv = 0$.) Define U_2 by

$$U_2 = \begin{cases} 1 + \theta - U_1 & \text{if } U_1 \geq \theta, \\ -1 + \theta - U_1 & \text{if } U_1 < \theta. \end{cases}$$

Then U_1, U_2, and $b =: (u+v)/2$ are as desired. □

LEMMA 2.8. *Any balanced $F \in \mathscr{D}(\mathbb{R})$ is a mixture of $G(u, v)$ distributions.*

PROOF. In the construction of the Skorokhod representation of a mean 0 random walk, Lemma 2.8 is proved for mean 0 distributions. The proof in Freedman (1971, pages 68–70) goes through word for word for balanced F if his formula $C(0) + A(1) = 0$ on page 69 is replaced by $A(1) = -C(0)$. □

PROOF OF (2.5). Let $F \in \mathscr{D}(\mathbb{R})$ be balanced. By (2.7) and (2.8), there exist random variables X and Y with $X + Y \sim F$, where X and Y are both mixtures of uniform random variables symmetric about 0.

Now suppose that $F \in \mathscr{S}_2$, so that there exist symmetric random variables X and Y such $Z =: X + Y$ has distribution F. Let $Z^+ =: (Z + |Z|)/2$ and $Z^- =: (-Z + |Z|)/2$. We will show that $E(Z^+) = E(Z^-)$, which will imply that F is balanced. For $T \in \mathbb{R}^+$, let $X_T =: (X \wedge T) \vee (-T)$ and $Y_T =: (Y \wedge T) \vee (-T)$. As $T \to \infty$, $(X_T + Y_T)^+$ converges monotonically upward to $(X + Y)^+ = Z^+$, and $(X_T + Y_T)^-$ converges monotonically upward to $(X + Y)^- = Z^-$. By the monotone convergence theorem

$$E(X_T + Y_T)^+ \to E(Z^+)$$

and

$$E(X_T + Y_T)^- \to E(Z^-)$$

as $T \to \infty$. But for each T, X_T and Y_T are bounded symmetric random variables. Thus

$$E(X_T + Y_T) = E(X_T) + E(Y_T) = 0$$

and therefore

$$E(X_T + Y_T)^+ = E(X_T + Y_T)^-.$$

Since the left side of the last equality converges to $E(Z^+)$ and the right side converges to $E(Z^-)$ as $T \to \infty$, it follows that $E(Z^+) = E(Z^-)$. □

REMARK 2.9. The proof of necessity is essentially the same as that given by Simons (1976) and is included here only for completeness.

REMARK 2.10. The symmetric random variables X and Y obtained in the proof of sufficiency are unimodal and identically distributed. They do not necessarily have means even when F has a mean.

COROLLARY 2.11. $\mathscr{S}_3 = \mathscr{D}(\mathbb{R})$.

PROOF. It follows from both Theorem 2.5 and from the Chen–Shepp example that there exist symmetric random variables X_0, Y_0, and Z_0 such that

$$X_0 + Y_0 + Z_0 \equiv 1.$$

(If the Chen–Shepp example is used, then the summand random variables may all be taken to be Cauchy.) Let $F \in \mathscr{D}(\mathbb{R})$. Let $W \sim F$ be independent of X_0, Y_0, and Z_0, and define

$$X =: WX_0, \qquad Y =: WY_0, \qquad Z =: WZ_0.$$

The random variables X, Y, and Z are clearly symmetric and satisfy $W = X + Y + Z$ so that $F \in \mathscr{S}_3$. □

REMARK 2.12. If X_0, Y_0, and Z_0 above are Cauchy, then the symmetric random variables X, Y, and Z will be unimodal but not necessarily identically distributed. (If X_0 is unimodal with mode 0 and W is independent of X_0, then WX_0 is unimodal.) However, if (X_1, Y_1, Z_1) is defined to be a random permutation of the triple (X, Y, Z), with the permutation being independent of (X, Y, Z) and with all of the six possible permutations having probability 1/6, then X_1, Y_1, and Z_1 are symmetric, unimodal, and identically distributed, and $X_1 + Y_1 + Z_1 = W \sim F$.

REMARK 2.13. Simons (1977) showed that the expectation (if it exists) of a sum of two random variables is determined by their marginal distributions, but that this does not hold for a sum of three random variables.

3. Sums of spherically symmetric random variables. This section will extend the one-dimensional results of Section 2 to higher dimensions.

THEOREM 3.1. *If $F \in \mathscr{D}(\mathbb{R}^n)$, then $F \in \mathscr{S}_2^{(n)}$ if and only if F is balanced.*

The "only if" part of Theorem 3.1 is an immediate consequence of Theorem 2.5. Indeed, if $\mathbf{X} + \mathbf{Y} = \mathbf{Z} \sim F$ for spherically symmetric $\mathbf{X}$ and $\mathbf{Y}$, then for each $\mathbf{t} \in \mathbb{R}^n$, $\mathbf{t}'\mathbf{X}$ and $\mathbf{t}'\mathbf{Y}$ are symmetric random variables and Theorem 2.5 implies that $\mathbf{t}'\mathbf{Z} = \mathbf{t}'\mathbf{X} + \mathbf{t}'\mathbf{Y}$ is a balanced random variable. The same reasoning works for $F \in \mathscr{C}_2^{(n)}$, so that we have

$$\mathscr{S}_2^{(n)} \subseteq \mathscr{C}_2^{(n)} \subseteq \{\text{balanced } F\text{s}\}.$$

