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IMPROVED ERDÖS–RÉNYI AND STRONG APPROXIMATION LAWS FOR INCREMENTS OF RENEWAL PROCESSES

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Let $X_1, X_2, \dots$ be an i.i.d. sequence with $EX_1 = \mu > 0$, $\mathrm{var}(X_1) = \sigma^2 > 0$, $E\exp(sX_1) < \infty$, $|s| < s_1$, and partial sums $S_0 = 0$, $S_n = X_1 + \cdots + X_n$. For $t \geq 0$, put $N(t) = \max\{n \geq 0: S_0, \dots, S_n \leq t\}$, i.e., $L(t) = N(t) + 1$ denotes the first-passage time of the random walk $\{S_n\}$. Starting from some analogous results for the partial sum sequence, this paper studies the almost sure limiting behaviour of $\sup_{0 \leq t \leq T-K_T}(N(t+K_T) - N(t))$ as $T \to \infty$, under various conditions on the real function K_T. Improvements of the Erdös–Rényi strong law for renewal processes (resp. first-passage times) are obtained as well as strong invariance principle type versions. An indefinite range between strong invariance and strong noninvariance is also treated.

1. Introduction. Consider a sequence $X_1, X_2, \dots$ of i.i.d. random variables satisfying $EX_1 = 0$, $EX_1^2 = 1$, and $\tilde{\varphi}(t) = E\exp(tX_1) < \infty$ for $|t| < t_0(>0)$. Set $\tilde{I} = \{\tilde{\varphi}'(t)/\tilde{\varphi}(t):\ t \in (0, t_0)\}$ and $\tilde{\rho}(a) = \inf \tilde{\varphi}(t)\exp(-ta)$. Stimulated by Révész's (1980, 1982) improved strong approximations for the increments of a standard Wiener process $\{W(t)\}_{t \geq 0}$, Csörgő and Steinebach (1981) obtained the following results for the increments of the partial sums $S_n = X_1 + \cdots + X_n$, $n = 1, 2, \dots, S_0 = 0$:

THEOREM A. *For $a \in \tilde{I}$, let $\tilde{C} = \tilde{C}(a)$ be the solution of* $\exp(-1/\tilde{C}) = \tilde{\rho}(a)$. *Then,*

$$\text{(A)}\qquad \lim_{N \to \infty} \max_{0 \leq n \leq N - [\tilde{C}\log N]} \frac{S_{n+[\tilde{C}\log N]} - S_n}{[\tilde{C}\log N]^{1/2}} - [\tilde{C}\log N]^{1/2} a = 0 \quad a.s.$$

Assertion (A), rewritten in the form

$$\text{(A}')\qquad \left| \max_{0 \leq n \leq N - [\tilde{C}\log N]} \frac{S_{n+[\tilde{C}\log N]} - S_n}{[\tilde{C}\log N]} - a \right| = o\left(\frac{1}{[\tilde{C}\log N]^{1/2}} \right) \quad a.s.,$$

yields a convergence rate statement in the original Erdös and Rényi (1970) law of large numbers dealing with maximum increments of the partial sums sequence over subintervals of size $K_N = [\tilde{C}\log N]$.

Moreover, for larger K_N, i.e., $K_N/\log N \to \infty$, it holds:

THEOREM B. *Let $K_N = [\tilde{K}_N]$ denote a nondecreasing integer sequence with $\tilde{K}_N/N \searrow 0$, $\tilde{K}_N/(\log N)^p \to 0$ for some $p > 2$ and $\tilde{K}_N/\log N \nearrow \infty$. Then, if*

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$a_N > 0$ is the unique positive solution of the equation

$$\rho^{K_N}\left(a_N K_N^{-1/2}\right) = K_N/N,$$

N sufficiently large, we have

$$\text{(B)} \qquad \lim_{N\to\infty}\left(\max_{0\le n\le N-K_N} \frac{S_{n+K_N}-S_n}{K_N^{1/2}} - a_N\right) = 0 \quad a.s.$$

In addition,

$$a_N \sim \left(2\log(N/K_N)\right)^{1/2} \sim (2\log N)^{1/2},$$

and, if $K_N/(\log N)^2 \to \infty$,

$$a_N - \left(2\log(N/K_N)\right)^{1/2} = o(1).$$

Hence, assertion (B) is equivalent to the convergence rate statement

$$\text{(B}'\text{)} \qquad \left|\max_{0\le n\le N-K_N} \frac{S_{n+K_N}-S_n}{a_N K_N^{1/2}} - 1\right| = o\left(\frac{1}{(\log N)^{1/2}}\right) \quad \text{a.s.}$$

where the denominator $a_N K_N^{1/2}$ can be replaced by $(2K_N\log(N/K_N))^{1/2}$, if $K_N/(\log N)^2 \to \infty$.

