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CLUMP COUNTS IN A MOSAIC

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A mosaic process is formed by centering independent and identically distributed random shapes at the points of a Poisson process in k-dimensional space. Clusters of overlapping shapes are called clumps. This paper provides approximations to the distribution of the number of clumps of a specified order within a large region. The approximations cover two different situations—"moderate-intensity" mosaics, in which the covered proportion of the region is neither very large nor very small; and "sparse" mosaics, in which the covered proportion is quite small. Both these mosaic types can be used to model observed phenomena, such as counts of bacterial colonies in a petri dish or dust particles on a membrane filter.

1. Introduction. A mosaic process (Boolean model) is a completely spatially random coverage process, in which independent and identically distributed random sets are centred at points of a homogeneous Poisson process in $\mathbb{R}^k$. The sets are permitted to overlap at will. Connected clusters of overlapping sets are called clumps. Our aim in this paper is to describe second-order properties of clump counts, using limit theory and approximate methods to overcome some of the mathematical difficulties which prevent an exact description of clumping.

Even first-order properties of random clumping are hard to describe with much precision. For example, there is no known exact formula for the expected number of clumps per unit content of $\mathbb{R}^k$, even in simple cases such as the distribution of fixed radius discs in the plane. Nevertheless, if we define the mean number of clumps per unit area to equal the limit of the mean number per unit area in a k-dimensional rectangular prism, as the side lengths of the prism diverge in an arbitrary way, we may prove by extending an argument of Grimmett (1976, 1981) (see also Kesten 1982, page 239ff) that the mean number equals $E\{C^{-1}I(C > 0)\}$, where C is the content of the clump containing the origin. Use of this formula is very restricted from a practical viewpoint, since we cannot work out the expectation. Armitage (1949) and Irwin, Armitage, and Davies (1949) have given approximate formulas in the case of certain fixed shapes in two-dimensional sparse mosaics; Mack (1954, 1956) has provided an upper bound to the mean number of clumps per unit content for convex shapes in two and three dimensions; and Kellerer (1983) has given an exact formula for the mean number of clumps minus voids in two dimension. First-order properties are often related to the so-called principal formulas of integral geometry; see for example Blaschke (1949, page 37), Santaló (1953, Chapter 8; 1976, Chapters 7 and 15), and Miles (1974). Kendall and Moran (1963, Chapter 5), Roach (1968,

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Chapter 4), and Santaló (1976, Chapter 6) have reviewed most of the first-order work on clumping.

Some of the difficulties connected with clumping arise because infinite clumps can occur in many realistic circumstances; see Gilbert (1968) and Hall (1985). This makes it difficult to define the centre of an arbitrary clump. The property of infinite clumping, or continuum percolation, has important applications in the physical sciences, and is being studied increasingly in the case of random-sized shapes; see for example Kertész and Vicsez (1982), Gawlinski and Redner (1983), and Phani and Dhar (1984).

Second-order properties seem to be even more elusive than first-order ones. We shall not attempt to discuss exact properties of the distribution of clump count about its mean. Instead, we shall examine clump count within large regions, and provide approximations to the distribution by proving limit theorems as the size of the region increases. There are two different mosaic types which are of interest in this context: "moderate-intensity" mosaics, in which the expected proportion of covered content remains roughly constant as the region increases; and "sparse" mosaics, in which the proportion of covered content decreases to zero. The opening paragraphs of Mack (1953) compare practical applications of these two situations. Moderate-intensity mosaics can arise when one is counting numbers of bacterial colonies, and sparse mosaics may be used to model dust-particle counting experiments. Each is a variant of the same basic stochastic model, which we now describe.

Let $\mathscr{P} \equiv \{\mathbf{X}_1, \mathbf{X}_2, \dots\}$ be a homogeneous Poisson point process with intensity λ in $\mathbb{R}^k$, and let S be a random subset of $\mathbb{R}^k$, where $k \geq 1$. For our purpose there is no real loss of generality in taking S to be either a random closed set, or a random open set; see Matheron (1975, pages 27 and 48) for definitions. This ensures that scalar quantities such as $\|S\|$, $\bar{s}(S)$, and $\text{rad}(S)$, to be introduced below, are well-defined random variables taking values on the extended real line. Let $S_1, S_2, \dots$ be independent copies of S, independent also of $\mathscr{P}$. For each $\delta > 0$ and $i \geq 1$, define

$$\mathbf{X}_i + \delta S_i \equiv \{\mathbf{X}_i + \delta \mathbf{x} \colon \mathbf{x} \in S_i\}.$$

We shall study the mosaic $\mathscr{C} = \mathscr{C}(\delta, \lambda)$ generated by overlapping random sets $\mathbf{X}_i + \delta S_i$, $i \geq 1$. The process $\mathscr{P}$ will be called the Poisson process *driving* $\mathscr{C}$, and the sets S_i will be termed random *shapes*, to help distinguish them from the random sets $\mathbf{X}_i + \delta S_i$. The set $\mathbf{X}_i + \delta S_i$ will be said to be *centred* at $\mathbf{X}_i$.

A moderate-intensity mosaic may be obtained by permitting $\delta = \delta(\lambda)$ to converge to zero as $\lambda \to \infty$, in such a manner that $\delta^k \lambda \to \rho$ where $0 < \rho < \infty$. Examining the mosaic pattern within a fixed region $\mathscr{R}$ for this process, is virtually the same as taking both δ and λ to be constant and viewing the resulting mosaic within a growing region

$$a\mathscr{R} \equiv \{a\mathbf{x} \colon \mathbf{x} \in \mathscr{R}\},$$

as $a \to \infty$. A sparse mosaic may be obtained by varying δ and λ such that $\delta^k \lambda \to 0$. If in addition $\delta \to 0$ and $\lambda \to \infty$, and we examine $\mathscr{C}$ within a fixed region $\mathscr{R}$, then we are effectively viewing within a growing region $a\mathscr{R}$ a mosaic in which δ is fixed and Poisson intensity decreases to zero.

It may be proved that under mild regularity conditions on the distribution of S, infinite clumps fail to occur if the mosaic is sparse enough; see Hall (1985). From that point of view at least, counting problems are easier for sparse than for moderate-intensity mosaics. In the moderate-intensity case we shall confine ourselves to counting singleton clumps (isolated sets). Our techniques extend to clumps of any fixed finite order, but calculation of the large-region variance is very tedious for clumps of order 2 or more. In many applications of mosaic process theory, overlapping sets are considered to "interact" in some sense. Thus, singleton clumps represent the inert part of the mosaic, and are important in their own right.

We shall count clumps of arbitrary order in sparse mosaics. The moderate-intensity case will be treated in Section 2, and sparse mosaics in Section 3. All proofs will be deferred to Section 4.

We conclude this section by introducing necessary notation. The set $\mathscr{R} \subseteq \mathbb{R}^k$ will always be assumed Riemann measurable (i.e., to have a Riemann integrable indicator function), to be bounded, and to have strictly positive $[k]$ content. Given a Lebesgue measurable set $\mathscr{S} \subseteq \mathbb{R}^k$, define $\|\mathscr{S}\|$ to be the $[k]$ Lebesgue measure of $\mathscr{S}$ and let

$$\bar{s}(\mathscr{S}) \equiv \inf\{\|\mathscr{T}\|: \mathscr{T} \text{ is a sphere and } \mathscr{S} \subseteq \mathscr{T}\}.$$

The "smallest sphere containing $\mathscr{S}$" is defined to be the closed sphere $\bar{\mathscr{T}}(\mathscr{S})$ for which $\|\mathscr{T}(\mathscr{S})\| = \bar{s}(\mathscr{S})$. The radius of $\mathscr{S}$, $\operatorname{rad}(\mathscr{S})$, is the radius of $\bar{\mathscr{T}}(\mathscr{S})$. When S is a random set, $\bar{s}(S)$ and $\operatorname{rad}(S)$ are random variables.

If $\operatorname{rad}(S) < \infty$ with probability 1 then we may always choose a proper random vector $\mathbf{Y}$ such that the sphere $\bar{\mathscr{T}}(\mathbf{Y} + S)$, of radius $\operatorname{rad}(S)$, is centred at the origin. Suppose $\mathbf{Y}_i + S_i$, $i \geq 1$, are independent copies of $\mathbf{Y} + S$, independent also of $\mathscr{P}$. Then the mosaic generated by sets $\mathbf{X}_i + \mathbf{Y}_i + S_i$, $i \geq 1$, has the same properties as that generated by sets $\mathbf{X}_i + S_i$, $i \geq 1$. (Proof: Conditional on $S_1, S_2, \ldots,$ the point process $\{\mathbf{X}_1 + \mathbf{Y}_1, \mathbf{X}_2 + \mathbf{Y}_2, \ldots\}$ is homogeneous Poisson with intensity λ.) For this reason there is no loss of generality in assuming that S has the property

$$\operatorname{rad}(S) = \sup\{|\mathbf{X}|: \mathbf{x} \in S\}.$$

On several occasions during our work we shall require the notion of an arbitrary set in the mosaic generated by sets $\mathbf{X}_i + S_i$, $i \geq 1$. One way of coping with arbitrariness is to use the concept of Palm measure; see, e.g., Papangelou (1974). Since our coverage process is completely spatially random and homogeneous, the following simpler procedure will give the same result. Let $\mathscr{P} \equiv \{\mathbf{Y}_1, \mathbf{Y}_2, \ldots\}$ be a homogeneous Poisson point process with intensity λ in $\mathbb{R}^k$, and let $S_1, S_2, \ldots$ be independent copies of the random set S, independent also of $\mathscr{P}$. Fix some point $\mathbf{x} \in \mathbb{R}^k$, and define $\mathbf{Z}_1 = \mathbf{x}$ and $\mathbf{Z}_i = \mathbf{Y}_{i-1}$ for $i \geq 2$. Then

$$\mathscr{P}' \equiv \mathscr{P} \cup \{\mathbf{x}\} = \{\mathbf{Z}_1, \mathbf{Z}_2, \ldots\}$$

is a Poisson process "conditional on some arbitrary point being sited at $\mathbf{x}$." The properties of $\mathbf{Z}_1 + S_1$ in the coverage process $\mathscr{C}'$ generated by $\mathbf{Z}_i + S_i$, $i \geq 1$, are defined to be those of an arbitrary set in the mosaic $\mathscr{C}$ generated by $\mathbf{X}_i + S_i$, $i \geq 1$. In particular, the probability $P(\mathbf{X}_1 + S_1 \text{ isolated})$ appearing in formulas (2.1) and (2.2) should be interpreted in this way. When in Section 3 we speak of the event that an arbitrary random set is part of a clump of order n, we are really referring to the set $\mathbf{Z}_1 + S_1$ just defined. Note that $\mathscr{C}'$ is a mosaic "conditional on some arbitrary set being centred at $\mathbf{x}$."

We shall also require the concept of two arbitrary sets centred within a given region $\mathscr{R}$. Consider the mosaic $\mathscr{C}$ driven by Poisson process $\mathscr{P}$. Let M denote the number of points of $\mathscr{P}$ within a given subset $\mathscr{R}$ of $\mathbb{R}^k$. Conditional on $M \geq 1$, call the M points $\mathbf{X}_1^*, \dots, \mathbf{X}_M^*$, numbered in completely random order, and let $\mathbf{X}_i^* + S_i^*$ denote the random set centred at $\mathbf{X}_i^*$. We define the probability

$$p = P(\mathbf{X}_1 + S_1, \mathbf{X}_2 + S_2 \text{ both isolated} | \mathbf{X}_1, \mathbf{X}_2 \in \mathscr{R})$$

appearing in (2.2), to be

$$p \equiv P(\mathbf{X}_1^* + S_1^*, \mathbf{X}_2^* + S_2^* \text{ both isolated} | M \geq 2).$$

2. Moderate-intensity mosaic. We begin by considering the mosaic generated by sets $\mathbf{X}_i + S_i$, $i \geq 1$. A set $\mathbf{X}_i + S_i$ will be called *isolated* if it has empty intersection with all other random sets. Let N denote the number of isolated random sets centred within the region $\mathscr{R}$. Since expectation is a linear operator,

$$E(N) = \lambda \|\mathscr{R}\| P(\mathbf{X}_1 + S_1 \text{ isolated}), \tag{2.1}$$

where $P(\mathbf{X}_1 + S_1 \text{ isolated})$ denotes the probability that an arbitrary random set is isolated. One way of computing the variance of N is to consider the driving Poisson process as the limit of uniform distributions. Given a very large region $\mathscr{A}_n$ containing $\mathscr{R}$, distribute n points $\mathbf{X}_1, \dots, \mathbf{X}_n$ independently and uniformly within $\mathscr{A}_n$. Let $S_1, \dots, S_n$ be independent and identically distributed random shapes, and let N_n equal the number of sets $\mathbf{X}_i + S_i$ $(1 \leq i \leq n)$ which are isolated and centred within $\mathscr{R}$. Then

$$N_n = \sum_{i=1}^{n} I(\mathbf{X}_i + S_i \text{ isolated and } \mathbf{X}_i \in \mathscr{R}),$$

where $I(E)$ denotes the indicator function of an event E. Thus,

$$\begin{aligned} E(N_n^2) &= nP(\mathbf{X}_1 + S_1 \text{ isolated and } \mathbf{X}_1 \in \mathscr{R}) \\ &\quad + (n^2 - n) P(\mathbf{X}_1 + S_1, \mathbf{X}_2 + S_2 \text{ both isolated}; \mathbf{X}_1, \mathbf{X}_2 \in \mathscr{R}) \\ &= \|\mathscr{R}\| \frac{n}{\|\mathscr{A}_n\|} P(\mathbf{X}_1 + S_1 \text{ isolated} | \mathbf{X}_1 \in \mathscr{R}) \\ &\quad + \|\mathscr{R}\|^2 (1 - n^{-1}) \left(\frac{n}{\|\mathscr{A}_n\|} \right)^2 \\ &\qquad \times P(\mathbf{X}_1 + S_1, \mathbf{X}_2 + S_2 \text{ both isolated } | \mathbf{X}_1, \mathbf{X}_2 \in \mathscr{R}). \end{aligned}$$