Before proceeding to the proof of sufficiency, let us examine what happens when one attempts a straightforward adaptation to $\mathbb{R}^n$ of the argument in Section 2. Lemma 2.8 remains true in $\mathbb{R}^n$ if the mean 0, two-point distributions $G(u, v)$ are replaced by mean $\mathbf{0}$, $(n + 1)$-point distributions. Indeed, see Lemma 3.17 below. Lemma 2.7 also generalizes. In the proof of Lemma 2.7, it is shown that, for any $G(u, v)$ distribution with $uv > 0$, there exists an interval $[-b, b]$ symmetric about 0 with a decomposition into subintervals $[-b, \theta b]$ and $[\theta b, b]$ which are symmetric about the points $-u/2$ and $v/2$, respectively. The ratio of the lengths of $[-b, \theta b]$ and $[\theta b, b]$ is necessarily equal to the ratio of the probabilities of the points $-u$ and v. The random variable bU_1 is uniform on $[-b, b]$, and bU_2 is obtained by reflecting bU_1 across the center of the subinterval containing bU_1. This trick generalizes to $\mathbb{R}^2$ as follows. If H is a mean $\mathbf{0}$ distribution on three noncolinear points $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ in $\mathbb{R}^2$, there exists a centrally symmetric parallelogram A and a decomposition of A into smaller parallelograms A_1, A_2, and A_3 whose centers are $\mathbf{x}_1/2$, $\mathbf{x}_2/x$, and $\mathbf{x}_3/2$, respec-

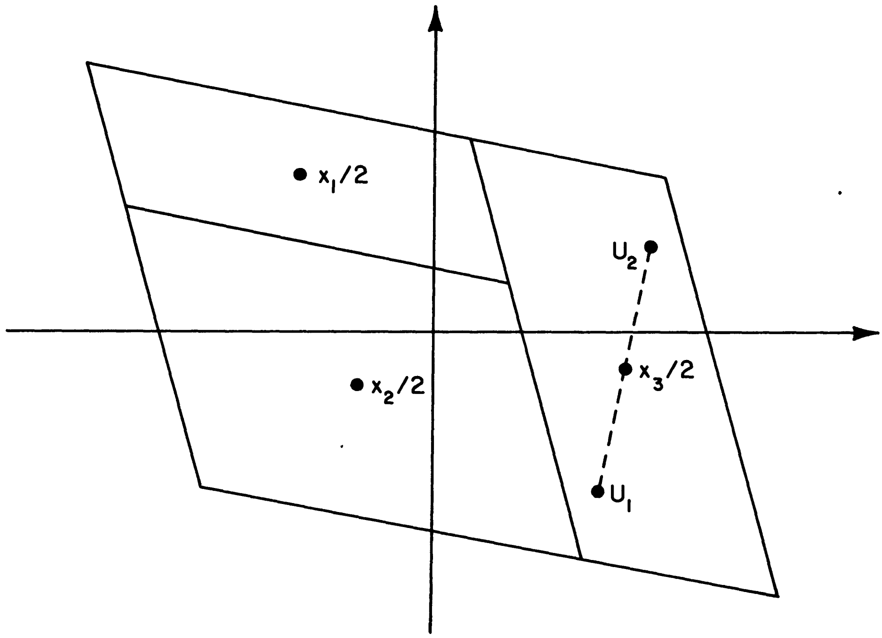

FIG. 1.

tively. See Figure 1. The ratios of the areas of the A_is are necessarily equal to the ratios of the corresponding probabilities. Let $\mathbf{U}_1$ be uniformly distributed on A and obtain $\mathbf{U}_2$ by reflecting $\mathbf{U}_1$ across the center of whichever A_i it is in. Then $\mathbf{U}_2$ is also uniform on A and $\mathbf{U}_1 + \mathbf{U}_2 \sim H$. It follows that $\mathscr{C}_2^{(2)}$ contains all mean $\mathbf{0}$ three-point distributions, and hence, by Lemma 3.17 and the fact that $\mathscr{C}_2^{(2)}$ is closed under mixtures, we have $\{\text{balanced } F\text{s}\} \subseteq \mathscr{C}_2^{(2)}$. All of this works in $\mathbb{R}^n$ if parallelograms are replaced by n-dimensional parallelopipeds.

Thus, adapting the arguments of Section 2 shows that $\mathscr{C}_2^{(n)} = \{\text{balanced } F\text{s}\}$. If one has argued along these lines, then the proof of Theorem 3.1 may be completed as follows. Let $\mathbf{U}_1$ and $\mathbf{U}_2$ be uniformly distributed on a symmetric parallelogram (or parallelopiped) as in the preceding paragraph, with $\mathbf{U}_1 + \mathbf{U}_2$ having the mean $\mathbf{0}$, $(n + 1)$-point distribution H. Lemma 3.14 below implies that there exists a random vector $\mathbf{V}$, independent of $(\mathbf{U}_1, \mathbf{U}_2)$, for which $\mathbf{X} =: \mathbf{U}_1 + \mathbf{V}$ is spherically symmetric. Then $\mathbf{Y} =: \mathbf{U}_2 - \mathbf{V}$ will also be spherically symmetric, since

$$\mathscr{L}(\mathbf{Y}) = \mathscr{L}(\mathbf{U}_2 - V) = \mathscr{L}(-\mathbf{U}_1 - \mathbf{V}) = \mathscr{L}(-\mathbf{X}) = \mathscr{L}(\mathbf{X}).$$

But $\mathbf{X} + \mathbf{Y} = \mathbf{U}_1 + \mathbf{U}_2 \sim H$, so that $\mathscr{S}_2^{(n)}$ contains any mean $\mathbf{0}$, $(n + 1)$-point distribution. Now take mixtures and apply Lemma 3.17 to get $\{\text{balanced } F\text{s}\} \subseteq \mathscr{S}_2^{(n)}$.