In a recent paper, Deheuvels, Devroye, and Lynch (1986) were able to show that a better rate $O(K_N^{-1}\log K_N)$, $K_N = [\tilde{C}\log N]$, can be obtained in (A′), the latter being best possible. Their approach has been based on a large deviation theorem of Petrov (1965) and a more precise estimate of the dependencies between overlapping subintervals. An improved version of (B′) seems to be unknown.

In what follows $X_1, X_2, \dots$ is a sequence of i.i.d. random variables satisfying

(i) $EX_1 = \mu > 0$,
(ii) $0 < \operatorname{var}(X_1) = \sigma^2 < \infty$,
(iii) $E|X_1|^3 < \infty$,
(iv) $\varphi(s) = E\exp(sX_1) < \infty$, $s_1 < s \le 0$ (for some $-\infty \le s_1 < 0$).

Put $I = \{\varphi'(s)/\varphi(s) \colon s_1 < s < 0\}$ and $\rho(1/a) = \inf\varphi(s)\exp(-s/a)$. Setting $S_0 = 0$, $S_n = X_1 + \cdots + X_n$, and $M_n = \max(S_0, S_1, \dots, S_n)$, we consider the processes $\{N(t)\}_{t\ge 0}$ and $\{L(t)\}_{t\ge 0}$, where

$$N(t) = \max\{n \ge 0 \colon M_n \le t\},$$
$$L(t) = \min\{n > 0 \colon S_n > t\}.$$

$\{N(t)\}$ was introduced by Heyde (1967) as a "*generalized* renewal process". For *nonnegative X_i's*,

$$N(t) = \max\{n \ge 0 \colon S_n \le t\},$$

and hence reduces to the "ordinary" number of renewals up to t under "failure-times" $X_1, X_2, \dots$. Clearly, $L(t) = N(t) + 1$, and $L(t)$ is known as the "first-passage time" (from $[0, t]$) of the random walk $\{S_n\}$.

In a joint work, Retka and Steinebach [see Retka (1982) and Steinebach (1984)] proved the following analogues of Theorems A and B for the "*ordinary* renewal process". Note that $X_i \geq 0$ implies (iv), with $s_1 = -\infty$, and $I = (\mu_1, \mu)$, where $\mu_1 = \operatorname{ess\,inf} X_1 \geq 0$:

THEOREM C. *For* $1/a \in I$ *let* $C = C(a)$ *be such that* $\exp(-1/C) = \rho^{\alpha}(1/a)$. *Then*

$$\text{(C)} \quad \lim_{T\to\infty}\left(\sup_{0\le t\le T-C\log T}\frac{N(t+C\log T)-N(t)}{(C\log T)^{1/2}}-(C\log T)^{1/2}a\right)=0 \quad a.s.$$

Thus, (C) yields a convergence rate statement

$$\text{(C}'\text{)} \quad \left|\sup_{0\le t\le T-C\log T}\frac{N(t+C\log T)-N(t)}{C\log T}-a\right|=o\left(\frac{1}{(\log T)^{1/2}}\right) \quad \text{a.s.}$$

in the Erdös–Rényi law for the ordinary renewal process [see e.g., Steinebach (1979) for the latter result].

Moreover, for subintervals of larger size, the following holds:

THEOREM D. *Suppose*

(iv′) $\hat{\varphi}(s) = E\exp\{-s(X_1-\mu)/\sigma\} < \infty$, $|s| < s_0$ (>0),

put $\hat{\rho}(a) = \inf \hat{\varphi}(s)\exp(-sa)$, *and let* $\{K_T\}_{T\geq 0}$ *denote a nondecreasing positive function satisfying*

(v) K_T/T *is nonincreasing*,
(vi) $K_T/\log T \nearrow \infty$,
(vii) $K_T/(\log T)^p \to 0$ *for some* $p > 2$.

Then, if $a_T > 0$ *is the unique positive solution of*

(viii) $\hat{\rho}^{K^{1/2}d_T}(\mu a_T/\sigma d_T) = K/T$,

where $d_T = (K^{1/2}/\mu) + a_T$, $K = [K_{[T]}]$, *T sufficiently large, we have*

$$\text{(D)} \quad \lim_{T\to\infty}\left(\sup_{0\le t\le T-K_T}\frac{N(t+K_T)-N(t)-K_T/\mu}{K_T^{1/2}}-a_T\right)=0 \quad a.s.$$

REMARK 1. (a) Similarly to Csörgő and Steinebach (1981), Remark 2, setting $\hat{\psi}(s) = \log\hat{\varphi}(s)$, $|s| < s_0$, we have

$$\hat{\psi}(s) = s^2/2 + O(s^3), \qquad \hat{\psi}'(s) = s + O(s^2) \qquad (s\to 0),$$

and

$$-\log\hat{\rho}(a) = a^2/2 + O(a^3) \qquad (a\to 0) \quad \text{by (iv}'\text{)}.$$