Letting $n \to \infty$ and $\mathscr{A}_n$ increase in such a manner that $n / \|\mathscr{A}_n\| \to \lambda$, we deduce a formula for $E(N^2)$. Thus,

$$\begin{aligned} \operatorname{var}(N) = {} & \lambda\|\mathscr{R}\|P(\mathbf{X}_1 + S_1 \text{ isolated}) \\ & +(\lambda\|\mathscr{R}\|)^2\big\{P(\mathbf{X}_1 + S_1, \mathbf{X}_2 + S_2 \text{ isolated} \mid \mathbf{X}_1, \mathbf{X}_2 \in \mathscr{R}) \\ & \qquad - P(\mathbf{X}_1 + S_1 \text{ isolated})^2\big\}. \end{aligned} \tag{2.2}$$

Now we turn to our mathematical model for a moderate-intensity mosaic, generated by random sets $\mathbf{X}_i + \delta S_i$. The driving Poisson process has intensity λ, and $\delta = \delta(\lambda) \to 0$ as $\lambda \to \infty$ in such a manner that $\delta^k\lambda \to \rho$ $(0 < \rho < \infty)$. As a prelude to describing the limiting behaviour of isolated clump count, we introduce the mosaic $\mathscr{C}_0$ in which the driving Poisson process has intensity ρ and all shapes are distributed as S (not δS). Let Q be the probability measure associated with $\mathscr{C}_0$, and let $S^{(1)}$ and $S^{(2)}$ be two independent copies of S, independent also of $\mathscr{C}_0$. We shall say that $S^{(1)}$ is isolated if it intersects no shape in $\mathscr{C}_0$, and that $\mathbf{x} + S^{(1)}$ and $S^{(2)}$ are both isolated if $(\mathbf{x} + S^{(1)}) \cap S^{(2)} = \varnothing$ and neither set intersects any shape in $\mathscr{C}_0$. The following integrals both are relevant to our formula for asymptotic variance of N:

$$I_1 \equiv \int_{\mathbb{R}^k} \big\{Q(\mathbf{x} + S^{(1)}, S^{(2)} \text{ both isolated}) - Q(S^{(1)} \text{ isolated})^2\big\}\, d\mathbf{x}$$

and

$$I_2 \equiv \int_{\mathbb{R}^k} Q\big\{(\mathbf{x} + S^{(1)}) \cap S^{(2)} \neq \varnothing\big\}\, d\mathbf{x}. \tag{2.3}$$

(Note that the integrand of I_2 could equally have been written

$$P\big\{(\mathbf{x} + S^{(1)}) \cap S^{(2)} \neq \varnothing\big\}.)$$

THEOREM 2.1. *If $E\{\bar{s}(S)\} < \infty$ then I_2 is finite, and if $E\{\bar{s}(S)^2\} < \infty$ then I_1 converges absolutely.*

We are now in a position to describe asymptotic properties of isolated clump count.

THEOREM 2.2. *Assume that $\delta \to 0$ as $\lambda \to \infty$, in such a manner that $\delta^k\lambda \to \rho$ where $0 < \rho < \infty$. Then*

$$(\lambda\|\mathscr{R}\|)^{-1}E(N) \to Q(S^{(1)} \textit{ isolated}). \tag{2.4}$$

If in addition $E\{\bar{s}(S)^2\} < \infty$ then

$$\begin{aligned} (\lambda\|\mathscr{R}\|)^{-1}\operatorname{var}(N) \to \kappa^2 \equiv {} & Q(S^{(1)} \textit{ isolated}) + \rho I_1 + 4\rho Q(S^{(1)} \textit{ isolated}) \\ & \times \int_{\mathbb{R}^k} Q\big\{(\mathbf{x} + S^{(1)}) \cap S^{(2)} \neq \varnothing;\, S_2^{(2)} \textit{ isolated}\big\}\, d\mathbf{x} \end{aligned} \tag{2.5}$$

and

$$\{N - E(N)\}/(\lambda\|\mathscr{R}\|)^{1/2} \to N(0, \kappa^2) \tag{2.6}$$

in distribution.

The probability $Q(S^{(1)}$ isolated), which occurs repeatedly in formulas (2.4) and (2.5), is perhaps most easily calculated in the case where S is a uniformly oriented (i.e., isotropic) convex set. For example, in that situation it may be deduced from the work of Mack (1954), for example, that for the case of $k = 2$ dimensions,

$$Q(S^{(1)} \text{ isolated}) = e^{-\alpha\rho}E\left[\exp\left\{-\rho\left(\|S\|_2 + (2\pi)^{-1}\beta\|\partial S\|_1\right)\right\}\right],$$

where $\alpha \equiv E(\|S\|_2)$ denotes the mean area of S, and $\beta \equiv E(\|\partial S\|_1)$ equals the mean perimeter of S.

Calculation of the integrals in the formula for κ^2 is more complicated. We shall treat only the case where shapes are fixed spheres of radius r, centred (without loss of generality) at the origin. Let v_k and s_k denote, respectively, the $[k]$ content and $[k-1]$ surface content of the $[k]$ unit sphere. In fact, $s_k = kv_k = 2\pi^{k/2}/\Gamma(k/2)$. Then for any $k \ge 1$,

$$Q(S^{(1)} \text{ isolated}) = \exp\left\{-\rho(2r)^k v_k\right\},$$

$$Q\left\{(\mathbf{x} + S^{(1)}) \cap S^{(2)} \ne \varnothing;\, S^{(2)} \text{ isolated}\right\} = Q\left\{(\mathbf{x} + S^{(1)}) \cap S^{(2)} \ne \varnothing\right\} \times Q(S^{(1)} \text{ isolated}),$$

$$Q\left\{(\mathbf{x} + S^{(1)}) \cap S^{(2)} \ne \varnothing\right\} = \begin{cases} 1 & \text{if } |\mathbf{x}| \le 2r, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Q(\mathbf{x} + S^{(1)}, S^{(2)} \text{ both isolated}) = \begin{cases} 0 \quad \text{if } |\mathbf{x}| \le 2r, \\ Q(S^{(1)} \text{ isolated})^2 \exp\left\{\rho(2r)^k B(|\mathbf{x}|/4r)\right\} \\ \quad \text{if } 2r < |\mathbf{x}| < 4r, \\ Q(S^{(1)} \text{ isolated})^2 \quad \text{if } |\mathbf{x}| \ge 4r, \end{cases}$$

where

$$B(x) \equiv \begin{cases} 2\pi^{(k-1)/2}\left\{\Gamma\left(\dfrac{1}{2} + \dfrac{k}{2}\right)\right\}^{-1} \displaystyle\int_x^1 (1 - y^2)^{(k-1)/2}\, dy & \text{if } 0 \le x < 1, \\ 0 \quad \text{otherwise,} \end{cases}$$

denotes the content of the lens of intersection of two unit $[k]$ spheres whose centres are distant $2x$ apart. Combining these results we see that

$$\kappa^2 = \exp\left\{-\rho(2r)^k v_k\right\} + \rho(4r)^k s_k \exp\left\{-2\rho(2r)^k v_k\right\} \times \int_{1/2}^1 x^{k-1}\left[\exp\left\{\rho(2r)^k B(x)\right\} - 1\right] dx + 3\rho(2r)^k v_k \exp\left\{-2\rho(2r)^k v_k\right\}.$$

The integral here may be calculated numerically, given values of k, ρ, and r.

3. Sparse mosaic. In this section it is notationally convenient to assume that the shape S is connected with probability one. In that case we shall say that a set $\mathbf{X}_i + S_i$ is part of a clump of order n if there exist sets $\mathbf{X}_{j_1} + S_{j_1}, \ldots, \mathbf{X}_{j_{n-1}} + S_{j_{n-1}}$ such that the set

$$(\mathbf{X}_i + S_i) \cup \bigcup_{l=1}^{n-1} (\mathbf{X}_{j_l} + S_{j_l})$$

is connected and has empty intersection with each set $\mathbf{X}_m + S_m$ whose index does not appear in the collection $\{i, j_1, \ldots, j_{n-1}\}$. Thus, a clump of order n cannot be part of a clump of order $n + 1$. (In this respect our definition of an nth order clump is different from that of Roach (1968).) Let $p(n)$ be the probability that an arbitrary random set is part of a clump of order n.

Our first result describes asymptotic properties of $p(n)$. By way of notation, let $S_0, S_1, S_2, \ldots$ be independent and identically distributed copies of S, and define

$$f(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}) = P\left\{\text{the set } S_0 \cup \bigcup_{j=1}^{n-1} (\mathbf{x}_j + S_j) \text{ is connected}\right\}.$$

Theorem 3.1. *Assume* $n \geq 2$ *and* $E\{\bar{s}(S)^n\} < \infty$. *Suppose* δ *and* λ *vary in such a manner that* $\mu \equiv \delta^k \lambda \to 0$. *Then*

$$(3.1)\quad p(n) = \frac{\mu^{n-1}}{(n-1)!} \int \cdots \int_{(\mathbb{R}^k)^{n-1}} f(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1})\, d\mathbf{x}_1 \cdots d\mathbf{x}_{n-1} + o(\mu^{n-1})$$

and

$$(3.2)\qquad \sum_{i=n+1}^{\infty} p(i) = o(\mu^{n-1})$$

as $\mu \to 0$. *The integral on the right-hand side of* (3.1) *is finite.*

The case $n = 1$, not covered by Theorem 3.1, is very easy, since it may be proved that if $E\{\bar{s}(S)^{2-1/k}\} < \infty$ then $p(1) \to 1$ and

$$(3.3)\qquad \sum_{n=2}^{\infty} p(n) = 1 - p(1) \to 0.$$

The equality in (3.3) follows from the fact that if $E\{\bar{s}(S)^{2-1/k}\} < \infty$ then for all sufficiently small η, the probability of an arbitrary set being part of an infinite clump equals zero; see Hall (1985).

Our next task is to count nth-order clumps. If

$$C \equiv \bigcup_{l=1}^{n} (\mathbf{X}_{jl} + S_{jl})$$

is a clump of order n, then following Mack (1954, 1956) we define the *right-hand centre* (r.h.c.) of C to be that vector out of $\mathbf{X}_{j1}, \ldots, \mathbf{X}_{jn}$ whose first coordinate is greatest. Let $N(n)$ equal the number of nth-order clumps whose r.h.c.'s lie within $\mathscr{R}$. (In this notation, the quantity N defined in Section 2 is $N(1)$.) Since expectation is a linear operator then

$$(3.4) \qquad E\{N(n)\} = \lambda\|\mathscr{R}\|n^{-1}p(n).$$

Suppose $\lambda \to \infty$, and $\delta = \delta(\lambda) \to 0$ so quickly that $\eta(\lambda) \equiv \delta^k\lambda \to 0$. Our thesis is that in this sparse mosaic, $N(n)$ is asymptotically Poisson-distributed. In some forms this Poisson approximation is really a normal approximation, since the Poisson mean is very large. Once again, the case $n = 1$ is most easily tackled separately, and we treat it first.

THEOREM 3.2. *Assume* $E\{\bar{s}(S)^2\} < \infty$. *Suppose* δ *and* λ *vary together such that* $\lambda \to \infty$ *and* $\eta \equiv \delta^k\lambda \to 0$. *Then* $(\lambda\|\mathscr{R}\|)^{-1}E\{N(1)\} \to 1$, $(\lambda\|\mathscr{R}\|)^{-1}\mathrm{var}\{N(1)\} \to 1$ *and*

$$\frac{\{N(1) - EN(1)\}}{(\lambda\|\mathscr{R}\|)^{1/2}} \to N(0,1)$$

in distribution as $\lambda \to \infty$.

Our calculations suggest that when $n \geq 2$, the moment condition $E\{\bar{s}(S)^{2n}\} < \infty$ is sufficient for a limit theorem for $N(n)$. However, the proof is dramatically shortened if we assume that the shape S is bounded with probability 1, and so we shall content ourselves with that case. Thus, we suppose that there exists $c > 0$ such that

$$(3.5) \qquad P(|\mathbf{x}| \leq c \text{ for all } \mathbf{x} \in S) = 1.$$

THEOREM 3.3. *Assume* $n \geq 2$ *and* (3.5) *holds. Suppose* δ *and* λ *vary together such that* $\lambda \to \infty$ *and* $\eta \equiv \delta^k\lambda \to 0$. *If* $\lambda\eta^{n-1} \to a < \infty$ *as* $\lambda \to \infty$, *then* $N(n)$ *is asymptotically Poisson-distributed with mean* $a\mu(n)$, *where*

$$\mu(n) \equiv \frac{\|\mathscr{R}\|}{n!} \int \cdots \int_{(\mathbb{R}^k)^{n-1}} f(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1})\, d\mathbf{x}_1 \cdots d\mathbf{x}_{n-1}.$$

If $\lambda\eta^{n-1} \to \infty$ *then* $\{N(n) - EN(n)\}/\{\mathrm{var}\, N(n)\}^{1/2}$ *is asymptotically normal* $N(0,1)$, *and*

$$\mathrm{var}\{N(n)\} = \lambda\eta^{n-1}\mu(n) + o(\lambda\eta^{n-1}).$$

The total number of clumps of all orders whose r.h.c.'s lie within $\mathscr{R}$, equals

$$N_{\mathrm{tot}} = \sum_{n=1}^{\infty} N(n),$$

and has mean

$$E(N_{\mathrm{tot}}) = \lambda\|\mathscr{R}\| \sum_{n=1}^{\infty} n^{-1}p(n).$$

If $\lambda\eta^m = O(1)$ as $\lambda \to \infty$ for some sufficiently large m, then we may deduce from

Theorems 3.1–3.3 that

$$\frac{\{N_{\text{tot}} - E(N_{\text{tot}})\}}{(\lambda\|\mathbb{R}\|)^{1/2}} = \frac{\{N(1) - EN(1)\}}{(\lambda\|\mathscr{R}\|)^{1/2}} + o_p(1)$$

in distribution. Thus, the three random variables: total number of clumps within $\mathscr{R}$, total number of isolated sets within $\mathscr{R}$, and total number of points from the driving Poisson process within $\mathscr{R}$, have essentially the same asymptotic behaviour. In each case,

$$\frac{(\text{observed number} - \text{expected number})}{(\lambda\|\mathscr{R}\|)^{1/2}} \to N(0,1)$$

in distribution.