The parallelopiped approach is geometrically appealing but notationally clumsy, and we have chosen to take a slightly different route. The proof of sufficiency is broken up into a sequence of lemmas. We begin with some analysis which culminates in Lemma 3.10. After that, the reasoning becomes more probabilistic. Readers wishing to skip the technicalities may proceed directly to Lemma 3.15, providing they are willing to accept Lemma 3.14 on faith.

Lemma 3.2 shows that a sufficiently gentle perturbation of a spherically symmetric Cauchy characteristic function (ch.f) is still a ch.f.

LEMMA 3.2. *Let* $b: \mathbb{R}^n \to \mathbb{C}$ *be a rapidly decreasing* C^∞ *function in the sense of Rudin* (1973, *page* 168), *for which* $b(\mathbf{0}) = 0$ *and* $b(-\mathbf{t}) = \overline{b(\mathbf{t})}$. *Then for sufficiently large* β,

$$\{1 - b(\mathbf{t}/\beta)\}e^{-\|\mathbf{t}\|} \tag{3.3}$$

is a ch.f.

PROOF. Let q be the Fourier transform of b:

$$q(\mathbf{x}) = (2\pi)^{-n}\int_{\mathbb{R}^n} \exp\{-i\mathbf{x}'\mathbf{t}\}b(\mathbf{t})\,d\mathbf{t}. \tag{3.4}$$

By Theorem 7.7 of Rudin (1973, page 170), q is also a rapidly decreasing C^∞ function. It follows from $b(-\mathbf{t}) = \overline{b(\mathbf{t})}$ that q is real. It follows from $b(\mathbf{0}) = 0$ that q is the density of a signed measure with net measure 0. Let m be the measure with density $|q|$. The fact that q is rapidly decreasing implies that

$$m\{\mathbf{y} \in \mathbb{R}^n: \|\mathbf{y}\| > r\} < Kr^{-(n+1)} \tag{3.5}$$

when $r > 1$ for some constant $K > 0$. Define q_β by $q_\beta(\mathbf{x}) =: \beta^n q(\beta\mathbf{x})$, so that q_β is the Fourier transform of $b(\mathbf{t}/\beta)$ and $m(\mathbb{R}^n)$ is the integral of $|q_\beta|$ over $\mathbb{R}^n$.

The Fourier transform of $e^{-\|\mathbf{t}\|}$ is the spherically symmetric n-dimensional Cauchy density

$$p(\mathbf{x}) =: C_n(1 + \|\mathbf{x}\|^2)^{-(n+1)/2}.$$

It is easy to show that for $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ so that $\|\mathbf{y}\| \le c(\varepsilon)(1 + \|\mathbf{x}\|)$ implies

$$|p(\mathbf{x} + \mathbf{y}) - p(\mathbf{x})| < \varepsilon p(\mathbf{x}).$$

(Just consider the cases $\|\mathbf{x}\| \le 1$ and $\|\mathbf{x}\| > 1$ separately.) Let $a =: c(\{2m(\mathbb{R}^n)\}^{-1})$.

The Fourier transform of (3.3) is

$$p - (p * q_\beta). \tag{3.6}$$

This function is the density of a real signed measure with net measure 1. If we can show that (3.6) is everywhere positive, then it will follow that (3.6) is a probability density, and the Fourier inversion theorem will imply that (3.3) is its ch.f. It is obvious that $(p * q_\beta)/p$ converges pointwise to 0 as $\beta \to \infty$. The idea is that $\log p$ is sufficiently flat (cf. the last inequality) and the tails of q are sufficiently thin [cf. (3.5)] so that this convergence is uniform.

Let $A_{\mathbf{x}} =: \{\mathbf{y}: \|\mathbf{y}\| \le a(1 + \|\mathbf{x}\|)\}$. Then

$$\begin{aligned}(p * q_\beta)(\mathbf{x}) &= \int_{\mathbb{R}^n} p(\mathbf{x}-\mathbf{y}) q_\beta(\mathbf{y})\, d\mathbf{y} \\ &= \int_{\mathbb{R}^n} \{p(\mathbf{x}-\mathbf{y}) - p(\mathbf{x})\} q_\beta(\mathbf{y})\, d\mathbf{y} \\ &\le \int_{A_{\mathbf{x}}} |p(\mathbf{x}-\mathbf{y}) - p(\mathbf{x})|\, |q_\beta(\mathbf{y})|\, d\mathbf{y} + \int_{A_{\mathbf{x}}^c} C_n |q_\beta(\mathbf{y})|\, dy \\ &< \{2m(\mathbb{R}^n)\}^{-1} p(\mathbf{x}) m(\mathbb{R}^n) + C_n \int_{\beta A_{\mathbf{x}}^c} |q(\mathbf{y})|\, d\mathbf{y} \\ &\le p(\mathbf{x})/2 + C_n m\{\|\mathbf{y}\| > \beta a(1 + \|\mathbf{x}\|)\}.\end{aligned}$$