(b) The definition of a_T differs from its counterpart in the partial sum case. However, for T fixed, differentiation w.r.t. a shows that the function

$$h(a) = K^{1/2}d(a)(-\log\hat{\rho}(b(a))),$$

where $d(a) = (K^{1/2}/\mu) + a$, $b(a) = \mu a/(\sigma d(a))$, is nonnegative and strictly convex. Since $-\log \hat{\rho}(0) = 0$, and $\log(T/K) \le h(a)$ for T sufficiently large, it is obvious that there is a unique positive solution a_T in (viii). Moreover,

$$b_T = b(a_T) \to 0 \qquad (T \to \infty).$$

(c) Using (viii) and Remarks 1(a) and 1(b), we further have

$$a_T \sim \left(2\sigma^2\mu^{-3}\log(T/K)\right)^{1/2} \qquad (T \to \infty),$$

and, if $K_T/(\log T)^2 \to \infty$,

$$a_T - \left(2\sigma^2\mu^{-3}\log(T/K)\right)^{1/2} = o(1) \qquad (T \to \infty).$$

By conditions (v)–(vii), $K = [K_{[T]}]$ can even be replaced by K_T.

Hence, assertion (D) is equivalent to

$$(\mathrm{D}') \qquad \left| \sup_{0 \le t \le T - K_T} \frac{N(t + K_T) - N(t) - K_T/\mu}{a_T K_T^{1/2}} - 1 \right| = o\left(\frac{1}{(\log T)^{1/2}}\right) \quad \text{a.s.},$$

yielding a convergence rate statement in an "extended version" of the Erdös–Rényi law for the ordinary renewal process [see e.g., Steinebach (1979)]. Unfortunately, the methods could not directly be applied to generalized renewal processes. However, partially following the lines of Deheuvels, Devroye, and Lynch (1986), we are now in a position to

(a) extend Theorems C and D to generalized renewal processes or first-passage times resp., and, moreover,
(b) replace the rate $o((\log T)^{-1/2})$ in (C′) by the possibly best rate

$$O\left((\log T)^{-1}\log\log T\right).$$

2. Results. Consider the processes $\{N(t)\}, \{L(t)\}$ as defined above, based upon an i.i.d. sequence $X_1, X_2, \ldots$ satisfying conditions (i)–(iv). For $0 < K < T$, set

$$D(T, K) = \sup_{0 \le t \le T-K} (N(t + K) - N(t)) = \sup_{0 \le t \le T-K} (L(t + K) - L(t)).$$

Then Theorems C and D can be extended as follows:

THEOREM 1. *For $0 < 1/a < \mu$, with $1/a \in I$, let $C = C(a)$ be such that $\exp(-1/C) = \rho^a(1/a)$. Then,*

$$(1) \qquad \left| \frac{D(T, C\log T)}{C\log T} - a \right| = O\left(\frac{\log\log T}{\log T}\right) \quad a.s.$$

THEOREM 2. *The assertions of Theorem D can be extended to generalized renewal processes, i.e.,*

$$(2) \qquad \left| \frac{D(T, K_T) - K_T/\mu}{a_T K_T^{1/2}} - 1 \right| = o\left(\frac{1}{(\log T)^{1/2}}\right),$$

and the denominator $a_T K_T^{1/2}$ *can be replaced by* $(2\sigma^2\mu^{-3}K_T\log(T/K_T))^{1/2}$, *if* $K_T/(\log T)^2 \to \infty$.

The proof of Theorem 1 is based upon

LEMMA 1. *Under the assumptions of Theorem* 1 *let* s_a *be such that* $1/a = \varphi'(s_a)/\varphi(s_a)$. *Then, for any* $\varepsilon > 0$, c_0, c_1, c_2, c_3 *fixed, there exist constants* $A_0, B_0 > 0$ *such that, for K sufficiently large,*

(3) $P\big(N(K + c_0) \geq Ka + \left(\tfrac{1}{2} + \varepsilon\right)a \log K/s_a + c_1\big) \leq A_0\rho^{Ka}(1/a)K^{-(1+\varepsilon)}$,

(4) $P\big(N(K + c_2) \geq Ka - \left(\tfrac{5}{2} + \varepsilon\right)a \log K/s_a + c_3\big) \geq B_0\rho^{Ka}(1/a)K^{1+\varepsilon}$.

The proof of Theorem 2 makes use of

LEMMA 2. *Under the assumptions of Theorem* 2, *for any* $\varepsilon > 0$, c_0, c_1, c_2, c_3 *fixed, there exist constants* $A_0, A_1, B_0, B_1 > 0$ *such that, for T sufficiently large,*

(5) $P\big(N(K + c_0) \geq K/\mu + K^{1/2}(a_T + \varepsilon) + c_1\big) \leq A_0\hat{\rho}^{K^{1/2}d_T}(b_T)e^{-A_1 a_T}$,

(6) $P\big(N(K + c_2) \geq K/\mu + K^{1/2}(a_T - \varepsilon) + c_3\big) \geq B_0 K^{-1}\hat{\rho}^{K^{1/2}d_T}(b_T)e^{B_1 a_T}$,

where $K = [K_{[T]}]$, $d_T = (K^{1/2}/\mu) + a_T$, $b_T = \mu a_T/(\sigma d_T)$.