4. Proofs.

PROOF OF THEOREM 2.1. Let $T^{(i)} = \bar{\mathscr{T}}(S^{(i)})$ be the smallest sphere containing $S^{(i)}$, let $R^{(i)}$ denote the radius of $T^{(i)}$, and assume (without loss of generality) that $T^{(i)}$ is centred at the origin with probability 1. Then

$$\begin{aligned} I_2 &\le \int_{\mathbb{R}^k} P\{(\mathbf{x} + T^{(1)}) \cap T^{(2)} \ne \varnothing\}\, d\mathbf{x} \\ &\le \int_{\mathbb{R}^k} P(R^{(1)} + R^{(2)} \ge |\mathbf{x}|)\, d\mathbf{x} \\ &= v_k E\{(R^{(1)} + R^{(2)})^k\} < \infty. \end{aligned}$$

Now we turn attention to I_1. We shall say that $\mathbf{x}^{(1)} + S^{(1)}$ and $\mathbf{x}^{(2)} + S^{(2)}$ are *virtually isolated* if no shapes from the mosaic intersect either $\mathbf{x}^{(1)} + S^{(1)}$ or $\mathbf{x}^{(2)} + S^{(2)}$, irrespective of whether or not $\mathbf{x}^{(1)} + S^{(1)}$ and $\mathbf{x}^{(2)} + S^{(2)}$ intersect one another. Then

$$\begin{aligned} Q(\mathbf{x} + S^{(1)}, S^{(2)} \text{ both isolated}) &\le Q(\mathbf{x} + S^{(1)}, S^{(2)} \text{ virtually isolated}) \\ &\le Q(\mathbf{x} + S^{(1)}, S^{(2)} \text{ both isolated}) \\ &\quad + Q\{(\mathbf{x} + S^{(1)}) \cap S^{(2)} \ne \varnothing\}. \end{aligned}$$

Therefore

$$\begin{aligned} &\int_{\mathbb{R}^k} \left| Q(\mathbf{x} + S^{(1)}, S^{(2)} \text{ both isolated}) - Q(S^{(1)} \text{ isolated})^2 \right| d\mathbf{x} \\ &\le \int_{\mathbb{R}^k} \left| Q(\mathbf{x} + S^{(1)}, S^{(2)} \text{ virtually isolated}) - Q(S^{(1)} \text{ isolated})^2 \right| d\mathbf{x} + I_2. \end{aligned} \tag{4.1}$$

In fact, the integrand of the quantity

$$I_1^* \equiv \int_{\mathbb{R}^k} \left\{ Q(\mathbf{x} + S^{(1)}, S^{(2)} \text{ virtually isolated}) - Q(S^{(1)} \text{ isolated})^2 \right\} d\mathbf{x}$$

is nonnegative. This is most easily seen in the case where $S \equiv \mathscr{S}_K$, where K is a positive integer-valued random variable and $\mathscr{S}_1, \mathscr{S}_2, \dots$ are fixed (i.e., nonrandom) shapes. From there the proof may be extended to other random shapes S. The argument is very adaptable; for example, it may be employed to show that the quantity Δ, to be introduced below, is nonnegative.

Since the integrand of I_1^* is nonnegative then in view of (4.1), absolute integrability of I_1 will follow if we prove that

$$I_1^* < \infty. \tag{4.2}$$

We may write

$$I_1^* = \int_{\mathbb{R}^k} E\left\{ \Delta(\mathbf{x} | S^{(1)}, S^{(2)}) \right\} d\mathbf{x}, \tag{4.3}$$

where the expectation is taken in the distribution of $S^{(1)}$ and $S^{(2)}$, and

$$\begin{aligned} \Delta(\mathbf{x} | S^{(1)}, S^{(2)}) \equiv {} & Q(\mathbf{x} + S^{(1)}, S^{(2)} \text{ virtually isolated} | S^{(1)}, S^{(2)}) \\ & - Q(\mathbf{x} + S^{(1)} \text{ isolated} | S^{(1)}) Q(S^{(2)} \text{ isolated} | S^{(2)}). \end{aligned}$$

Suppose $\text{rad}(S^{(i)}) = r^{(i)}$ for $i = 1$ and 2. The coverage process $\mathscr{C}_0$ may be regarded as the superposition of two *independent* mosaics, the first comprised of shapes whose radii are no more than $(|\mathbf{x}| - r^{(1)} - r^{(2)})/2$, and the second of shapes whose radii exceed $(|\mathbf{x}| - r^{(1)} - r^{(2)})/2$. Denote these by $\mathscr{C}_{01}$ and $\mathscr{C}_{02}$, respectively, and let Q_i be the probability measure associated with $\mathscr{C}_{0i}$. No random set from $\mathscr{C}_{01}$ can intersect both $\mathbf{x} + S^{(1)}$ and $S^{(2)}$. Therefore, if $\text{rad}(S^{(i)}) = r^{(i)}$ for $i = 1$ and 2, then

$$\begin{aligned} & Q(\mathbf{x} + S^{(1)}, S^{(2)} \text{ virtually isolated} | S^{(1)}, S^{(2)}) \\ & \quad = Q_1(\mathbf{x} + S^{(1)} \text{ isolated} | S^{(1)}) Q_1(S^{(2)} \text{ isolated} | S^{(2)}) \\ & \qquad \times Q_2(\mathbf{x} + S^{(1)}, S^{(2)} \text{ virtually isolated} | S^{(1)}, S^{(2)}). \end{aligned}$$

Thus,

$$\begin{aligned} 0 & \le \Delta(\mathbf{x} | S^{(1)}, S^{(2)}) \\ & = Q_1(\mathbf{x} + S^{(1)} \text{ isolated} | S^{(1)}) Q_1(S^{(2)} \text{ isolated} | S^{(2)}) \\ & \quad \times \left\{ Q_2(\mathbf{x} + S^{(1)}, S^{(2)} \text{ virtually isolated} | S^{(1)}, S^{(2)}) \right. \\ & \qquad \left. - Q_2(\mathbf{x} + S^{(1)} \text{ isolated} | S^{(1)}) Q_2(S^{(2)} \text{ isolated} | S^{(2)}) \right\} \\ & \le 1 - Q_2(S^{(1)} \text{ isolated} | S^{(1)}) Q_2(S^{(2)} \text{ isolated} | S^{(2)}). \end{aligned}$$

But $Q_2(S^{(i)} \text{ isolated} | S^{(i)}) = \exp(-\nu^{(i)})$, where $\nu^{(i)}$ equals the mean number of shapes in $\mathscr{C}_{02}$ which intersect $S^{(i)}$, conditional on $S^{(i)}$. Therefore

$$0 \le \Delta(\mathbf{x}|S^{(1)}, S^{(2)}) \le \nu^{(1)} + \nu^{(2)}. \tag{4.4}$$

Let $\mathscr{C}_{03}$ denote the mosaic in which the driving Poisson process has intensity $\rho P\{R > (|\mathbf{x}| - r^{(1)} - r^{(2)})/2\}$, and the shapes are all distributed as random radius spheres, all independent of $S^{(1)}$ and $S^{(2)}$. Sphere radius, A, is given the distribution of R conditional on $R > (|\mathbf{x}| - r^{(1)} - r^{(2)})/2$, where $R = \text{rad}(S)$. We shall take R, $R^{(1)}$, and $R^{(2)}$ to be independent and identically distributed random variables. In this notation,

$$\begin{aligned}
&E(\nu^{(i)}|R^{(i)} = r^{(i)})\\
&\quad\le \big(\text{expected number of spheres from } \mathscr{C}_{03}\\
&\qquad\quad \text{which intersect a fixed sphere of radius } r^{(i)}\big)\\
&\quad= \rho P\left\{R > \frac{|\mathbf{x}| - r^{(1)} - r^{(2)}}{2}\right\} v_k E\{(r^{(i)} + A)^k\}\\
&\quad\le 2^{k-1}\rho v_k\big[(r^{(i)})^k P(r^{(1)} + r^{(2)} + 2R > |\mathbf{x}|)\\
&\qquad\qquad + E\{R^k I(r^{(1)} + r^{(2)} + 2R > |\mathbf{x}|)\}\big].
\end{aligned}$$

Combining this estimate with (4.3) and (4.4), we conclude that

$$\begin{aligned}
I_1^* &\le 2^k\rho v_k \int_{\mathbb{R}^k}\big[E\{(R^{(1)})^k I(R^{(1)} + R^{(2)} + 2R > |\mathbf{x}|)\}\\
&\qquad\qquad + E\{R^k I(R^{(1)} + R^{(2)} + 2R > |\mathbf{x}|)\}\big]\,d\mathbf{x}\\
&\le 2^{2k}\rho v_k^2 E\{(R^{(1)} + R^{(2)} + R)^{2k}\} < \infty,
\end{aligned} \tag{4.5}$$

establishing (4.2) and completing the proof of Theorem 2.1.

PROOF OF THEOREM 2.2. Result (2.4) follows easily from (2.1).

PROOF OF (2.5). Let M equal the number of points $\mathbf{X}_i$ centred within $\mathscr{R}$. Then M is Poisson-distributed with parameter $\mu = \lambda\|\mathscr{R}\|$. Given that $M \ge 2$, let $\mathbf{X}_1^* + \delta S_1^*$ and $\mathbf{X}_2^* + \delta S_2^*$ denote any two different sets $\mathbf{X}_i + \delta S_i$ for which $\mathbf{X}_i \in \mathscr{R}$, chosen at random from among all such sets. Result (2.5) will follow from (2.2) and (2.4) if we prove that

$$\begin{aligned}
&\lambda\|\mathscr{R}\|\big\{P(\mathbf{X}_1^* + \delta S_1^*, \mathbf{X}_2^* + \delta S_2^* \text{ both isolated}|M \ge 2)\\
&\qquad\qquad - P(\mathbf{X}_1 + \delta S_1 \text{ isolated})^2\big\}\\
&\to \rho\bigg[\int_{\mathbb{R}^k}\big\{Q(\mathbf{x} + S^{(1)}, S^{(2)} \text{ both isolated}) - Q(S^{(1)} \text{ isolated})^2\big\}\,d\mathbf{x}\\
&\qquad + 4Q(S^{(1)} \text{ isolated})\int_{\mathbb{R}^k} Q\{(\mathbf{x} + S^{(1)}) \cap S^{(2)} \ne \varnothing;\, S^{(2)} \text{ isolated}\}\,d\mathbf{x}\bigg].
\end{aligned} \tag{4.6}$$

Let $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ denote independent random variables uniformly distributed on $\mathscr{R}$, and recall that $S^{(1)}$ and $S^{(2)}$ are independent copies of S, all being independent of one another and of everything defined before. Now,

$$\begin{aligned}
&P(\mathbf{X}_1^* + \delta S_1^*, \mathbf{X}_2^* + \delta S_2^* \text{ both isolated} | M \geq 2) + o(\lambda^{-1}) \\
&\quad = \sum_{m=2}^{\infty} P(\mathbf{X}_1^* + \delta S_1^*, \mathbf{X}_2^* + \delta S_2^* \text{ both isolated} | M = m) P(M = m) \\
&\quad = \sum_{m=2}^{\infty} P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated} | M = m-2) P(M = m) \\
(4.7) &\quad = \sum_{m=2}^{\infty} P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated} | M = m-2) \\
&\qquad\quad \times P(M = m-2) \\
&\qquad + \sum_{m=2}^{\infty} P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated} | M = m-2) \\
&\qquad\quad \times \{P(M = m) - P(M = m-2)\} \\
&\quad = P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated}) + A,
\end{aligned}$$

where

$$\begin{aligned}
A \equiv &\sum_{m=0}^{\infty} P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated} | M = m) \\
&\times P(M = m) \left\{ \frac{\mu^2}{(m+1)(m+2)} - 1 \right\}.
\end{aligned}$$

The following lemma, to be proved shortly, estimates A.

Lemma 4.1. *If $E\{\bar{s}(S)^2\} < \infty$ then*

$$\begin{aligned}
A &\sim A_1 \\
&\equiv \lambda^{-1} 4\rho \|\mathscr{R}\|^{-1} Q(S^{(1)} \textit{ isolated}) \int_{\mathbb{R}^k} Q\{(\mathbf{x} + S^{(1)}) \cap S^{(2)} \neq \varnothing ; S^{(2)} \textit{ isolated}\}\, d\mathbf{x}
\end{aligned}$$

as $\lambda \to \infty$.

Combining (4.7), Lemma 4.1, and the fact that

$$P(\mathbf{X}_1 + \delta S_1 \text{ isolated}) = P(\mathbf{X}^{(i)} + \delta S^{(i)} \text{ isolated})$$

for $i = 1$ and 2, we obtain:

$$\begin{aligned}
&P(\mathbf{X}_1^* + \delta S_1^*, \mathbf{X}_2^* + \delta S_2^* \text{ both isolated} | M \geq 2) - P(\mathbf{X}_1 + \delta S_1 \text{ isolated})^2 \\
(4.8) &\quad = P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated}) \\
&\qquad - \prod_{i=1}^{2} P(\mathbf{X}^{(i)} + \delta S^{(i)} \text{ isolated}) + A_1 + o(\lambda^{-1}).
\end{aligned}$$

The desired result (4.6) follows from (4.8) and the limit theorem,

$$(4.9)\qquad \begin{aligned}\lambda\Big\{&P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated})\\ &\qquad - \prod_{i=1}^{2} P(\mathbf{X}^{(i)} + \delta S^{(i)} \text{ isolated})\Big\}\\ &\to \rho\|\mathscr{R}\|^{-1}\int_{\mathbb{R}^k}\big\{Q(\mathbf{x} + S^{(1)}, S^{(2)} \text{ both isolated})\\ &\qquad - Q(S^{(1)} \text{ isolated})^2\big\}\,d\mathbf{x}.\end{aligned}$$

The remainder of our proof of formula (2.5) consists of deriving (4.9).