If $\beta a > 1$, then by (3.5) the last term is less than

$$C_n K\{\beta a(1 + \|\mathbf{x}\|)\}^{-(n+1)}. \tag{3.7}$$

If $\beta a \ge (K/2)^{1/(n+1)}$, then (3.7) is less than $p(\mathbf{x})/2$ and $(p * q_\beta)(\mathbf{x})$ is less than $p(\mathbf{x})$. This implies that, for $\beta > \max\{a^{-1}, a^{-1}(K/2)^{1/(n+1)}\}$, (3.6) is everywhere positive, and we are done. □

REMARK 3.8. Lemma 3.2 also holds if (3.3) is replaced by

$$\{1 - b(\mathbf{t})\}\exp(-\beta\|\mathbf{t}\|),$$

since this is just a change of scale.

DEFINITION 3.9. Let $B(\mathbf{x}, r)$ be the closed ball in $\mathbb{R}^n$ with center $\mathbf{x}$ and radius r.

LEMMA 3.10. *Let* $f_0: \mathbb{R}^n \to \mathbb{C}$ *be a* ch.f. *with support inside* $B(\mathbf{0}, r)$. *Let* $g: \mathbb{R}^n \to \mathbb{C}$ *be a* C^∞ ch.f. *with no zeros inside* $B(\mathbf{0}, r)$. *Then for all sufficiently large* β,

$$f_0(\mathbf{t})\exp\{-\beta\|\mathbf{t}\|\}/g(\mathbf{t}) \tag{3.11}$$

is a ch.f.

PROOF. Let $\varepsilon > 0$ be small enough so that g has no zeroes in $B(\mathbf{0}, r + \varepsilon)$. Let $h: \mathbb{R}^n \to \mathbb{R}$ be a spherically symmetric C^∞ "hat function" which equals 1 inside $B(\mathbf{0}, r)$ and equals 0 outside $B(\mathbf{0}, r + \varepsilon)$. Write (3.11) as

$$f_0(\mathbf{t})\left[1 - h(\mathbf{t})\left\{1 - g(\mathbf{t})^{-1}\right\}\right]\exp\{-\beta\|\mathbf{t}\|\} \tag{3.12}$$

and apply Remark 3.8 to the last two factors of (3.12). □

REMARK 3.13. Let $\mathbf{X} = (X_1, X_2, \dots, X_n)'$ be a random vector in $\mathbb{R}^n$ whose coordinates X_i are iid from a distribution on $\mathbb{R}$ whose characteristic function has compact support. Then the ch.f. of $\mathbf{X}$ will have compact support in $\mathbb{R}^n$. If, in addition, M is a random $n \times n$ orthogonal matrix whose distribution is the Haar measure on the group and if M is independent of $\mathbf{X}$, then the ch.f. of $\mathbf{Y} =: M\mathbf{X}$ will be spherically symmetric with compact support.

LEMMA 3.14. *Let* $\mathbf{W}$ *be a random vector with all moments. Then there exists an independent random vector* $\mathbf{V}$ *such that the sum* $\mathbf{W} + \mathbf{V}$ *is spherically symmetric.*

PROOF. Denote the ch.f. of $\mathbf{W}$ by g, which is C^{∞} since $\mathbf{W}$ has all moments. Let f_0 be a spherically symmetric ch.f. with support inside a ball $B(\mathbf{0}, r)$ not containing any zeroes of g. Choose a sufficiently large β and let $\mathbf{V}$ be a random vector, independent of $\mathbf{W}$, with ch.f. (3.11). Then $\mathbf{W} + \mathbf{V}$ has the spherically symmetric ch.f. $f_0(\mathbf{t})\exp\{-\beta\|\mathbf{t}\|\}$. □

LEMMA 3.15. *Any mean* $\mathbf{0}$, *two-point distribution* H *on* $\mathbb{R}^n$ *is in* $\mathscr{S}_2^{(n)}$.

PROOF. We may assume without loss of generality that H puts all mass on the first coordinate axis. By Lemma 2.7 there exist random vectors $\mathbf{U}_1$ and $\mathbf{U}_2$ which are uniformly distributed on a symmetric subinterval of the first coordinate axis and for which $\mathbf{U}_1 + \mathbf{U}_2 \sim H$. Let $\mathbf{V}$, independent of $(\mathbf{U}_1, \mathbf{U}_2)$, be such that $\mathbf{X} =: \mathbf{U}_1 + \mathbf{V}$ is spherically symmetric. Then $\mathbf{Y} =: \mathbf{U}_2 - \mathbf{V}$ is also spherically symmetric, since $\mathscr{L}(\mathbf{Y}) = \mathscr{L}(\mathbf{X})$. But

$$\mathbf{X} + \mathbf{Y} = \mathbf{U}_1 + \mathbf{U}_2 \sim H,$$

so that $H \in \mathscr{S}_2^{(n)}$. □

LEMMA 3.16. *Any mean* $\mathbf{0}$ *distribution* H *on* $\mathbb{R}^n$ *which puts all probability on a finite set of points* $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ *is in* $\mathscr{S}_2^{(n)}$.