3. Proofs. To prove Lemmas 1 and 2, we further need the following corollaries to results of Petrov (1965) and Pollaczek (1952) [see Deheuvels, Devroye, and Lynch (1986)]:

LEMMA 3. *With the notation of Theorem* 1, *let* $\{y_n\}$ *be a real sequence satisfying* $ny_n^2 \to 0$ *as* $n \to \infty$. *Then, uniformly over all sequences* $\{z_n\}$ *with* $|z_n| \leq |y_n|$, *we have*

(7) $$P(S_n \leq n(1/a + z_n)) \sim C_0 n^{-1/2}\rho^n(1/a)e^{nz_n s_a},$$

where $C_0 > 0$ *depends on the distribution of* X_1.

LEMMA 4. *For the partial sums* $S_n = X_1 + \cdots + X_n$ *of an i.i.d. sequence, we have*

(8) $$\frac{1}{n}P(S_n \leq 0) \leq P(S_1 \leq 0, \dots, S_n \leq 0) \leq P(S_n \leq 0).$$

Let us now turn to the

PROOF OF LEMMA 1. (a) Put $n_K = [Ka + (\frac{1}{2} + \varepsilon)a \log K/s_a + c_1]$. Then, using Lemma 3,

$$\begin{aligned} &P\big(N(K + c_0) \geq Ka + \left(\tfrac{1}{2} + \varepsilon\right)a \log K/s_a + c_1\big) \\ &\quad\leq P\big(S_{n_K} \leq K + c_0\big) = P\big(S_{n_K} \leq n_K(1/a + z_K)\big) \\ &\quad\sim C_0 n_K^{-1/2}\rho^{n_K}(1/a)e^{n_K z_K s_a}, \end{aligned}$$

where $z_K = (K + c_0)/n_K - 1/a \sim -(\frac{1}{2} + \varepsilon)\log K/(Kas_a)$. Hence $n_K z_K^2 = O(K^{-1}(\log K)^2) \to 0$, and $n_K z_K \sim -(\frac{1}{2} + \varepsilon)\log K/s_a$, which implies (3).

(b) Put $m_K = [Ka - (\frac{5}{2} + \varepsilon)a \log K/s_a + c_3 + 1]$. Then, by Lemmas 3 and 4, for $m_K \geq 1$,

$$\begin{aligned} P\big(N(K + c_2) &\geq Ka - \big(\tfrac{5}{2} + \varepsilon\big)a \log K/s_a + c_3\big) \\ &\geq P\big(S_1 \leq K + c_3, \dots, S_{m_K} \leq K + c_3\big) \\ &\geq P\left(S_1 \leq \frac{K + c_3}{m_K}, \dots, S_{m_K} \leq m_K \frac{K + c_3}{m_K}\right) \\ &\geq \frac{1}{m_K} P\big(S_{m_K} \leq K + c_3\big) \sim C_0 m_K^{-3/2} \rho^{m_K}(1/a) e^{m_K \tilde{z}_K s_a}, \end{aligned}$$

where $\tilde{z}_K \sim (\frac{5}{2} + \varepsilon)\log K/(Kas_a)$. This implies (4). □

Proof of Theorem 1. Setting $K_T = C \log T$, we prove

$$\limsup_{T \to \infty} \frac{K_T}{\log K_T}\left(\frac{D(T, K_T)}{K_T} - a\right) \leq \frac{a}{2s_a} \quad \text{a.s.}, \tag{9}$$

$$\liminf_{T \to \infty} \frac{K_T}{\log K_T}\left(\frac{D(T, K_T)}{K_T} - a\right) \geq -\frac{5a}{2s_a} \quad \text{a.s.} \tag{10}$$

(a) Set $T_j = \sup\{T: K_T \leq j\}$. Then, for $T_{j-1} < T \leq T_j$, we have $j - 1 < K_T \leq j$ and $\exp(j - 1)/C < T \leq \exp(j/C)$. Following the lines in Steinebach (1981), we further estimate

$$\begin{aligned} \frac{D(T, K_T)}{\log K_T} &= \sup_{0 \leq t \leq T - K_T}\left(\frac{N(t + K_T) - N(t)}{\log K_T}\right) \\ &\leq \sup_{0 \leq t \leq T_j - 1}\left(\frac{N([t] + j + 1) - N([t])}{\log K_{T_{j-1}}}\right) = \frac{D_1(T_j, j)}{\log K_{T_{j-1}}}, \end{aligned}$$

and, by Lemma 1 and our choice of $C = C(a)$, for $\varepsilon > 0$,

$$\begin{aligned} P\big(D_1(T_j, j) &\geq K_{T_{j-1}}a + \big(\tfrac{1}{2} + \varepsilon\big)a \log K_{T_{j-1}}/s_a\big) \\ &\leq T_j P\big(N(j + 1) \geq (j - 2)a + \big(\tfrac{1}{2} + \varepsilon\big)a \log(j - 2)/s_a\big) \\ &\leq A_0 T_j \rho^{ja}(1/a) j^{-(1+\varepsilon)} \leq A_0 j^{-(1+\varepsilon)}. \end{aligned}$$