If the mosaic is rescaled by the factor δ^{-1} along each dimension, it becomes a new coverage process in which the driving Poisson process has intensity $\delta^k\lambda$ and each shape is distributed as S. Under this transformation, $\mathbf{X}^{(1)}$ becomes uniformly distributed on $\delta^{-1}\mathscr{R}$. We shall use $P^\dagger$ instead of P for the probability measure associated with this rescaled mosaic. Thus, the left-hand side of (4.9) equals

$$(4.10)\qquad \begin{aligned}\lambda\|\delta^{-1}\mathscr{R}\|^{-2}&\int\int_{(\delta^{-1}\mathscr{R})^2}\Big\{P^\dagger(\mathbf{x}^{(1)} + S^{(1)}, \mathbf{x}^{(2)} + S^{(2)} \text{ both isolated})\\ &\qquad - \prod_{i=1}^{2} P^\dagger(\mathbf{x}^{(i)} + S^{(i)} \text{ isolated})\Big\}\,d\mathbf{x}^{(1)}\,d\mathbf{x}^{(2)}\\ &= \lambda\|\delta^{-1}\mathscr{R}\|^{-2}\int_{\mathscr{R}_\delta}\|\delta^{-1}\mathscr{R}\cap(\delta^{-1}\mathscr{R} - \mathbf{x})\|\\ &\qquad \times\big\{P^\dagger(\mathbf{x} + S^{(1)}, S^{(2)} \text{ both isolated}) - P^\dagger(S^{(1)} \text{ isolated})^2\big\}\,d\mathbf{x},\end{aligned}$$

where $\mathscr{R}_\delta = \{\mathbf{x} - \mathbf{y}\colon \mathbf{x}, \mathbf{y} \in \delta^{-1}\mathscr{R}\}$.

Since

$$\begin{aligned}&P^\dagger(\mathbf{x} + S^{(1)}, S^{(2)} \text{ both isolated})\\ &\quad = P^\dagger(\mathbf{x} + S^{(1)}, S^{(2)} \text{ virtually isolated})\\ &\qquad - P^\dagger(\mathbf{x} + S^{(1)}, S^{(2)} \text{ virtually isolated but not both isolated}),\end{aligned}$$

then by (4.10), the left-hand side of (4.9) equals

$$\delta^k\lambda\|\mathscr{R}\|^{-1}\left\{\int_{\mathscr{R}_\delta} p_1(\mathbf{x})\,d\mathbf{x} - \int_{\mathscr{R}_\delta} p_2(\mathbf{x})\,d\mathbf{x}\right\},$$

where

$$\begin{aligned}p_1(\mathbf{x}) &= \|\delta^{-1}\mathscr{R}\|^{-1}\|\delta^{-1}\mathscr{R}\cap(\delta^{-1}\mathscr{R} - \mathbf{x})\|\\ &\qquad \times\big\{P^\dagger(\mathbf{x} + S^{(1)}, S^{(2)} \text{ virtually isolated}) - P^\dagger(S^{(1)} \text{ isolated})^2\big\}\end{aligned}$$

and

$$\begin{aligned}p_2(\mathbf{x}) &= \|\delta^{-1}\mathscr{R}\|^{-1}\|\delta^{-1}\mathscr{R}\cap(\delta^{-1}\mathscr{R} - \mathbf{x})\|\\ &\qquad \times P^\dagger(\mathbf{x} + S^{(1)}, S^{(2)} \text{ virtually isolated but not both isolated}).\end{aligned}$$

Likewise, the right-hand side of (4.9) equals

$$\rho\|\mathscr{R}\|^{-1}\left\{\int_{\mathbb{R}^k} q_1(\mathbf{x})\,d\mathbf{x} - \int_{\mathbb{R}^k} q_2(\mathbf{x})\,d\mathbf{x}\right\},$$

where

$$q_1(\mathbf{x}) = Q(\mathbf{x} + S^{(1)}, S^{(2)} \text{ virtually isolated}) - Q(S^{(1)} \text{ isolated})^2$$

and

$$q_2(\mathbf{x}) = Q(\mathbf{x} + S^{(1)}, S^{(2)} \text{ virtually isolated but not both isolated}).$$

In consequence, (4.9) will follow if we prove that

$$\int_{\mathscr{R}_\delta} p_i(\mathbf{x})\,d\mathbf{x} \to \int_{\mathbb{R}^k} q_i(\mathbf{x})\,d\mathbf{x} \tag{4.11}$$

as $\delta \to 0$, for $i = 1$ and 2.

Note that $0 \le p_i(\mathbf{x}) \le 1$, and $p_i(\mathbf{x}) \to q_i(\mathbf{x})$ as $\delta \to 0$, for $i = 1$ and 2 and for each fixed $\mathbf{x}$. Therefore by dominated convergence,

$$\int_{\mathscr{A}} p_i(\mathbf{x})\,d\mathbf{x} \to \int_{\mathscr{A}} q_i(\mathbf{x})\,d\mathbf{x}$$

as $\delta \to 0$, whenever $\mathscr{A}$ is a bounded, measurable subset of $\mathbb{R}^k$. Since $\mathscr{R}$ is Riemann measurable then for each $t > 0$, $\mathscr{R}_\delta$ contains the closed sphere

$$\mathscr{T}(t) \equiv \{\mathbf{x}\colon |\mathbf{x}| \le t\}$$

for all sufficiently small δ. Combining these facts, we see that

$$\begin{aligned}&\limsup_{\delta\to 0}\left|\int_{\mathscr{R}_\delta} p_i(\mathbf{x})\,d\mathbf{x} - \int_{\mathscr{R}_\delta} q_i(\mathbf{x})\,d\mathbf{x}\right| \\ &\quad\le \limsup_{t\to\infty}\limsup_{\delta\to 0}\int_{\mathbb{R}^k\setminus\mathscr{T}(t)} |p_i(\mathbf{x})|\,d\mathbf{x} + \limsup_{t\to\infty}\int_{\mathbb{R}^k\setminus\mathscr{T}(t)} |q_i(\mathbf{x})|\,d\mathbf{x}.\end{aligned} \tag{4.12}$$

The functions p_2 and q_2 are obviously nonnegative, and arguments given early in the proof of Theorem 2.1 show that p_1 and q_1 are nonnegative. Therefore the absolute value signs on the right-hand side of (4.12) are not really necessary. It also follows from the proof of Theorem 2.1 that q_1 and q_2 are integrable, so that the last term on the right-hand side of (4.12) equals zero. A similar argument may be used to show that the other term on the right is zero. (Note that the factor

$$\|\delta^{-1}\mathscr{R}\|^{-1}\|\delta^{-1}\mathscr{R} \cap (\delta^{-1}\mathscr{R} - \mathbf{x})\|$$

appearing in $p_1(\mathbf{x})$ and $p_2(\mathbf{x})$ does not exceed 1.) This proves (4.11).

It remains only to derive Lemma 4.1.

PROOF OF LEMMA 4.1. Markov's inequality implies that for any ε, $l > 0$,

$$P(|M - \mu| > \mu^{1/2+\varepsilon}) \le E(|M - \mu|/\mu^{1/2+\varepsilon})^l = O(\mu^{-\varepsilon l}).$$

Therefore if $\nu \equiv \mu^{1/2+\varepsilon}$ for some positive ε,

$$A = \sum_{|m-\mu|\le\nu} P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated} | M = m) \tag{4.13}$$
$$\times P(M = m)\left\{\frac{\mu^2}{(m+1)(m+2)} - 1\right\} + o(\mu^{-1}).$$

Writing $n = m - \mu$, we have

$$\frac{\mu^2}{(m+1)(m+2)} = 1 - \frac{2n+3}{\mu} + 3\left(\frac{n}{\mu}\right)^2 + O\left\{\frac{|n|^3}{\mu^3} + \frac{|n|+1}{\mu^2}\right\}.$$

Substituting into (4.13), and choosing $\varepsilon \in (0, \frac{1}{6})$, we see that

$$A = \sum_{|m-\mu|\le\nu} P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated} | M = m)$$
$$\times P(M = m)\left\{-\frac{2n+3}{\mu} + 3\left(\frac{n}{\mu}\right)^2\right\} + o(\mu^{-1})$$
$$= \sum_{m=0}^{\infty} P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated} | M = m)$$
$$\times P(M = m)\left\{-\frac{2n+3}{\mu} + 3\left(\frac{n}{\mu}\right)^2\right\} + o(\mu^{-1}) \tag{4.14}$$
$$= -2\mu^{-1} \sum_{m=0}^{\infty} P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated} | M = m)$$
$$\times P(M = m)(m - \mu)$$
$$- 3\mu^{-1} P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated})$$
$$+ 3\mu^{-2} \sum_{m=0}^{\infty} P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated} | M = m)$$
$$\times P(M = m)(m - \mu) + o(\mu^{-1}).$$

In view of (4.9),

$$P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated}) \to Q(S^{(1)} \text{ isolated})^2,$$

and similarly it may be shown that

$$\sum_{m=0}^{\infty} P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated} | M = m) P(M = m)(m - \mu)^2$$
$$\sim \sum_{m=0}^{\infty} Q(S^{(1)} \text{ isolated})^2 P(M = m)(m - \mu)^2$$
$$= \mu Q(S^{(1)} \text{ isolated})^2.$$

Combining the results from (4.14) down, we obtain:

$$A = 2A_2 + o(\mu^{-1}), \tag{4.15}$$

where

$$A_2 = \sum_{m=0}^{\infty} P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated} | M = m) \times \{P(M = m) - P(M = m-1)\}.$$

An application of Abel's method of summation shows that

$$\begin{aligned} A_2 = \sum_{m=0}^{\infty} P(M = m) \times \{ &P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated} | M = m) \\ &- P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated} | M = m+1)\}. \end{aligned} \tag{4.16}$$

Let $\mathbf{X}^{(3)} + \delta S^{(3)}$ denote an independent copy of $\mathbf{X}^{(1)} + \delta S^{(1)}$, independent of everything defined so far. Then

$$\begin{aligned} &P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated} | M = m+1) \\ &\quad = P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated, and } \mathbf{X}^{(3)} + \delta S^{(3)} \\ &\qquad \text{intersects neither } \mathbf{X}^{(1)} + \delta S^{(1)} \text{ nor } \mathbf{X}^{(2)} + \delta S^{(2)} | M = m). \end{aligned}$$

Substituting into (4.16), we see that

$$\begin{aligned} A_2 &= \sum_{m=0}^{\infty} P(M = m) P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated, and } \mathbf{X}^{(3)} + \delta S^{(3)} \text{ intersects at least one of } \mathbf{X}^{(1)} + \delta S^{(1)} \text{ and } \mathbf{X}^{(2)} + \delta S^{(2)} | M = m) \\ &= P(\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated, and } \mathbf{X}^{(3)} + \delta S^{(3)} \text{ intersects at least one of } \mathbf{X}^{(1)} + \delta S^{(1)} \text{ and } \mathbf{X}^{(2)} + \delta S^{(2)}) \\ &\sim 2P\{\mathbf{X}^{(1)} + \delta S^{(1)}, \mathbf{X}^{(2)} + \delta S^{(2)} \text{ both isolated, and} (\mathbf{X}^{(3)} + \delta S^{(3)}) \cap (\mathbf{X}^{(2)} + \delta S^{(2)}) \neq \varnothing\} \end{aligned}$$

$$\begin{aligned} &\sim 2P(\mathbf{X}^{(1)} + \delta S^{(1)} \text{ isolated}) \times P\{(\mathbf{X}^{(3)} + \delta S^{(3)}) \cap (\mathbf{X}^{(2)} + \delta S^{(2)}) \neq \varnothing \text{ and } \mathbf{X}^{(2)} + \delta S^{(2)} \text{ isolated}\} \\ &= 2P(\mathbf{X}^{(1)} + \delta S^{(1)} \text{ isolated}) \delta^k \|\mathscr{R}\|^{-1} \|\delta^{-1}\mathscr{R}\|^{-1} \\ &\quad \times \iint_{(\delta^{-1}\mathscr{R})^2} P\{(\mathbf{x}^{(3)} + \delta S^{(3)}) \cap (\mathbf{x}^{(2)} + \delta S^{(2)}) \neq \varnothing \text{ and } \mathbf{x}^{(2)} + \delta S^{(2)} \text{ isolated}\}\, d\mathbf{x}^{(1)}\, d\mathbf{x}^{(2)} \\ &\sim 2Q(S^{(1)} \text{ isolated}) \\ &\quad \times \delta^k \|\mathscr{R}\|^{-1} \int_{\mathbb{R}^k} Q\{(\mathbf{x} + S^{(3)}) \cap S^{(2)} \neq \varnothing;\, S^{(2)} \text{ isolated}\}\, d\mathbf{x}. \end{aligned} \tag{4.17}$$

That is,

$$A_2 = \tfrac{1}{2}A_1 + o(\lambda^{-1}).$$

Lemma 4.1 follows on combining (4.15) and (4.17).

PROOF OF (2.6). Our proof of the central limit theorem (2.6) contains two steps. First we show that the error committed by truncating the radius of random shapes may be made arbitrarily small by taking the truncation point to be sufficiently large. Then we prove the central limit theorem in the case of bounded shapes.