Proof by induction on k. Suppose (3.16) is true for distributions putting mass on k or fewer points, $k \geq 2$. Let H be a distribution for which $H\{\mathbf{x}_1, \dots, \mathbf{x}_{k+1}\} = 1$, and let $\mathbf{Z} \sim H$. Let

$$\tilde{\mathbf{x}} =: E(\mathbf{Z} | \mathbf{Z} \neq \mathbf{x}_1).$$

Let G be the mean $\mathbf{0}$, two-point distribution on $\{\mathbf{x}_1, \tilde{\mathbf{x}}\}$. Let F be the distribution $\mathscr{L}(\mathbf{Z} - \tilde{\mathbf{x}} | \mathbf{Z} \neq \mathbf{x}_1)$. By the induction hypothesis, there exist spherically symmetric $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{V}_1$, and $\mathbf{V}_2$ such that

$$\mathbf{W}_1 + \mathbf{W}_2 \sim G \quad \text{and} \quad \mathbf{V}_1 + \mathbf{V}_2 \sim F.$$

We may assume without loss of generality that $(\mathbf{W}_1, \mathbf{W}_2)$ is independent of $(\mathbf{V}_1, \mathbf{V}_2)$, and that $\mathbf{V}_1$ and $\mathbf{V}_2$ are identically distributed (c.f. Remark 2.12). Define $\mathbf{V}_2^*$ by

$$\mathbf{V}_2^* =: \begin{cases} -\mathbf{V}_1 & \text{if } \mathbf{W}_1 + \mathbf{W}_2 = \mathbf{x}_1, \\ \mathbf{V}_2 & \text{if } \mathbf{W}_1 + \mathbf{W}_2 = \tilde{\mathbf{x}}. \end{cases}$$

Then $\mathbf{V}_2^*$ has distribution $\mathscr{L}(\mathbf{V}_2)$ and is independent of $(\mathbf{W}_1, \mathbf{W}_2)$, since the conditional distribution of $\mathbf{V}_2^*$ given $(\mathbf{W}_1, \mathbf{W}_2)$ is either $\mathscr{L}(-\mathbf{V}_1)$ or $\mathscr{L}(\mathbf{V}_2)$, and

$\mathscr{L}(-\mathbf{V}_1) = \mathscr{L}(\mathbf{V}_1) = \mathscr{L}(\mathbf{V}_2)$. Set

$$\mathbf{X} =: \mathbf{W}_1 + \mathbf{V}_1 \quad \text{and} \quad \mathbf{Y} =: \mathbf{W}_2 + \mathbf{V}_2^*.$$

Then $\mathbf{X} + \mathbf{Y} \sim$ H, and $\mathbf{X}$ and $\mathbf{Y}$ are spherically symmetric since they are sums of independent, spherically symmetric random vectors. □

LEMMA 3.17. *If H is balanced on $\mathbb{R}^n$, then H is a mixture of mean $\mathbf{0}$ distributions with support on at most $n + 1$ points.*

PROOF. Lemma 3.17 for mean $\mathbf{0}$ H is a special case of Theorem 7 of Mulholland and Rogers (1958). To finish the proof, it will suffice to show that any balanced H is a countable mixture of mean $\mathbf{0}$ distributions. Inductively define an increasing sequence $\{g_i\}_{i=0}^{\infty}$ of "mean $\mathbf{0}$ subdensity functions with respect to H" as follows. Let $g_0(\mathbf{x}) \equiv 0$. Given g_i, let $\mathscr{G}_{i+1}$ be the set of measurable functions given by

$$\mathscr{G}_{i+1} =: \left\{ g \colon \mathbb{R}^n \to [0,1] \,\middle|\, g(\mathbf{x}) \ge g_i(\mathbf{x}) \text{ for all } \mathbf{x},\ g \text{ has compact support, and } \int_{\mathbb{R}^n} g(\mathbf{x}) H(d\mathbf{x}) > 1 - 2^{-i} \right\}.$$

Let $\mathscr{M}_{i+1}$ be the set of "means" of functions in $\mathscr{G}_{i+1}$:

$$\mathscr{M}_{i+1} =: \left\{ \mathbf{z} \in \mathbb{R}^n \,\middle|\, \mathbf{z} = \int_{\mathbb{R}^n} \mathbf{x} g(\mathbf{x}) H(d\mathbf{x}) \text{ for some } g \in \mathscr{G}_{i+1} \right\}.$$

We wish to choose g_{i+1} to be an element of $\mathscr{G}_{i+1}$ with mean $\mathbf{0}$. This is obviously possible precisely when $\mathbf{0} \in \mathscr{M}_{i+1}$. Since $\mathscr{G}_{i+1}$ is closed under the taking of convex linear combinations, it follows that $\mathscr{M}_{i+1}$ is a convex set in $\mathbb{R}^n$. If $\mathbf{0} \notin \mathscr{M}_{i+1}$, then the separating hyperplane theorem implies that there exists a nonzero vector $\mathbf{t}$ such that $\mathbf{t}'\mathbf{Z} \ge 0$ for all $\mathbf{Z} \in \mathscr{M}_{i+1}$. Since the subdistribution with density $1 - g_i$ with respect to H is itself balanced, it is easy to show that the existence of such a $\mathbf{t}$ is impossible if $\mathbf{0} \notin \mathscr{M}_{i+1}$. Thus, we can define the entire sequence $\{g_i\}_{i=0}^{\infty}$. Let $f_i =: g_i - g_{i-1}$ and let $\alpha_i =: \int f_i \, dH$. Then H can be written as the countable mixture

$$H = \sum_{i=1}^{\infty} \alpha_i F_i,$$

where F_i is the probability distribution with density $(\alpha_i)^{-1} f_i$ with respect to H. □

Combining Lemmas 3.16 and 3.17 completes the proof of Theorem 3.1.