Hence, the Borel–Cantelli lemma yields

$$\limsup_{j \to \infty} \frac{K_{T_{j-1}}}{\log K_{T_{j-1}}}\left(\frac{D_1(T_j, j)}{K_{T_{j-1}}} - a\right) \leq \big(\tfrac{1}{2} + \varepsilon\big)a/s_a \quad \text{a.s.}$$

Since, for $T_{j-1} < T \le T_j$ and j sufficiently large

$$\frac{D(T, K_T)}{\log K_T} - \frac{K_T a}{\log K_T} \le \frac{D_1(T_j, j)}{\log K_{T_{j-1}}} - \frac{K_{T_{j-1}} a}{\log K_{T_{j-1}}},$$

assertion (9) is proved.

(b) To prove (10), we first estimate, for T sufficiently large and with $K = [K_{[T]}]$,

$$D(T, K_T) \ge \sup_{0 \le t \le [T] - (K+2)} (N(t+K) - N(t)) = D_0([T], K)$$

and

$$\frac{D(T, K_T)}{\log K_T} - \frac{K_T a}{\log K_T} \ge \left(\frac{\log K}{\log K_T}\right)\left(\frac{D_0([T], K)}{\log K} - \frac{Ka}{\log K}\right) + a\frac{K - K_T}{\log K_T}.$$

Since $K/K_T \to 1$, $(K - K_T)/\log K_T \to 0$ $(T \to \infty)$, it suffices to prove

$$\liminf_{n \to \infty}\left(\frac{D_0(n, K)}{\log K} - \frac{Ka}{\log K}\right) \ge -\frac{5a}{2s_a} \quad \text{a.s.}$$

for integer n. By the method of random renewal epochs, as used in Steinebach (1981), Section 4, it is then enough to show

$$(11) \quad P\Big(\max_{i=0,\dots,M(n)} (N(t_i + K) - N(t_i)) \le Ka - \left(\tfrac{5}{2} + \varepsilon\right)a \log K/s_a \text{ i.o.}\Big) = 0,$$

where $M(n) = \max\{i \ge 0\colon\ t_i + K + 2 \le n\}$, $t_i = t_{i,n}$ as defined in Steinebach (1981), Lemma 5. Let n_j denote the smallest integer n with $K = [K_n] = j$. Then, for $n_j \le n < n_{j+1}$, $[K_n] = j$ and $M(n) \ge M(n_j)$. Hence it is enough to prove (11) for the subsequence $\{n_j\}$. In the vein of Steinebach (1981), Section 4, we estimate, setting $i_n = [\delta n/K]$, $0 < \delta < 1$,

$$P\Big(\max_{i=0,\dots,M(n)} (N(t_i + K) - N(t_i)) \le Ka - \left(\tfrac{5}{2} + \varepsilon\right)a \log K/s_a\Big)$$

$$\le P\Big(\max_{i=0,\dots,i_n} (N(t_i + K) - N(t_i)) \le Ka - \left(\tfrac{5}{2} + \varepsilon\right)a \log K/s_a\Big)$$

$$+ P(M(n) < i_n).$$

Using Lemma 1, the first probability in the last line can be bounded by

$$\exp\{-\tilde{B}_0 \rho^{Ka}(1/a) K^{1+\varepsilon} n/K\} \le \exp\{-\tilde{B}_0 K^{\varepsilon}\},$$

for some $\tilde{B}_0 > 0$, remembering $\rho^{Ka}(1/a) \ge 1/n$, $K = [K_n]$. For $n = n_j$, this is summable in j, since $K = j$. By Lemma 5 in Steinebach (1981), we have $\sum P(M(n) < i_n) < \infty$, thus also for the subseries $n = n_j$. Now use of the Borel–Cantelli lemma completes the proof. □

The proofs of Lemma 2 and Theorem 2 are slightly different.