STEP (i). Fix $r > 0$, and define

$$\begin{aligned} N^{(1)} = \big(&\text{number of sets } \mathbf{X}_i + S_i \text{ for which } \mathbf{X}_i \in \mathscr{R}, \operatorname{rad}(S_i) \le r, \text{ and} \\ &(\mathbf{X}_i + S_i) \cap (\mathbf{X}_j + S_j) = \varnothing \text{ for all sets } \mathbf{X}_j + S_j \\ &\text{with } j \ne i \text{ and } \operatorname{rad}(S_j) \le r\big), \\ N^{(2)} = \big(&\text{number of sets } \mathbf{X}_i + S_i \text{ for which } \mathbf{X}_i \in \mathscr{R}, \operatorname{rad}(S_i) > r \text{ and} \\ &(\mathbf{X}_i + S_i) \cap (\mathbf{X}_j + S_j) = \varnothing \text{ for all } j \ne i\big), \end{aligned}$$

and

$$\begin{aligned} N^{(3)} = \big(&\text{number of sets } \mathbf{X}_i + S_i \text{ for which } \mathbf{X}_i \in \mathscr{R}, \operatorname{rad}(S_i) \le r \text{ and} \\ &(\mathbf{X}_i + S_i) \cap (\mathbf{X}_j + S_j) = \varnothing \text{ for all sets } \mathbf{X}_j + S_j \\ &\text{with } j \ne i \text{ and } \operatorname{rad}(S_j) \le r, \\ &\text{but } (\mathbf{X}_i + S_i) \cap (\mathbf{X}_j + S_j) \ne \varnothing \text{ for some } j \text{ with } \operatorname{rad}(S_j) > r\big). \end{aligned}$$

Then

$$N^{(1)} + N^{(2)} = N + N^{(3)}. \tag{4.18}$$

The variable $N^{(1)}$ equals the number of isolated shapes which would result if we ignored all shapes of radius r or more.

We shall prove that

$$\lim_{r\to\infty} \limsup_{\lambda\to\infty} \lambda^{-1}\operatorname{var}(N^{(i)}) = 0 \tag{4.19}$$

for $i = 2$ and 3, and that

$$\lim_{r\to\infty} \limsup_{\lambda\to\infty} \lambda^{-1}|\operatorname{var}(N) - \operatorname{var}(N^{(1)})| = 0. \tag{4.20}$$

Actually, result (4.20) follows from (4.19), since after a little algebra and an application of the Cauchy–Schwarz inequality it may be proved that

$$\begin{aligned} |\operatorname{var}(N) - \operatorname{var}(N^{(1)})| \le 4\big[&\{\operatorname{var}(N^{(2)}) + \operatorname{var}(N^{(3)})\} \\ &\times \{\operatorname{var}(N) + \operatorname{var}(N^{(2)}) + \operatorname{var}(N^{(3)})\}\big]^{1/2}. \end{aligned}$$

Together, (4.19) and (4.20) imply that our central limit theorem for N will follow if we prove it instead for $N^{(1)}$. (That proof constitutes Step (ii) below.)

The methods used to derive (4.19) are similar for $i = 2$ and $i = 3$, and so we consider only $i = 3$. Using the argument leading to (2.2), we obtain:

$$
\begin{aligned}
\operatorname{var}(N^{(3)}) = {} & \lambda\|\mathscr{R}\|P\{\operatorname{rad}(S_1) \le r; (\mathbf{X}_1 + \delta S_1) \cap (\mathbf{X}_i + \delta S_i) = \varnothing \\
& \qquad \text{for all } i \ne 1 \text{ with } \operatorname{rad}(S_i) \le r; \\
& \qquad (\mathbf{X}_1 + \delta S_1) \cap (\mathbf{X}_i + \delta S_i) \ne \varnothing \text{ for some } i \ne 1\} \\
& + (\lambda\|\mathscr{R}\|)^2\Big[P\big\{\operatorname{rad}(S_1), \operatorname{rad}(S_2) \text{ both } \le r; (\mathbf{X}_i + \delta S_i) \\
& \qquad \cap \big(\mathbf{X}_j + \delta S_j\big) = \varnothing \text{ for all } j \ne i \text{ with } \operatorname{rad}\big(S_j\big) \le r, \\
& \qquad \text{for both } i = 1 \text{ and } 2; (\mathbf{X}_i + \delta S_i) \cap \big(\mathbf{X}_{j_i} + \delta S_{j_i}\big) \ne \varnothing \\
& \qquad \text{for some } j_i \ne i, \text{ for both } i = 1 \text{ and } 2 | \mathbf{X}_1, \mathbf{X}_2 \in \mathscr{R}\big\} \\
& \quad - P\{\operatorname{rad}(S_1) \le r; (\mathbf{X}_1 + \delta S_1) \cap (\mathbf{X}_i + \delta S_i) = \varnothing \\
& \qquad \text{for all } i \ne 1 \text{ with } \operatorname{rad}(S_i) \le r; \\
& \qquad (\mathbf{X}_1 + \delta S_1) \cap (\mathbf{X}_i + \delta S_i) \ne \varnothing \text{ for some } i \ne 1\}^2\Big].
\end{aligned} \tag{4.21}
$$

Let $\mathbf{X}^{(1)} + \delta S^{(i)}$ and M have the meanings ascribed to them during our proof of (2.5). Define E to be the event:

$\operatorname{rad}(S^{(1)}), \operatorname{rad}(S^{(2)})$ both $\le r$; $(\mathbf{X}^{(i)} + \delta S^{(i)}) \cap (\mathbf{X}_j + \delta S_j) = \varnothing$ for all j with $\operatorname{rad}(S_j) \le r$, for both $i = 1$ and 2; $(\mathbf{X}^{(i)} + \delta S^{(i)}) \cap (\mathbf{X}_{j_i} + \delta S_{j_i}) \ne \varnothing$ for some j_i, for both $i = 1$ and 2.

Let E_1 denote the event: $E \cap \{(\mathbf{X}^{(1)} + \delta S^{(1)}) \cap (\mathbf{X}^{(2)} + \delta S^{(2)}) = \varnothing\}$. Employing the argument leading to (4.7), we may deduce that

$$
\begin{aligned}
& P\big\{\operatorname{rad}(S_1), \operatorname{rad}(S_2) \text{ both } \le r; (\mathbf{X}_i + \delta S_i) \cap \big(\mathbf{X}_j + \delta S_j\big) = \varnothing \text{ for all } j \ne i \\
& \quad \text{with } \operatorname{rad}\big(S_j\big) \le r, \text{ for both } i = 1 \text{ and } 2; (\mathbf{X}_i + \delta S_i) \cap \big(\mathbf{X}_{j_i} + \delta S_{j_i}\big) \ne \varnothing \\
& \quad \text{for some } j_i \ne i, \text{ for both } i = 1 \text{ and } 2 | \mathbf{X}_1, \mathbf{X}_2 \in \mathscr{R}\} \\
& = P(E_1) + A,
\end{aligned} \tag{4.22}
$$

where

$$
A \equiv \sum_{m=0}^{\infty} P(E_1 | M = m)\left\{\frac{\mu^2}{(m+1)(m+2)} - 1\right\}
$$

and $\mu = \lambda\|\mathscr{R}\|$. We may estimate A using techniques from the proof of Lemma 4.1. Indeed, we may obtain the following analogue of (4.14):

$$
\begin{aligned}
A = {} & -2\mu^{-1} \sum_{m=0}^{\infty} P(E_1 | M = m) P(M = m)(m - \mu) \\
& - 3\mu^{-1} P(E_1) + 3\mu^{-2} \sum_{m=0}^{\infty} P(E_1 | M = m) P(M = m)(m - \mu)^2 + o(\mu^{-1}).
\end{aligned}
$$

Since

$$\lim_{r\to\infty} \limsup_{\lambda\to\infty} P(E_1) = 0$$

and

$$\sum_{m=0}^{\infty} P(E_1|M = m)P(M = m)(m - \mu)^2 \le \left\{P(E_1)E(M - \mu)^4\right\}^{1/2},$$

then we may write

$$A = -2\mu^{-1} \sum_{m=0}^{\infty} P(E_1|M = m)P(M = m)(m - \mu) + A_1, \tag{4.23}$$

where

$$\lim_{r\to\infty} \limsup_{\lambda\to\infty} \lambda|A_1| = 0.$$

An argument like that leading to (4.17) reveals that an estimate similar to (4.17) applies to the series on the right-hand side of (4.23). Thus, we may conclude that

$$\lim_{r\to\infty} \limsup_{\lambda\to\infty} \lambda|A| = 0. \tag{4.24}$$

Combining (4.21) and (4.22), we obtain:

$$\begin{aligned}(\lambda\|\mathscr{R}\|)^{-1}\mathrm{var}(N^{(3)}) \le P\{&(\mathbf{X}_1 + \delta S_1) \cap (\mathbf{X}_i \cap \delta S_i) \ne \varnothing \\ &\text{for some } i \ne 1 \text{ with } \mathrm{rad}(S_i) > r\} \\ &+\lambda\|\mathscr{R}\|\left\{P(E_1) + A - P(E^{(1)})P(E^{(2)})\right\},\end{aligned}$$

where $E^{(i)}$ denotes the event

$$\mathrm{rad}(S^{(i)}) \le r;\ (\mathbf{X}^{(i)} + \delta S^{(i)}) \cap \left(\mathbf{X}_j + \delta S_j\right) = \varnothing \text{ for all } j \text{ with}$$

$$\mathrm{rad}\left(S_j\right) \le r;\ (\mathbf{X}^{(i)} + \delta S^{(i)}) \cap \left(\mathbf{X}_j + \delta S_j\right) \ne \varnothing \text{ for some } j.$$

Note that $E_1 \subseteq E = E^{(1)} \cap E^{(2)}$. From this inequality and (4.24) we see that (4.19) in the case $i = 3$ will follow if we prove that

$$\lim_{r\to 0} \limsup_{\lambda\to\infty} \lambda\left\{P(E^{(1)} \cap E^{(2)}) - P(E^{(1)})P(E^{(2)})\right\} \le 0. \tag{4.25}$$

We shall estimate the probabilities in (4.25) by first conditioning on $S^{(1)}$ and $S^{(2)}$. Choose versions of $S^{(i)}$ for which $\mathrm{rad}(S^{(i)}) \le r$, for $i = 1$ and 2. Let $P^{\dagger}$ denote the probability measure associated with the mosaic in which the driving Poisson process has intensity $\delta^k\lambda$, and each shape is distributed as S. Let $F^{(i)}(\mathbf{x})$ be the event:

$$(\mathbf{x} + S^{(i)}) \cap \left(\mathbf{X}_j + S_j\right) = \varnothing \quad \text{for all } j \text{ with } \mathrm{rad}\left(S_j\right) \le r;$$

$$(\mathbf{x} + S^{(i)}) \cap \left(\mathbf{X}_j + S_j\right) \ne \varnothing \quad \text{for some } j.$$

Then

$$P(E^{(1)} \cap E^{(2)}|S^{(1)}, S^{(2)}) - P(E^{(1)}|S^{(1)})P(E^{(2)}|S^{(2)})$$
$$= \|\delta^{-1}\mathscr{R}\|^{-2}\int\int_{(\delta^{-1}\mathscr{R})^2}\big[P^{\dagger}\{F^{(1)}(\mathbf{x}^{(1)}) \cap F^{(2)}(\mathbf{x}^{(2)})|S^{(1)}, S^{(2)}\}$$
$$-P^{\dagger}\{F^{(1)}(\mathbf{x}^{(1)})|S^{(1)}\}P^{\dagger}\{F^{(2)}(\mathbf{x}^{(2)})|S^{(2)}\}\big]\,d\mathbf{x}^{(1)}\,d\mathbf{x}^{(2)}$$
$$\leq \|\delta^{-1}\mathscr{R}\|^{-1}\int_{\mathbb{R}^k}\big|P^{\dagger}\{F^{(1)}(\mathbf{x}) \cap F^{(2)}(\mathbf{0})|S^{(1)}, S^{(2)}\}$$
$$-P^{\dagger}\{F^{(1)}(\mathbf{x})|S^{(1)}\}P^{\dagger}\{F^{(2)}(\mathbf{0})|S^{(2)}\}\big|\,d\mathbf{x}.$$

The term within modulus signs may be shown to be nonnegative, for all values of $S^{(1)}$ and $S^{(2)}$. Taking expectations with respect to $S^{(1)}$ and $S^{(2)}$, we obtain:

$$P(E^{(1)} \cap E^{(2)}) - P(E^{(1)})P(E^{(2)})$$
$$\leq \delta^k\|\mathscr{R}\|^{-1}\int_{\mathbb{R}^k}\big[P^{\dagger}\{F^{(1)}(\mathbf{x}) \cap F^{(2)}(\mathbf{0})\} - P^{\dagger}\{F^{(1)}(\mathbf{x})\}P^{\dagger}\{F^{(2)}(\mathbf{0})\}\big]\,d\mathbf{x}.$$

Therefore

$$P^{-1}\|\mathscr{R}\|\limsup_{\lambda\to\infty}\lambda\{P(E^{(1)} \cap E^{(2)}) - P(E^{(1)})P(E^{(2)})\}$$
$$\leq \int_{\mathbb{R}^k}\Big[Q\{F^{(1)}(\mathbf{x}) \cap F^{(2)}(\mathbf{0})\} - Q\{F^{(1)}(\mathbf{0})\}^2\Big]\,d\mathbf{x}$$
$$\leq \int_{\mathscr{T}(t)} Q\{F^{(1)}(\mathbf{x})\}\,d\mathbf{x}$$
$$+\int_{\mathbb{R}^k\setminus\mathscr{T}(t)}\Big[Q\{F^{(1)}(\mathbf{x}) \cap F^{(2)}(\mathbf{0})\} - Q\{F^{(1)}(\mathbf{0})\}^2\Big]\,d\mathbf{x},$$

where $\mathscr{T}(t)$ denotes the sphere $\{\mathbf{x}\colon |\mathbf{x}| \leq t\}$. The first integral on the right-hand side converges to zero as $r \to \infty$, for each $t > 0$, while the second integral may be made arbitrarily small uniformly in $r > 0$ by choosing t sufficiently large. These two observations are enough to prove (4.25), and that result gives us (4.19) in the case $i = 3$.