Readers who share the authors' preference for probabilistic methods over characteristic function methods may wonder whether there is a more probabilistic proof of Theorem 3.1. The answer seems to be no, as the following argument indicates. Fix $n \ge 2$ and let G be the mean $\mathbf{0}$, two-point distribution putting probability $1/2$ at each of $2\mathbf{e}_1$ and $-2\mathbf{e}_1$, where $\mathbf{e}_1$ is the unit vector in the first coordinate direction. Let $\mathbf{X}_0$ and $\mathbf{Y}_0$ be spherically symmetric random vectors with $\mathbf{X}_0 + \mathbf{Y}_0 \sim G$. Define $(\mathbf{X}, \mathbf{Y})$ to be a random choice of $(\mathbf{X}_0, \mathbf{Y}_0)$, $(\mathbf{Y}_0, \mathbf{X}_0)$, $(-\mathbf{X}_0, -\mathbf{Y}_0)$, and $(-\mathbf{Y}_0, -\mathbf{X}_0)$, with each choice having probability 1/4 and with the choice being independent of $(\mathbf{X}_0, \mathbf{Y}_0)$. Then $\mathbf{X}$ and $\mathbf{Y}$ are also spherically symmetric and $\mathbf{X} + \mathbf{Y} \sim G$. In addition, the randomization implies that (i) $\mathscr{L}(\mathbf{X}, \mathbf{Y}) = \mathscr{L}(\mathbf{Y}, \mathbf{X})$ and (ii) $\mathscr{L}(\mathbf{X}, \mathbf{Y}) = \mathscr{L}(-\mathbf{X}, -\mathbf{Y})$. Let $\mathbf{W} =: (\mathbf{X} + \mathbf{Y})/2$, so that (iii) $\mathbf{X} - \mathbf{W} = \mathbf{W} - \mathbf{Y}$.

Proposition 3.18. *If* $\mathbf{X}$, $\mathbf{Y}$, *and* $\mathbf{W}$ *are as above, then* $\mathbf{X} - \mathbf{W}$ *is independent of* $\mathbf{W}$ *and the common* ch.f. f *of* $\mathbf{X}$ *and* $\mathbf{Y}$ *has support inside* $B(\mathbf{0}, \pi/2)$.

Proof. Let A be a Borel subset of $\mathbb{R}^n$. Then

$$\begin{aligned} P\{\mathbf{X} - \mathbf{W} \in A, \mathbf{W} = \mathbf{e}_1\} &= P\{\mathbf{W} - \mathbf{Y} \in A, \mathbf{W} = \mathbf{e}_1\} \quad \text{by (iii)} \\ &= P\{\mathbf{W} - \mathbf{X} \in A, \mathbf{W} = \mathbf{e}_1\} \quad \text{by (i)} \\ &= P\{(-\mathbf{W}) - (-\mathbf{X}) \in A, (-\mathbf{W}) = \mathbf{e}_1\} \quad \text{by (ii)} \\ &= P\{\mathbf{X} - \mathbf{W} \in A, \mathbf{W} = -\mathbf{e}_1\}. \end{aligned}$$

It follows that the conditional distribution of $\mathbf{X} - \mathbf{W}$, given $\mathbf{W}$, does not depend on $\mathbf{W}$, so that $\mathbf{X} - \mathbf{W}$ and $\mathbf{W}$ are independent.

The ch.f. of $\mathbf{W}$ is $\cos(t_1)$, where t_1 is the first coordinate of $\mathbf{t}$. If g is the ch.f. of $\mathbf{X} - \mathbf{W}$, then the ch.f. f of $\mathbf{X}$ satisfies

$$f(\mathbf{t}) = g(\mathbf{t})\cos(t_1),$$

by the independence of $\mathbf{X} - \mathbf{W}$ and $\mathbf{W}$. But $\cos(t_1)$ equals zero on the $(n-1)$-dimensional hyperplane $t_1 = \pi/2$, and f is a spherically symmetric function, since $\mathbf{X}$ is a spherically symmetric random vector. The second part of the proposition follows. □

If we had wanted to prove Lemma 3.15 only for a symmetric two-point distribution G, we could have simplified the proof slightly by starting with a random vector $\mathbf{W}$ satisfying $2\mathbf{W} \sim G$ and then using Lemma 3.10 to assure the existence of an independent $\mathbf{V}$ such that $\mathbf{X} =: \mathbf{W} + \mathbf{V}$ and $\mathbf{Y} =: \mathbf{W} - \mathbf{V}$ would be spherically symmetric. Proposition 3.18 shows that this is essentially the *only* way of obtaining spherically symmetric $\mathbf{X}$ and $\mathbf{Y}$ with $\mathbf{X} + \mathbf{Y} \sim G$, and that distributions whose characteristic functions have compact support are necessarily involved in any such construction. This leads the authors to suspect that any proof of Theorem 3.1 will involve something at least as probabilistically mysterious as the division of a characteristic function with compact support by another characteristic function.