PROOF OF LEMMA 2. (a) Let s_T be such that $b_T = \hat{\varphi}'(s_T)$, hence $\hat{\rho}(b_T) = \hat{\varphi}(s_T)\exp(-s_T b_T)$. Put $n_T = [K/\mu + K^{1/2}(a_T + \varepsilon) + c_1]$. Then,

$$\begin{aligned} P\big(N(K + c_0) \geq K/\mu + K^{1/2}(a_T + \varepsilon) + c_1\big) &\leq P\big(S_{n_T} \leq K + c_0\big) \\ &\leq P\left(-\frac{S_{n_T} - n_T\mu}{\sigma} \geq K^{1/2}\frac{\mu(a_T + \varepsilon)}{\sigma} + c_0\right) \\ &\leq \hat{\varphi}^{n_T}(s_T)\exp\big(-s_T\big(K^{1/2}d_T b_T + K^{1/2}\tilde{\varepsilon} + \tilde{c}_0\big)\big) \\ &\leq \hat{\rho}^{K^{1/2}d_T}(b_T)\exp\big(s_T b_T\big(n_T - K^{1/2}d_T\big) - K^{1/2}s_T\tilde{\varepsilon} - \tilde{c}_0 s_T\big). \end{aligned}$$

Since $s_T \sim b_T \to 0$, $n_T - K^{1/2}d_T \sim K^{1/2}\varepsilon$, $K^{1/2}s_T \sim K^{1/2}b_T \sim \mu^2 a_T/\sigma = o(K^{1/2})$, this proves (5).

(b) Put $m_T = [K/\mu + K^{1/2}(a_T - \varepsilon) + c_3 + 1]$. Then, taking s_T as defined in (a),

$$\begin{aligned} &P\big(N(K + c_2) \geq K/\mu + K^{1/2}(a_T - \varepsilon) + c_3\big) \\ &\quad\geq \frac{1}{m_T}P\big(S_{m_T} \leq K + c_3\big) \\ &\quad\geq \frac{1}{m_T}P\left(-\frac{S_{m_T} - m_T\mu}{\sigma} \geq K^{1/2}\frac{\mu(a_T - \varepsilon)}{\sigma} + \tilde{c}_2\right). \end{aligned}$$

Using associated probability measures $\hat{P}_{s,T}$, defined by

$$\hat{P}_{s,T}(A) = \int_A \exp\big(s\hat{S}_{m_T}\big)/\hat{\varphi}^{m_T}(s)\,dP,$$

where $\hat{S}_{m_T} = -(S_{m_T} - m_T\mu)/\sigma$, we have

$$\begin{aligned} &P\left(\hat{S}_{m_T} - m_T b_T \geq K^{1/2}\frac{\mu(a_T - \varepsilon)}{\sigma} - m_T b_T + \tilde{c}_2\right) \\ &\quad= P\big(\hat{S}_{m_T} - m_T b_T \geq \big(K^{1/2}d_T - m_T\big)b_T - K^{1/2}\tilde{\varepsilon} + \tilde{c}_2\big) \\ &\quad\geq \hat{\rho}^{m_T}(b_T)\exp\big(K^{1/2}s_T\tilde{\varepsilon}/2\big)\hat{P}_{s_T,T}\left(-\frac{3\tilde{\varepsilon}}{4} \leq \frac{\hat{S}_{m_T} - m_T b_T}{K^{1/2}} \leq -\frac{\tilde{\varepsilon}}{2}\right). \end{aligned}$$

For the last inequality, note $K^{1/2}d_T - m_T \sim K^{1/2}\varepsilon$ and $b_T \to 0$. Taylor series expansion of the cumulant generating function shows that $(\hat{S}_{m_T} - m_T b_T)/K^{1/2}$ is asymptotically normal under $\hat{P}_{s_T,T}$, $T \to \infty$. Since, for large T, $m_T \leq d_T$, $m_T \sim K/\mu$, and $K^{1/2}s_T \sim \mu^2 a_T/\sigma$, this completes the proof of (6). □

PROOF OF THEOREM 2. (a) Set $T_j = \sup\{T\colon K = [K_{[T]}] = j\}$. Using the same notations as in the proof of Theorem 1, for $T_{j-1} < T \leq T_j$, we have $[K_{[T]}] = j$ and

$$\frac{D(T, K_T) - K_T/\mu}{K_T^{1/2}} - a_T \leq \frac{D_1(T_j, j) - j/\mu}{j^{1/2}} - a_{T_j} + \big(a_{T_j} - a_T\big).$$

Hence, for the lim sup-part, we prove, for $\varepsilon > 0$,

$$\limsup_{j\to\infty}\left(\frac{D_1(T_j, j) - j/\mu}{j^{1/2}} - a_{T_j}\right) \le \varepsilon \quad \text{a.s.,} \tag{12}$$

$$\lim_{j\to\infty} \sup\{a_{T_j} - a_T\colon T_{j-1} < T \le T_j\} = 0. \tag{13}$$

Assertion (12) is a consequence of Lemma 2 and the choice of a_T, since

$$P\big(D_1(T_j, j) \ge j/\mu + j^{1/2}(a_{T_j} + \varepsilon)\big) \le A_0 T_j(j/T_j)\exp\big(-A_1 a_{T_j}\big).$$