STEP (ii). In view of Step (i), it suffices to consider the case where the shape S is uniformly bounded: For some $c > 0$,

$$P(|\mathbf{x}| \leq c \text{ for all } \mathbf{x} \in S) = 1. \tag{4.26}$$

Let d be a very large positive constant. Divide $\mathbb{R}^k$ into a regular lattice of cubes of side length $cd\delta$, with nearest faces of adjacent cubes distant $4c\delta$ apart. Those regions which lie in between cubes will be called "spacings." They may be regarded as being a disjoint union of rectangular prisms, no prism having a side length exceeding $cd\delta$. Let $\mathscr{B}_1$ equal the union of all those cubes contained wholly within $\mathscr{R}$; let $\mathscr{B}_2$ be the union of all those spacing prisms wholly within $\mathscr{R}$; and let $\mathscr{B}_3 = \mathscr{R}\setminus(\mathscr{B}_1 \cap \mathscr{B}_2)$. Write N_i for the number of isolated sets centred within

$\mathscr{B}_i$. Then

$$N_1 = \sum_{j=1}^{\nu} M_j,$$

where M_j denotes the number of isolated sets centred within the jth cube $\mathscr{D}_j$ $(1 \le j \le \nu)$ of side length $cd\delta$. Since no random set in the mosaic can intersect both a set centred in $\mathscr{D}_i$ and a set centred in $\mathscr{D}_j$, for $i \ne j$ (note condition (4.26)), then the variables M_j are independent as well as identically distributed. Thus,

$$\begin{aligned}\operatorname{var}(N_1) = \nu\Big[\lambda\|\mathscr{D}_1\|P(\mathbf{X}_1 + \delta S_1 \text{ isolated}) + (\lambda\|\mathscr{D}_1\|)^2 \\ \times\big\{P(\mathbf{X}_1 + \delta S_1, \mathbf{X}_2 + \delta S_2 \text{ both isolated}|\mathbf{X}_1, \mathbf{X}_2 \in \mathscr{D}_1) \\ - P(\mathbf{X}_1 + \delta S_1 \text{ isolated})^2\big\}\Big].\end{aligned}$$

Since $\lambda\|\mathscr{D}_1\| \to \rho(cd)^k$ as $\lambda \to \infty$, then

$$\begin{aligned}\operatorname{var}(N_1) = \nu\Big[\rho(cd)^k Q(\mathbf{X}_1 + S_1 \text{ isolated}) + \big\{\rho(cd)^k\big\}^2 \\ \times\big\{Q(\mathbf{X}_1 + S_1, \mathbf{X}_2 + S_2 \text{ both isolated}|\mathbf{X}_1, \mathbf{X}_2 \in \mathscr{E}) \\ - Q(\mathbf{X}_1 + S_1 \text{ isolated})^2\big\}\Big] + o(\nu),\end{aligned}$$

where $\mathscr{E}$ is an arbitrary $[k]$ cube of side length cd with the same orientation as $\mathscr{D}_1$. As $\lambda \to \infty$, ν is asymptotic to a constant multiple of λ. Therefore $\lambda^{-1}\operatorname{var}(N_1)$ has a finite limit σ^2, say, as $\lambda \to \infty$.

Observe that $M_1 \le M_0$, where M_0 equals the total number of points from the driving Poisson process which are centred within $\mathscr{D}_1$. Since M_0 has a Poisson distribution with parameter $\lambda(cd\delta)^k = \rho(cd)^k + o(1)$, then for a positive constant C,

$$E|M_1 - E(M_1)|^3 \le 4\big\{E(M_0^3) + (EM_0)^3\big\} \le C$$

uniformly in λ. Thus,

$$\sum_i \frac{E|M_i - E(M_i)|^3}{\lambda^{3/2}} = \nu\frac{E|M_1 - E(M_1)|^3}{\lambda^{3/2}} = O(\lambda^{-1/2})$$

as $\lambda \to \infty$. Lyapounov's central limit theorem (see Chung (1974, page 200)) now tells us that

$$\frac{\{N_1 - E(N_1)\}}{\lambda^{1/2}} \to N(0, \sigma^2) \tag{4.27}$$

in distribution.

Our goal of a central limit theorem for N will follow from (4.27), provided we prove that

$$\lim_{d\to\infty} \limsup_{\lambda\to\infty} \lambda^{-1}\operatorname{var}(N_i) = 0 \tag{4.28}$$

for $i = 2$ and 3, and

$$\lim_{d\to\infty} \limsup_{\lambda\to\infty} \lambda^{-1}|\operatorname{var}(N) - \operatorname{var}(N_1)| = 0. \tag{4.29}$$

Since

$$|\operatorname{var}(N) - \operatorname{var}(N_1)| \le 4\{\operatorname{var}(N_2) + \operatorname{var}(N_3)\}^{1/2}\{\operatorname{var}(N) + \operatorname{var}(N_2) + \operatorname{var}(N_3)\}^{1/2},$$

then (4.29) follows from (4.28) and (2.5). We shall complete the proof by deriving (4.28).

The variables N_2 and N_3 may each be written in the form

$$N_i = \sum_{j=1}^{m} \sum_{l=1}^{n_{ij}} M_{ijl},$$

where $m \le 2^k$, the M_{ijl}'s represent numbers of isolated shapes centred within respective disjoint regions $\mathscr{A}_{ijl}$ of dimension no more than $(cd\delta) \times \cdots \times (cd\delta)$, and for fixed i and j, the variables $M_{ij1}, \dots, M_{ijn_{ij}}$ are stochastically independent. Thus,

$$\operatorname{var}(N_i) \le C \sum_{j=1}^{m} \sum_{l=1}^{n_{ij}} \operatorname{var}(M_{ijl}),$$

where the constant C depends only on k. Using formula (2.2) to evaluate $\operatorname{var}(M_{ijl})$, we see that

$$\begin{aligned}\operatorname{var}(M_{ijl}) \le \lambda\|\mathscr{A}_{ijl}\| + (\lambda\|\mathscr{A}_{ijl}\|)^2 &\times\Big\{P(\mathbf{X}_1 + \delta S_1, \mathbf{X}_2 + \delta S_2 \text{ both isolated} | \mathbf{X}_1, \mathbf{X}_2 \in \mathscr{A}_{ijl}) \\ &\qquad - P(\mathbf{X}_1 + \delta S_1 \text{ isolated})^2\Big\}\end{aligned}$$

and

$$\begin{aligned}\operatorname{var}(N_i) \le C\Big[1 + \lambda \sup_{j,l}\|\mathscr{A}_{ijl}\| &\times\Big\{P(\mathbf{X}_1 + \delta S_1, \mathbf{X}_2 + \delta S_2 \text{ both isolated} | \mathbf{X}_1, \mathbf{X}_2 \in \mathscr{A}_{ijl}) \\ &\qquad - P(\mathbf{X}_1 + \delta S_1 \text{ isolated})^2\Big\}\Big]\lambda\|\mathscr{B}_i\|.\end{aligned} \tag{4.30}$$

It may be proved that $\|\mathscr{B}_2\| \le C_1/d$, for a constant C_1 depending on neither d nor δ, and $\|\mathscr{B}_3\| \to 0$ as $\lambda \to \infty$. Furthermore,

$$\begin{aligned}&\limsup_{\lambda\to\infty} \lambda \sup_{j,l}\|\mathscr{A}_{2jl}\|\Big\{P(\mathbf{X}_1 + \delta S_1, \mathbf{X}_2 + \delta S_2 \text{ both isolated} | \mathbf{X}_1, \mathbf{X}_2 \in \mathscr{A}_{2jl}) \\ &\qquad\qquad - P(\mathbf{X}_1 + \delta S_1 \text{ isolated})^2\Big\} \\ &\le \rho \sup_{\mathscr{B}}\|\mathscr{B}\|\Big\{Q(\mathbf{X}_1 + S_1, \mathbf{X}_2 + S_2 \text{ both isolated} | \mathbf{X}_1, \mathbf{X}_2 \in \mathscr{B}) \\ &\qquad\qquad - Q(\mathbf{X}_1 + S_1 \text{ isolated})^2\Big\},\end{aligned} \tag{4.31}$$

where the last supremum is taken over rectangular boxes $\mathscr{B} \subseteq \mathbb{R}^k$; and

$$\lambda \sup_{j,l} \|\mathscr{A}_{3jl}\| \le \lambda (cd\delta)^k \to \rho (cd)^k$$

as $\lambda \to \infty$. Therefore result (4.28) will follow from (4.30) if we prove that the supremum on the right-hand side of (4.31) is finite. In fact, that sup is dominated by

$$\begin{aligned} &4 \sup_{\mathscr{B}} \|\mathscr{B}\|^{-1} \int\int_{\mathscr{B}^2} Q\{(\mathbf{x}_1 + S_1) \cap (\mathbf{x}_2 + S_2) \ne \varnothing\}\, d\mathbf{x}_1\, d\mathbf{x}_2 \\ &\quad + \sup_{\mathscr{B}} \|\mathscr{B}\|^{-1} \int\int_{\mathscr{B}^2} \{Q(\mathbf{x}^{(1)} + S^{(1)}, \mathbf{x}^{(2)} + S^{(2)} \text{ virtually isolated}) \\ &\qquad\qquad - Q(S^{(1)} \text{ isolated})^2\}\, d\mathbf{x}^{(1)}\, d\mathbf{x}^{(2)} \\ &\quad \le 4 \int_{\mathbb{R}^k} Q\{(\mathbf{x} + S^{(1)}) \cap S^{(2)} \ne \varnothing\}\, d\mathbf{x} \\ &\qquad + \int_{\mathbb{R}^k} \{Q(\mathbf{x} + S^{(1)}, S^{(2)} \text{ virtually isolated}) - Q(S^{(1)} \text{ isolated})^2\}\, d\mathbf{x} \\ &\quad < \infty. \end{aligned}$$

PROOF OF THEOREM 3.1. The proof consists of two steps. First we show that, under our assumption $E\{\bar{s}(S)^n\} < \infty$, there is no real loss of generality in supposing that shapes are bounded with probability 1. In the second step we establish the theorem for bounded shapes.

STEP (i). Let $p(n, r)$ denote the probability that the clump containing an arbitrary shape is of order $\ge n$, and contains at least one shape of radius $> \delta r$. Step (i) consists of proving that

$$\lim_{r \to \infty} \limsup_{\eta \to 0} \eta^{-(n-1)} p(n, r) = 0. \tag{4.32}$$

Let R have the distribution of rad(S). Consider the mosaic $\mathscr{C}_1$ in which the driving Poisson process has intensity λ, and shapes are $[k]$ spheres centred at the origin with radius distribution $\delta([R] + 1)$, where $[R]$ denotes the integer part of R. Let $p_1(n, r)$ be the probability that the clump containing an arbitrary shape in $\mathscr{C}_1$ is of order $\ge n$, and contains a sphere of radius $> \delta r$. Then $p(n, r) \le p_1(n, r)$. Therefore (4.32) will follow if we prove it for $p_1(n, r)$ instead of $p(n, r)$.

We shall bound the mosaic $\mathscr{C}_1$ using a multitype branching process. There will be a countable infinity of types, indexed by positive integers which equal sphere radii divided by δ. Individuals in the process will be represented by points in $\mathbb{R}^k$. The individual in the zeroth generation is the centre of our arbitrary sphere in $\mathscr{C}_1$. Its type is the radius of the sphere divided by δ, and so is distributed as $N \equiv [R] + 1$. Given individuals $\mathbf{Z}_{n1}, \dots, \mathbf{Z}_{nM_n}$ in the nth generation, we define the $(n + 1)$th generation as follows. Suppose $\mathbf{Z}_{nl}$ is of type i. Let $\mathscr{P}_{nl}$ be a Poisson process in $\mathbb{R}^k$ of intensity λ, independent of the previous history and also of $\mathscr{P}_{nl'}$ for $l' \ne l$. Centre spheres at the points of $\mathscr{P}_{nl}$, their radii being distributed independently and identically as δN. The progeny of $\mathbf{Z}_{nl}$ of type j in the $(n + 1)$th generation, are those points of $\mathscr{P}_{nl}$ whose associated spheres are of radius δj and intersect the sphere of radius δi centred at $\mathbf{Z}_{nl}$. Let $p_2(n, r)$ be the probability that the total number of individuals in all generations of this branching process is at least n, and that at least one of these individuals is of a type $> r$. Then $p_1(n, r) \le p_2(n, r)$, and so it suffices to prove (4.32) with $p_2(n, r)$ replacing $p(n, r)$. A key assumption in that proof is that $E(N^{kn}) < \infty$, which follows from the moment condition $E\{\bar{s}(S)^n\} < \infty$.

Suppose there is a total of at least n individuals (including the zeroth individual) in all generations of our branching process. Then at least one of the following occurs:

(a) there is at least one individual in the nth generation;
(b) for some $0 \le i \le n - 1$, at least one individual in the ith generation has $\ge n$ children;
(c) the number M_i of individuals in the ith generation satisfies $0 \le M_i \le (n - 1)^i$ for $1 \le i \le n - 1$, $\sum_{i=1}^{n-1} M_i \ge n - 1$, and $M_n = 0$.