We now generalize Corollary 2.11 of the previous section.

Corollary 3.19. $\mathscr{S}_3^{(n)} = \mathscr{D}(\mathbb{R}^n)$.

Proof. It follows from both Theorem 3.1 and from the generalization of the Chen–Shepp example in Section 4 that there exist spherically symmetric random vectors $\mathbf{X}_0$, $\mathbf{Y}_0$, and $\mathbf{Z}_0$ such that

$$\mathbf{X}_0 + \mathbf{Y}_0 + \mathbf{Z}_0 = (1, 0, \dots, 0)'.$$

Choose $F \in \mathscr{D}(\mathbb{R}^n)$ and let $\mathbf{W} \sim F$ be independent of $\mathbf{X}_0$, $\mathbf{Y}_0$, and $\mathbf{Z}_0$. Let M be a random $n \times n$ matrix, also independent of $\mathbf{X}_0$, $\mathbf{Y}_0$, and $\mathbf{Z}_0$, whose first column is $\mathbf{W}$ and for which $\|\mathbf{W}\|^{-1}M$ is an orthogonal matrix when $\mathbf{W} \neq \mathbf{0}$. When $\mathbf{W} = \mathbf{0}$ set M equal to the $n \times n$ matrix of all 0s. If $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ are the coordinate unit vectors in $\mathbb{R}^n$, a suitable M matrix may be constructed by taking the columns of $\|\mathbf{W}\|^{-1}M$ to be the orthonormal basis of $\mathbb{R}^n$ obtained by applying the Gram–Schmidt procedure to the spanning sequence $\mathbf{W}, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$. Then

$$\mathbf{X} =: M\mathbf{X}_0, \qquad \mathbf{Y} =: M\mathbf{Y}_0, \quad \text{and} \quad \mathbf{Z} =: M\mathbf{Z}_0.$$

are spherically symmetric random vectors, and

$$\mathbf{X} + \mathbf{Y} + \mathbf{Z} = \mathbf{W} \sim F. \qquad \square$$

4. Sums of symmetric Cauchy random vectors. As a final curiosity, we show that the Ferguson–Chen–Shepp argument can be generalized to higher dimensions.

THEOREM 4.1. *For any positive integer n, there exist n-dimensional Cauchy random vectors* $\mathbf{X}$ *and* $\mathbf{Y}$, *spherically symmetric about the origin, such that the sum* $\mathbf{X} + \mathbf{Y}$ *has an n-dimensional Cauchy distribution which is spherically symmetric about the point* $(1, 0, \dots, 0)'$.

PROOF. In this proof, $\mathbf{t}$, $\mathbf{s}$ and $\mathbf{x}$ will be points in $\mathbb{R}^n$ with first coordinates t_1, s_1, and x_1.

Let $A(\mathbf{x}) =: I_{\{\|\mathbf{x}\| \leq 1\}}$ and for each $\alpha \in [-1, 1]$, define $\varphi(\cdot, \alpha)$ on $\mathbb{R}^n$ by

$$\varphi(\mathbf{t}, \alpha) =: \int_{\mathbb{R}^n} \frac{e^{i(\mathbf{t}'\mathbf{x})} - 1 - iA(\mathbf{x})(\mathbf{t}'\mathbf{x})}{\|\mathbf{x}\|^{n+1}} (1 + \alpha \operatorname{sgn}(x_1))\, d\mathbf{x}.$$

For each α, $\varphi(\cdot, \alpha)$ is the logarithm of the characteristic function (hereafter abbreviated log ch.f.) of an infinitely divisible distribution. Indeed, for each $\mathbf{x}$, the integrand is the log ch.f. of a shifted Poisson random vector with "jumps" of size and direction $\mathbf{x}$, jumping intensity $\|\mathbf{x}\|^{-(n+1)}(1 + \alpha \operatorname{sgn}(x_1))$, and deterministic shift $-A(\mathbf{x})\|\mathbf{x}\|^{-(n+1)}(1 + \alpha \operatorname{sgn}(x_1))\mathbf{x}$. Thus, φ is the log ch.f. of a shifted compound Poisson random vector.

Define $\psi(\cdot)$ on $\mathbb{R}^n$ by

$$\psi(\mathbf{t}) =: \int_{\mathbb{R}^k} \frac{e^{i(\mathbf{t}'\mathbf{x})} - 1 - iA(\mathbf{x})(\mathbf{t}'\mathbf{x})}{\|\mathbf{x}\|^{n+1}} \operatorname{sgn}(x_1)\, d\mathbf{x}.$$

If $c \in \mathbb{R}$, straightforward calculation shows that

$$\varphi(c\mathbf{t}, \alpha) = |c|\varphi(\mathbf{t}, 0) + c\alpha\psi(\mathbf{t}) - it_1 k_1 \alpha c \log|c|$$

and that

$$\varphi(\mathbf{t}, 0) = -k_2|\mathbf{t}|,$$

for some positive constants k_1 and k_2. The last formula implies that $\varphi(\cdot, 0)$ is the log ch.f. of an n-dimensional Cauchy distribution centered at the origin.