Similar to Csörgő and Steinebach (1981), we estimate by (vi) and (vii),

$$A_1 a_{T_j} \ge \log\big(K_{T_j}^3\big) = \log(j^3).$$

Hence, assertion (11) follows by using the Borel–Cantelli lemma. Moreover, similar to the partial sum case, for $T_{j-1} < T \le T_j$, we have

$$0 \le a_{T_j} - a_T = O\left(\frac{\log T_j}{j}\right),$$

the latter expression tending to zero by assumption (vi). Assertions (12) and (13) yield

$$\limsup_{T\to\infty}\left(\frac{D(T, K_T) - K_T/\mu}{K_T^{1/2}} - a_T\right) \le 0 \quad \text{a.s.}$$

(b) The proof of

$$\liminf_{T\to\infty}\left(\frac{D(T, K_T) - K_T/\mu}{K_T^{1/2}} - a_T\right) \ge 0 \quad \text{a.s.}$$

follows in a similar vein as part (b) of Theorem 1. We estimate, for large T,

$$\frac{D(T, K_T) - K_T/\mu}{K_T^{1/2}} - a_T \ge \left(\frac{K}{K_T}\right)^{1/2}\left(\frac{D_0([T], K) - K/\mu}{K^{1/2}} - a_{[T]}\right) + \frac{K - K_T}{\mu K_T^{1/2}} + \frac{K^{1/2}a_{[T]} - K_T^{1/2}a_T}{K_T^{1/2}},$$

$K = [K_{[T]}]$. Assumption (v) yields $K - K_T \to 0$ $(T \to \infty)$, and hence also $(K/K_T)^{1/2} \to 1$. Moreover, similar to part (a)

$$K^{1/2}a_{[T]} - K_T^{1/2}a_T = O\big((\log T)/K_T^{1/2}\big).$$

Hence it suffices to prove, for integer n,

$$\liminf_{n\to\infty}\left(\frac{D_0(n, K)}{K^{1/2}} - a_n\right) \ge -\varepsilon \quad \text{a.s.,} \tag{14}$$

$K = [K_n]$. Again using random renewal epochs t_i as in part (b) of Theorem 1, we obtain by Lemma 2 and our choice of a_n

$$P\Big(\max_{i=0,\dots,i_n}(N(t_i+K)-N(t_i))\le K/\mu+K^{1/2}(a_n-\varepsilon)\Big)$$
$$\le \exp\big(-\tilde{B}_0K^{-2}e^{B_1a_n}\big)=O\Big(\frac{1}{n^2}\Big),$$

$\tilde{B}_0>0$, $i_n=[\delta_1 n/K]$, since $a_n\sim(2\sigma^2\mu^{-3}\log(n/K))^{1/2}$, and $K_n^2/a_n^{4p}=o(1)$. Thus, the Borel–Cantelli lemma yields (14), which completes the proof. □

4. Strong invariance principles. Theorems A, C, and 1 give strong noninvariance principles, since the functional dependence $C=C(\alpha)$, given there, uniquely determines the underlying distribution via its moment-generating function [see Erdös and Rényi (1970) and Steinebach (1981)]. On the other hand, Theorems B, D, and 2 give strong invariance principles, if $K_N(K_T)$ is such that $K_N/(\log N)^2$ resp. $K_T/(\log T)^2$ tends to infinity. For smaller $K_N(K_T)$, the latter theorems treat an indefinite range between strong invariance and noninvariance, since (via ρ) $a_N(a_T)$ may depend upon a number of cumulants of the distribution.

Now, suppose the underlying probability space $(\Omega,\mathscr{A},P)$ to be rich enough that the Kómlos–Major–Tusnády (1976) strong approximation holds, i.e., one can define a standard Wiener process $\{W(t)\}_{t\ge0}$ on it such that

$$\sup_{0\le t\le T}\left|\frac{S_{[t]}-[t]\mu}{\sigma}-W(t)\right|=O(\log T)\quad\text{a.s.,}\tag{15}$$

where again $S_{[t]}$ denotes the $[t]$th partial sum of an i.i.d. sequence $X_1, X_2,\dots$ satisfying conditions (i), (ii), and (iv′). Assertion (15) is a slightly extended version [see Csörgő and Révész (1981)] of the following original form:

$$\max_{1\le k\le n}|S_k-T_k|=O(\log n)\quad\text{a.s.,}\tag{16}$$

where T_k denotes the kth partial sum of an i.i.d. sequence $Y_1, Y_2,\dots$ of normal rv's with same mean μ and variance σ^2.