Let A, B, and C denote the events described in (a), (b), and (c), respectively, and let D be the event that some individual in the branching process is of a type $> r$. Then

$$p_2(n, r) = P\{(A \cup B \cup C) \cap D\} \le P(A) + P(B) + P(C \cap D).$$

Therefore (4.32) will follow if we prove that

$$P(A) + P(B) = o(\eta^{n-1}) \tag{4.33}$$

as $\eta \to 0$, and

$$\lim_{r \to \infty} \limsup_{\eta \to 0} \eta^{-(n-1)} P(C \cap D) = 0. \tag{4.34}$$

The probability of event A is dominated by the expected number of individual in the nth generation. (Use Markov's inequality.) To calculate this mean, let

$$\begin{aligned} \eta\mu_{ij} &\equiv (\text{expected number of children of type } j \\ &\qquad \text{born to an individual of type } i) \\ &= \eta(i + j)^k P(N = j) v_k \\ &\le \eta i^k j^k P(N = j) 2^k v_k. \end{aligned} \tag{4.35}$$

Define $\mathbf{M}$ to be the matrix whose (i, j)th element is $\eta\mu_{ij}$, and let $\eta^n \mu_{ij}^{(n)}$ denote the (i, j)th element of $\mathbf{M}^n$. It may be proved by induction that

$$\mu_{ij}^{(n)} \le i^k j^k P(N = j)(2^k v_k)^n \{E(N^{2k})\}^{n-1}$$

for $n \ge 1$. Therefore the mean number of individuals in the nth generation equals

$$\eta^n \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P(N = i)\mu_{ij}^{(n)} \le (\eta 2^k v_k)^n \{E(N^k)\}^2 \{E(N^{2k})\}^{n-1}.$$

(See Athreya and Ney (1972, page 184).) Combining the results in this paragraph, we conclude that

$$P(A) = O(\eta^n) \tag{4.36}$$

as $\eta \to 0$.

Let E_m denote the event that some individual in the mth generation has $\geq n$ children. We shall prove by induction over m that

$$P(E_m) = o(\eta^{n-1}) \tag{4.37}$$

for all $m \geq 0$. As an immediate corollary, we have

$$P(B) = o(\eta^{n-1}). \tag{4.38}$$

Suppose (4.37) is true for $m \leq m_0 - 1$, and let M equal the number of individuals in the mth generation. Set $n_0 = (n-1)^{m_0}$. Then, by our induction hypothesis,

$$P(M > n_0) = o(\eta^{n-1}).$$

Number the members of the m_0th generation from 1 to M in random order, and let K_i equal the number of progeny of individual i. Then

$$I(E_{m_0+1}) \leq \sum_{i=1}^{\min(M,\, n_0)} I(K_i \geq n) + I(M > n_0),$$

and so

$$P(E_{m_0+1}) \leq n_0 P(K_1 \geq n) + P(M > n_0).$$

Therefore (4.37) for $m = m_0 + 1$ will follow if we prove that

$$P\{K(m) \geq n\} = o(\eta^{n-1}) \tag{4.39}$$

for each $m \geq 0$, where $K(m)$ has the distribution of the number of progeny born to a single, arbitrary individual in the mth generation, given that there is at least one individual in the mth generation. Furthermore, demonstrating (4.39) for $m = 0$ will establish (4.37) for $m = 0$. This will complete our inductive proof of (4.37).

If a child is chosen at random among from the progeny of a type i individual, the chance that he is of type j equals

$$p_{ij} = \frac{\eta\mu_{ij}}{\sum_{j=1}^{\infty}\eta\mu_{ij}} = \frac{\mu_{ij}}{\mu_{i\cdot}},$$

where

$$\mu_{i\cdot} = \sum_{j=1}^{\infty} \mu_{ij} \geq i^k v_k.$$

Therefore

$$0 \leq p_{ij} \leq 2^k j^k P(N = j).$$

(Note formula (4.35).) It may now be proved by induction that

$$0 \leq p_{ij}^{(m)} \leq 2^{km}\{E(N^k)\}^{m-1} j^k P(N = j)$$

for $m \geq 1$, where $p_{ij}^{(m)}$ is the (i, j)th element in the mth power of the matrix (p_{ij}).

The probability that an individual chosen at random from the mth generation has $\geq n - 1$ children, given that there is at least one individual in the mth generation, equals

$$P\{K(m) > n-1\} = \sum_{i=1}^{\infty} P(N=i) \sum_{j=1}^{\infty} p_{ij}^{(m)} \sum_{l=n-1}^{\infty} \frac{(\eta u_{j\cdot})^l}{l!} \exp(-\eta\mu_{j\cdot})$$

$$(4.40) \qquad \leq \sum_{i=1}^{\infty} P(N=i) \sum_{j=1}^{\infty} p_{ij}^{(m)} \frac{(\eta\mu_{j\cdot})^{n-1}}{(n-1)!}$$

$$= \frac{\eta^{n-1}}{(n-1)!} \Sigma(m, n),$$

where

$$\Sigma(m, n) \equiv \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P(N=i) p_{ij}^{(m)} \mu_{j\cdot}^{n-1}.$$

Note that $\Sigma(m, n)$ does not depend on η, and that

$$\Sigma(m, n) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P(N=i) 2^k j^k P(N=j) \{2^k v_k E(N^k) j^k\}^{n-1}$$

$$= 2^{kn} v_k^{n-1} \{E(N^k)\}^{n-1} E(N^{kn}) < \infty.$$

The finiteness of $\Sigma(m, n)$ implies that

$$P\{K(m) = n-1\} = \sum_{i=1}^{\infty} P(N=i) \sum_{j=1}^{\infty} p_{ij}^{(m)} \frac{(\eta\mu_{j\cdot})^{n-1}}{(n-1)!} \exp(-\eta\mu_{j\cdot})$$

$$(4.41) \qquad = \frac{\eta^{n-1}}{(n-1)!} \Sigma(m, n) + o(\eta^{n-1})$$

as $\eta \to 0$. Combining (4.40) and (4.41), we conclude that

$$P\{K(m) \geq n\} = P\{K(m) \geq n-1\} - P\{K(m) = n-1\} = o(\eta^{n-1}),$$

establishing (4.39).

Result (4.33) follows from (4.36) and (4.38). We shall complete Step (i) by proving (4.34).

Our multitype branching process may be viewed as a vector-valued Markov chain. Denote the state of the ith generation by the infinite-component vector $\mathbf{M}_i = (M_{i1}, M_{i2}, \ldots)$, where M_{ij} equals the number of type j individuals in generation i. Let $\mathbf{m}_i = (m_{i1}, m_{i2}, \ldots)$ be a particular value of $\mathbf{M}_i$, in which all but a finite number of the m_{ij}'s are zero. The first step in proving (4.34) is to bound the probability $P(\mathbf{M}_i = \mathbf{m}_i \text{ for } 0 \leq i \leq n-1)$.

Conditional on $\mathbf{M}_i = \mathbf{m}_i$, the number of individuals of type l in the $(i+1)$th generation is Poisson-distributed with mean

$$\sum_{j=1}^{\infty} m_{ij}\eta\mu_{jl} = \eta v_k P(N = l) \sum_{j=1}^{\infty} m_{ij}(j+l)^k$$
$$\leq \eta 2^k v_k l^k P(N = l) \sum_{j=1}^{\infty} m_{ij} j^k.$$

Therefore

$$\begin{aligned}
\pi(\mathbf{m}_i, \mathbf{m}_{i+1}) &\equiv P(\mathbf{M}_{i+1} = \mathbf{m}_{i+1} | \mathbf{M}_i = \mathbf{m}_i) \\
&= \sum_{l=1}^{\infty} P(M_{i+1,l} = m_{i+1,l} | \mathbf{M}_i = \mathbf{m}_i) \\
&= \prod_{l=1}^{\infty} \left[\frac{1}{m_{i+1,l}!} \left\{ \eta v_k P(N = l) \sum_{j=1}^{\infty} m_{ij}(j+l)^k \right\}^{m_{i+1,l}} \right. \\
&\qquad \left. \times \exp\left\{ -\eta v_k P(N = l) \sum_{j=1}^{\infty} m_{ij}(j+l)^k \right\} \right] \\
&\leq \left(\eta 2^k v_k \sum_{j=1}^{\infty} m_{ij} j^k \right)^{m_{i+1,\cdot}} \exp\left(-\eta v_k \sum_{j=1}^{\infty} m_{ij} j^k \right) \\
&\qquad \times \prod_{l=1}^{\infty} \{ l^k P(N = l) \}^{m_{i+1,l}},
\end{aligned} \tag{4.42}$$

where $m_{i\cdot} \equiv \sum_{j=1}^{\infty} m_{ij}$ for $i \geq 1$.

We shall assume throughout that for each $1 \leq i \leq n-1$, $m_{i\cdot} \leq (n-1)^i$, and also that $1 + \sum_{i=1}^{n-1} m_{i\cdot} \geq n$; note the definition of event C at (c) above. The symbol γ will denote a positive generic constant depending only on n and the distribution of N. (In particular, γ does not depend on choice of the vectors $\mathbf{m}_i$.) Set $m_i \equiv \min(n-1, m_{i\cdot})$, and note that

$$\left(\eta v_k \sum_{j=1}^{\infty} m_{ij} j^k \right)^{m_{i+1,\cdot} - m_{i+1}} \exp\left(-\eta v_k \sum_{j=1}^{\infty} m_{ij} j^k \right) \leq \gamma$$

for $0 \leq i \leq n-2$. Therefore by (4.42),

$$\pi(\mathbf{m}_i, \mathbf{m}_{i+1}) \leq \gamma \left(\eta \sum_{j=1}^{\infty} m_{ij} j^k \right)^{m_{i+1}} \prod_{l=1}^{\infty} \{ l^k P(N = l) \}^{m_{i+1,l}}. \tag{4.43}$$

Let $l_i \equiv \max\{l\colon m_{il} > 0\}$, and observe that

$$\sum_{j=1}^{\infty} m_{ij} j^k \leq m_{i\cdot} l_i^k \leq (n-1)^i l_i^k.$$

Substituting into (4.43), we conclude that for $0 \leq i \leq n-2$,

$$\pi(\mathbf{m}_i, \mathbf{m}_{i+1}) \leq \gamma (\eta l_i^k)^{m_{i+1}} \prod_{l=1}^{\infty} \{ l^k P(N = l) \}^{m_{i+1,l}}.$$

Consequently if $0 < \eta \le 1$,

$$P(\mathbf{M}_i = \mathbf{m}_i \text{ for } 0 \le i \le n-1) = P(\mathbf{M}_0 = \mathbf{m}_0) \prod_{i=0}^{n-2} \pi(\mathbf{m}_i, \mathbf{m}_{i+1})$$

$$\le \gamma \eta^{m.} P(N = l_0) \left(\prod_{i=0}^{n-2} l_i^{k m_{i+1}} \right) \prod_{l=1}^{\infty} \{ l^k P(N = l) \}^{m_{.l}}$$

$$\le \gamma \eta^{n-1} l_0^{k(n-1)} P(N = l_0) \left(\prod_{i=1}^{n-2} l_i \right)^{k(n-1)} \tag{4.44}$$

$$\times \prod_{l=1}^{\infty} \{ l^k P(N = l) \}^{m_{.l}}$$

$$\le \gamma \eta^{n-1} l_0^{kn} P(N = l_0) \prod_{l=1}^{\infty} \{ l^{kn} P(N = l) \}^{m_{.l}}, \tag{4.45}$$

where $m_{.} = \sum_{i=1}^{n-1} m_i \ge n-1$ and $m_{.l} = \sum_{i=1}^{n-1} m_{il}$. (Note that $\mathbf{m}_0$ is a vector consisting entirely of zeros except for a single one in position l_0, and that $P(\mathbf{M}_0 = \mathbf{m}_0) = P(N = l_0)$.) Let n_l equal the total number of individuals in generations $0, 1, \ldots, n-1$ who are of type l. Then by (4.45),

$$P(\mathbf{M}_i = \mathbf{m}_i \text{ for } 0 \le i \le n-1) \le \gamma \eta^{n-1} \prod_{l=1}^{\infty} \{ l^{kn} P(N = l) \}^{n_l}. \tag{4.46}$$

Adding formula (4.46) over all vectors $\mathbf{m}_i$ for which both $m_{i.} = m_i^{(0)}$ (a predetermined number satisfying $m_i^{(0)} \le (n-1)^i$ for $1 \le i \le n-1$ and $1 + \sum_{i=1}^{n-1} m_i^{(0)} \ge n$) and

$$\sup_{j > r} \sup_{0 \le i \le n-1} m_{ij} > 0,$$

we conclude that

$$\begin{aligned} q \equiv P\big(&\text{there are precisely } m_i^{(0)} \text{ individuals in generation } i, \text{ for} \\ &0 \le i \le n-1, \text{ and at least one individual in the first } n-1 \\ &\text{generations is of a type } > r\big) \\ \le{}& \gamma \eta^{n-1} \sum_{l > r} l^{kn} P(N = l). \end{aligned} \tag{4.47}$$

Therefore the probability q satisfies

$$\lim_{r \to \infty} \limsup_{\eta \to 0} \eta^{-(n-1)} q = 0. \tag{4.48}$$

The probability $P(C \cap D)$ appearing in (4.34) is dominated by a finite sum of probabilities like q, the number of terms in the series depending on neither η nor r. Therefore (4.34) follows from (4.48). This completes Step (i).

Before passing to Step (ii), we note that the techniques used in Step (i) may be employed to prove two other results which are of use to us. The first of these,

$$\limsup_{\eta \to 0} \eta^{-(n-1)} p(n) < \infty, \tag{4.49}$$

follows immediately from (4.33) and (4.46), since

$$p(n) \le P(A) + P(B) + P\left\{\text{the number } M_i \text{ of individuals in the } i\text{th generation satisfies } 0 \le M_i \le (n-1)^i \text{ for } 1 \le i \le n-1 \text{ and } \sum_{i=1}^{n-1} M_i \ge n-1\right\}.$$

The second result is (3.2). To prove it, observe that the sum on the left-hand side of (3.2) is dominated by the probability that our multitype branching process contains a total of $n+1$ or more individuals in all generations. That probability does not exceed $P(A \cap B \cap C')$, where events A and B are defined as before, and C' is the event,

the number M_i of individuals in the ith generation satisfies

$$0 < M_i \le (n-1)^i \text{ for } 1 \le i \le n-1 \text{ and } \sum_{i=1}^{n-1} M_i \ge n.$$

In view of (4.33), result (3.2) will follow if we show that

$$P(C') = O(\eta^n) \tag{4.50}$$

as $\eta \to 0$.