Let $\mathbf{U}$ be a random vector in $\mathbb{R}^n$ with log ch.f. $\varphi(\cdot, \alpha)$. For each $\theta \in [0, 2\pi)$, define the n-dimensional random vectors

$$\mathbf{V}_\theta =: (\cos\theta)\mathbf{U} \quad \text{and} \quad \mathbf{W}_\theta =: (\sin\theta)\mathbf{U}.$$

Then $(\mathbf{V}_\theta, \mathbf{W}_\theta)$ is an infinitely divisible $2n$-dimensional random vector with log ch.f.

$$\tilde{\varphi}(t, s, \theta, \alpha) =: \log E\left[\exp\{i(\mathbf{t}'\mathbf{V}_\theta) + i(\mathbf{s}'\mathbf{W}_\theta)\}\right] = \varphi(\mathbf{t}\cos\theta + \mathbf{s}\sin\theta, \alpha).$$

Let $\lambda(\cdot)$ be a measurable function from $[0, 2\pi)$ to $[-1, 1]$. Taking a "continuous convolution" of the infinitely divisible distributions associated with the $\tilde{\varphi}(\cdot, \cdot, \theta, \lambda(\theta))$s produces the log ch.f.

$$\zeta(\mathbf{t}, \mathbf{s}, \lambda) =: \int_0^{2\pi} \tilde{\varphi}(\mathbf{t}, \mathbf{s}, \theta, \lambda(\theta))\, d\theta.$$

Let $\mathbf{X}$ and $\mathbf{Y}$ be n-dimensional random vectors such that the $2n$-dimensional random vector $(\mathbf{X}, \mathbf{Y})$ has log ch.f. $\zeta(\cdot, \cdot, \lambda)$. Then $\mathbf{X}$ by itself has log ch.f.

$$\begin{aligned}\log E\left[\exp\{i(\mathbf{t}'\mathbf{X})\}\right] = \zeta(\mathbf{t}, \mathbf{0}, \lambda) &= \int_0^{2\pi} \varphi(\mathbf{t}\cos\theta, \lambda(\theta))\, d\theta \\ &= \int_0^{2\pi} |\cos\theta| \varphi(\mathbf{t}, 0) + (\cos\theta)\lambda(\theta)\psi(t) \\ &\quad - it_1 k_1 \lambda(\theta)(\cos\theta)\log|\cos\theta|\, d\theta,\end{aligned}$$

and Y has log ch.f.

$$\begin{aligned}\log E\left[\exp\{i(\mathbf{s}'\mathbf{y})\}\right] = \zeta(\mathbf{0}, \mathbf{s}, \lambda) &= \int_0^{2\pi} \varphi(\mathbf{s}\sin\theta, \lambda(\theta))\, d\theta \\ &= \int_0^{2\pi} |\sin\theta| \varphi(\mathbf{s}, 0) + (\sin\theta)\lambda(\theta)\psi(\mathbf{s}) \\ &\quad - is_1 k_1 \lambda(\theta)(\sin\theta)\log|\sin\theta|\, d\theta.\end{aligned}$$

The sum $\mathbf{X} + \mathbf{Y}$ has log ch.f.

$$\begin{aligned}\log E\left[\exp\{i(\mathbf{t}'(\mathbf{X} + \mathbf{Y}))\}\right] = \zeta(\mathbf{t}, \mathbf{t}, \lambda) &= \int_0^{2\pi} \varphi(\mathbf{t}(\cos\theta + \sin\theta), \lambda(\theta))\, d\theta \\ &= \int_0^{2\pi} |\cos\theta + \sin\theta| \varphi(\mathbf{t}, 0) + (\cos\theta + \sin\theta)\lambda(\theta)\psi(\mathbf{t}) \\ &\quad - it_1 k_1 \lambda(\theta)(\cos\theta + \sin\theta)\log|\cos\theta + \sin\theta|\, d\theta.\end{aligned}$$

If $\lambda(\cdot)$ is chosen to be orthogonal in $L^2[0, 2\pi)$ to $\sin\theta$, $\cos\theta$, $(\sin\theta)\log|\sin\theta|$, and $(\cos\theta)\log|\cos\theta|$, but *not* to $(\cos\theta + \sin\theta)\log|\cos\theta + \sin\theta|$, then

$$\zeta(\mathbf{t}, \mathbf{0}, \lambda) = -k_3|t|, \qquad \zeta(\mathbf{0}, \mathbf{s}, \lambda) = -k_3|\mathbf{s}|,$$

and

$$\zeta(\mathbf{t}, \mathbf{t}, \lambda) = -k_4|\mathbf{t}| + it_1 k_5,$$

where $k_3 > 0$, $k_4 > 0$, and $k_5 \neq 0$ are constants. Thus, $\mathbf{X}$ and $\mathbf{Y}$ satisfy the conditions in the theorem, except that $\mathbf{X} + \mathbf{Y}$ is symmetric about $(k_5, 0, \dots, 0)$. The random vectors $k_5^{-1}\mathbf{X}$ and $k_5^{-1}\mathbf{Y}$ are as desired. □

REMARK 4.2. Calculation of the log ch.f. of $a\mathbf{X} + b\mathbf{Y}$ shows that any such linear combination has an n-dimensional Cauchy distribution which is spherically symmetric about some point on the first coordinate axis.

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