For $K_N/(\log N)^2\to\infty$, an extension of Theorem B has been derived by Csörgő and Steinebach (1981), using (15) and corresponding results for the Wiener process. In a similar vein, an extension of Theorems D and 2 would be desirable for $K_T/(\log T)^2\to\infty$. But, using (15), one only gets the following approximation:

THEOREM 3. *Under the aforementioned assumptions,*

$$\sup_{0\le t\le T}\left|\frac{N(t)-t/\mu}{\sigma/\mu^{3/2}}-W(t)\right|=O\Big((T\log\log T)^{1/4}(\log T)^{1/2}\Big)\quad a.s.\tag{17}$$

OUTLINE OF PROOF. Assertion (17) can easily be derived by decomposing as follows:

$$\frac{N(t)-t/\mu}{\sigma/\mu^{3/2}} - W(t) = \left(\frac{N(t)-S_{N(t)}/\mu}{\sigma/\mu^{3/2}} - \mu^{1/2}W(N(t))\right)$$
$$-\left(\frac{S_{N(t)}-t}{\sigma/\mu^{1/2}}\right) + \mu^{1/2}(W(N(t)) - W(t/\mu)).$$

Using (15) and the strong law of large numbers for $\{N(t)\}_{t\geq 0}$, the first summand can be shown to be of order $O(\log T)$ a.s. uniformly in $0 \leq t \leq T$. Since $\varphi(s) = E\exp(sX_1) < \infty$, $|s| < s_1$, by direct estimates

$$\max(X_1, \dots, X_n) = O(\log N) \quad \text{a.s.},$$

which further implies

$$\sup_{0\leq t\leq T}\left|\frac{S_{N(t)}-t}{\sigma/\mu^{1/2}}\right| \leq \sup_{0\leq t\leq T}\left|\frac{X_{N(t)+1}}{\sigma/\mu^{1/2}}\right| = O(\log T) \quad \text{a.s.},$$

again using the SLLN for $\{N(t)\}_{t\geq 0}$. However, in view of the law of iterated logarithm for $\{N(t)\}_{t\geq 0}$ and the results of Csörgö and Révész (1979),

$$\sup_{0\leq t\leq T}\mu^{1/2}\left|W(N(t)) - W\left(\frac{t}{\mu}\right)\right| = O\left((T\log\log T)^{1/4}(\log T)^{1/2}\right) \quad \text{a.s.},$$

which cannot be improved.

In view of Theorem 3 and the final remark in the proof, an extension of Theorems D and 2 via strong approximations can only be stated for "very large increments", i.e., $K_T/(T\log\log T)^{1/2}\log T \to \infty$. Such an approach has recently been given by Horváth (1984).

It is clear, however, that a strong invariance principle must hold for all K_T with $K_T/(\log T)^2 \to \infty$. For further considerations, the following lemma should be helpful:

LEMMA 5. *Suppose the underlying probability space to be rich enough that* (16) *holds. For* $t \geq 0$ *set*

$$M(t) := \max\{n \geq 0\colon T_1, \dots, T_n \leq t\}.$$

Then, we have

$$\sup_{0\leq t\leq T}|N(t) - M(t)| = O(\log T) \quad a.s. \tag{18}$$

PROOF. To prove (18) we combine the K–M–T strong approximation (16), the SLLN for $\{N(t)\}$ and $\{M(t)\}$ and a version of the Erdös–Rényi strong law for $\{M(t)\}$, i.e., for each $C > 0$,

$$\lim_{T\to\infty} \sup_{0\le t\le T} \frac{M(t + C\log T) - M(t)}{C\log T} = a(C) \quad \text{a.s.,} \tag{19}$$

[see e.g., Theorem 1, for $\varphi(s) = \exp(s\mu + s^2\sigma^2/2)$]. Suppose

$$\max_{1\le k\le N} |S_k - T_k| \le C_0 \log N \quad \forall\, N \ge N_0(\omega), \qquad C_0 > 0, \tag{20}$$

$$\delta_0 T \le N(T) \le \delta_1 T \quad \forall\, T \ge T_0(\omega), \qquad \delta_0 < \frac{1}{\mu} < \delta_1. \tag{21}$$

Let $N(t) = k \le N(T)$, i.e., $S_1, \dots, S_k \le t$, $S_{k+1} > t$. If $\delta_0 T_0 \ge N_0$, in view of (20) and (21),

$$T_1, \dots, T_k \le t + C_0 \log N(T) \le t + C_0 \log(\delta_1 T),$$

and

$$T_{k+1} > t - C_0 \log(N(T) + 1) \ge t - C_0 \log(\delta_1 T + 1).$$

Moreover, if T_0 is such that $C_0\log(\delta_1 T + 1) \le 2C_0 \log T$, for all $T \ge T_0$ we obtain

$$M(t + 2C_0 \log T) \ge k = N(t),$$
$$M(t - 2C_0 \log T) \le k = N(t).$$

Hence, setting $C = 2C_0$, for $T \ge T_0$,

$$\sup_{0\le t\le T} \frac{|N(t) - M(t)|}{\log T}$$
$$\le \sup_{0\le t\le C\log T} \frac{|N(t) - M(t)|}{\log T} + \sup_{0\le t\le T} \frac{M(t + C\log T) - M(t)}{\log T}.$$

By (19) and the SLLN for $\{N(t)\}$ and $\{M(t)\}$, this completes the proof.

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