To prove (4.50), note that if $\mathbf{M}_i = \mathbf{m}_i$ for $1 \le i \le n-1$, if $m_{1\cdot} \le n-1$, and $\sum_{i=1}^{n-1} m_{i\cdot} \ge n$, then

$$m_{\cdot} \equiv \sum_{i=1}^{n-1} \min(n-1, m_{i\cdot}) \ge n.$$

Therefore the factor $\eta^{m_{\cdot}}$ appearing in (4.44) may be simplified to η^n, instead of η^{n-1}, in (4.45). Likewise, the factor η^{n-1} in (4.46) may be replaced by η^n. Adding (4.46) over all numbers $m_i^{(0)}$ satisfying $m_i^{(0)} \le (n-1)^i$ for $1 \le i \le n-1$ and $\sum_{i=1}^{n-1} m_i^{(0)} \ge n$, we obtain (4.50).

STEP (ii). We shall assume initially that the shape S is essentially bounded; that is, for some $c > 0$,

$$P(|\mathbf{x}| \le c \text{ for all } \mathbf{x} \in S) = 1.$$

Define $x \equiv 2(n-1)c$, and let $\mathscr{T}(t)$ denote the closed $[k]$ sphere of radius t centred at the origin. If our arbitrary shape $\mathbf{X}_i + \delta S_i$ is part of a clump

$$(\mathbf{X}_i + \delta S_i) \cup \bigcup_{l=1}^{n-1} (\mathbf{X}_{j_l} + \delta S_{j_l})$$

of order n, then all the points $\mathbf{X}_{j_l}$ must lie within the (closed) sphere $\mathbf{X}_i + \mathscr{T}(x\delta)$ centred at $\mathbf{X}_i$ and of radius $x\delta$.

Let the random variable M have the Poisson distribution with mean $\mu \equiv \lambda\|\mathscr{T}(x\delta)\| = \eta\|\mathscr{T}(x)\|$, and distribute M points $\mathbf{Y}_1, \ldots, \mathbf{Y}_M$ independently and uniformly within $\mathscr{T}(x\delta)$. In view of the conclusion of the previous paragraph, we have:

$$p(n) = P(\text{the clump containing } S_0 \text{ and formed from the random sets } \delta S_0, \mathbf{Y}_1 + \delta S_1, \ldots, \mathbf{Y}_M + \delta S_M, \text{ is of size } n)$$

$$= \sum_{m=n-1}^{\infty} P(M=m) \|\mathscr{T}(x)\|^{-m} \int \cdots \int_{\{\mathscr{T}(x)\}^m} g_n(\mathbf{y}_1, \ldots, \mathbf{y}_m)\, d\mathbf{y}_1 \cdots d\mathbf{y}_m,$$

where

$$g_n(\mathbf{y}_1, \ldots, \mathbf{y}_m) \equiv P(\text{the clump containing } S_0 \text{ and formed from the random sets } S_0, \mathbf{y}_1 + S_1, \ldots, \mathbf{y}_m + S_m, \text{ is of size } n).$$

However,

$$P(M = n-1) = \frac{\mu^{n-1}}{(n-1)!} e^{-\mu} = \frac{\{\eta \|\mathscr{T}(x)\|\}^{n-1}}{(n-1)!} + O(\eta^n),$$

and

$$0 \le \sum_{m=n}^{\infty} P(M=m) \|\mathscr{T}(x)\|^{-m} \int \cdots \int_{\{\mathscr{T}(x)\}^m} g_n(\mathbf{y}_1, \ldots, \mathbf{y}_m)\, d\mathbf{y}_1 \cdots d\mathbf{y}_m$$

$$\le \sum_{m=n}^{\infty} P(M=m)$$

$$\le \frac{\mu^n}{n!} = O(\eta^n)$$

as $\eta \to 0$. Therefore

$$p(n) = \frac{\eta^{n-1}}{(n-1)!} \int \cdots \int_{\{\mathscr{T}(x)\}^{n-1}} g_n(\mathbf{y}_1, \ldots, \mathbf{y}_{n-1})\, d\mathbf{y}_1 \cdots d\mathbf{y}_{n-1} + O(\eta^n).$$

Since $g_n(\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}) \equiv f(\mathbf{y}_1, \ldots, \mathbf{y}_{n-1})$, and this function vanishes outside the set $\{\mathscr{T}(x)\}^{n-1}$, then result (3.1) is proved.

It remains only to extend this conclusion to the case of general shapes satisfying $E\{\bar{s}(S)^n\} < \infty$. We may write

$$p(n) = p(n, r) + q(n, r)$$

where $p(n, r)$ was defined in Step (i), and $q(n, r)$ equals the probability that an arbitrary shape is part of a clump of order n which contains no shapes of radius $> r$. The result proved in the previous paragraph establishes that

$$\eta^{-(n-1)} q(n, r) = \frac{1}{(n-1)!} \int \cdots \int_{(\mathbb{R}^k)^{n-1}} f(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1} | r)\, d\mathbf{x}_1 \cdots d\mathbf{x}_{n-1} + o(1),$$

where

$$f(\mathbf{x}_1,\ldots,\mathbf{x}_{n-1}|r) \equiv P\left\{\text{the set } S_0 \cup \bigcup_{i=1}^{n-1} (\mathbf{x}_i + S_i) \text{ is connected,} \right.$$
$$\left. \text{and rad}(S_i) \le r \text{ for } 1 \le i \le n-1\right\}.$$

Step (i) shows that $\eta^{-(n-1)}p(n, r)$ may be made arbitrarily small uniformly in η by choosing r large. Therefore result (3.1) for general shapes will follow if we prove that

$$\text{(4.51)} \qquad \begin{aligned} I(r) &\equiv \int \cdots \int_{(\mathbb{R}^k)^{n-1}} f(\mathbf{x}_1,\ldots,\mathbf{x}_{n-1}|r)\, d\mathbf{x}_1 \cdots d\mathbf{x}_{n-1} \\ &\to \int \cdots \int_{(\mathbb{R}^k)^{n-1}} f(\mathbf{x}_1,\ldots,\mathbf{x}_{n-1})\, d\mathbf{x}_1 \cdots d\mathbf{x}_{n-1} \end{aligned}$$

as $r \to \infty$, and that the limit is finite.

For each $\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}$, the function $f(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}|r)$ *increases* to $f(\mathbf{x}_1,\ldots,\mathbf{x}_{n-1})$ as $r \to \infty$. Therefore (4.51) will follow by dominated convergence if we show that

$$\sup_{r>0} I(r) < \infty.$$

But for each $r > 0$,

$$\frac{1}{(n-1)!} I(r) \le \limsup_{\eta \to 0} \eta^{-(n-1)} p(n),$$

and so

$$\sup_{r>0} I(r) \le (n-1)! \limsup_{\eta \to 0} \eta^{-(n-1)} p(n) < \infty,$$

using (4.49).

Proof of Theorem 3.2. Result (3.1) implies that $(\lambda\|\mathscr{R}\|)^{-1}E\{N(1)\} \to 1$. To prove the remainder of the theorem, let M equal the total number of points from the driving Poisson process which fall within $\mathscr{R}$, and set $N' \equiv M - N(1)$. Then

$$|\text{var}(M) - \text{var}(N(1))| \le 4\{\text{var}(N')\}\{\text{var}(N(1)) + \text{var}(N')\}^{1/2},$$

$(\lambda\|\mathscr{R}\|)^{-1}\text{var}(M) = 1$ and $\{M - E(M)\}/(\lambda\|\mathscr{R}\|)^{1/2} \to N(0,1)$ in distribution as $\lambda \to \infty$. Therefore the result $(\lambda\|\mathscr{R}\|)^{-1}\text{var}(N(1)) \to 1$, and the central limit theorem, will follow if we prove that $\lambda^{-1}\text{var}(N') \to 0$. This may be done using techniques from the proofs of Theorems 2.1 and 2.2, since

$$\begin{aligned} \text{var}(N') &= \lambda\|\mathscr{R}\| P(\mathbf{X}_1 + \delta S_1 \text{ not isolated}) \\ &\quad + (\lambda\|\mathscr{R}\|)^2\big\{P(\mathbf{X}_1 + \delta S_1, \mathbf{X}_2 + \delta S_2 \text{ both not isolated}|\mathbf{X}_1, \mathbf{X}_2 \in \mathscr{R}) \\ &\qquad - P(\mathbf{X}_1 + \delta S_1 \text{ not isolated})^2\big\}. \end{aligned}$$

Proof of Theorem 3.3. Our proof is similar to Step (ii) in the proof of Theorem 2.2. Define the regions $\mathscr{B}_1$, $\mathscr{B}_2$, $\mathscr{B}_3$ and $\mathscr{D}_1, \dots, \mathscr{D}_\nu$ as in that argument, except that here we stipulate that nearest faces of adjacent cubes be distant $4nc\delta$ (instead of $4c\delta$) apart. Let N_i equal the number of clumps of order n whose r.h.c.'s lie within $\mathscr{B}_i$. Then $N(n) = N_1 + N_2 + N_3$. In Step (i) of the proof below we shall establish a Poisson limit (if $\lambda\eta^{n-1} \to a$) or a normal limit (if $\lambda\eta^{n-1} \to \infty$) for N_1. Then in Step (ii) we shall prove that for large values of d, N_2, and N_3 are negligible in comparison with N_1.

Step (i). Let $\mathscr{D}_i'$ be the $[k]$ cube of side length $(d + 4n)c\delta$, concentric to $\mathscr{D}_i$. Any clump of order n whose r.h.c. lies within $\mathscr{D}_i$, is composed entirely of sets centred within $\mathscr{D}_i'$. Let M_i equal the number of clumps of order n centred within $\mathscr{D}_i$, and K_i equal the total number of points of the driving Poisson process within $\mathscr{D}_i'$. The following results may be proved in succession after a little algebra:

$$E\{M_i^2 I(M_i \ge 2)\} \le E\{K_i^2 I(K_i \ge 2n)\} = O(\eta^{2n}),$$

$$P(M_i = 1) = P(M_i = 1, K_i = n) + O(\eta^{n+1}) = \eta^n b + O(\eta^{n+1}), \tag{4.52}$$

$$E(M_i) = \eta^n b + O(\eta^{n+1}), \tag{4.53}$$

$$\operatorname{var}(M_i) = \eta^n b + O(\eta^{n+1}), \tag{4.54}$$

and if $x = x(\eta)$ is any sequence diverging to $+\infty$ as $\eta \to 0$,

$$E\{(M_i - EM_i)^2 I(|M_i - EM_i| > x)\} = O(\eta^{2n}). \tag{4.55}$$

In identities (4.53) and (4.54), b is defined by

$$b \equiv \frac{1}{n!}\int \cdots \int_{\mathscr{E}^n} P\left\{\bigcup_{i=1}^{n} (\mathbf{x}_i + S_i) \text{ connected}\right\} d\mathbf{x}_1 \cdots d\mathbf{x}_n,$$

where $\mathscr{E}$ is any cube of side length $(d + 4n)c$ with the same orientation as $\mathscr{D}_1$. It may be proved that

$$b \sim (d^k/\|\mathscr{R}\|)\mu(n) \tag{4.56}$$

as $d \to \infty$. Note that $N_1 \equiv \sum_{i=1}^{\nu} M_i$, where the summands are independent and identically distributed, and that $\nu \sim \lambda\eta^{-1}\|\mathscr{R}\|/\{(d + 4n)c\}^k$ as $\eta \to 0$.

Case (a). $\lambda\eta^{n-1} \to a < \infty$. In this situation,

$$\sum_{i=1}^{\nu} E\{M_i I(M_i \ge 2)\} = O(\lambda\eta^{-1} \cdot \eta^{2n}) \to 0$$

by (4.52), and

$$\sum_{i=1}^{\nu} E(M_i) = \nu\eta^n b + O(\nu\eta^{n+1}) \to a\mu_1$$

by (4.53), where

$$\mu_1 \equiv \|\mathscr{R}\| b/\{(d+4n)c\}^k. \tag{4.57}$$

These two properties imply Poisson convergence for a sum of independent, identically distributed, nonnegative integer-valued random variables M_i; see, for example, Theorem 5, page 132 of Gnedenko and Kolmogorov (1968). Therefore:

$$N_1 \equiv \sum_{i=1}^{\nu} M_i \text{ is asymptotically Poisson-distributed as } \eta \to 0, \text{ with mean } a\mu_1. \tag{4.58}$$

Note that by (4.56) and (4.57),

$$\mu_1 \to \mu(n) \tag{4.59}$$

as $d \to \infty$.

CASE (b). $\lambda\eta^{n-1} \to \infty$. Here,

$$\begin{aligned} \operatorname{var}(N_1) &= \sum_{i=1}^{\nu} \operatorname{var}(M_i) = \nu\eta^n b + O(\nu\eta^{n+1}) \\ &= \lambda\eta^{n-1}\mu_1 + o(\lambda\eta^{n-1}), \end{aligned} \tag{4.60}$$

using (4.54). If we take $x = \varepsilon\{\operatorname{var}(N_1)\}^{1/2}$ in (4.55), we may deduce that Lindeberg's condition holds for the series $\sum_{i=1}^{\nu}(M_i - EM_i)$:

$$\begin{aligned} &\{\operatorname{var}(N_1)\}^{-1} \sum_{i=1}^{\nu} E\left[(M_i - EM_i)^2 I\{|M_i - EM_i| > \varepsilon(\operatorname{var} N_1)^{1/2}\}\right] \\ &= O\{(\nu\eta^n)^{-1}(\nu\eta^{2n})\} = O(\eta^n) \to 0 \end{aligned}$$

as $\eta \to 0$. Lindeberg's central limit theorem (see, for example, Chung (1974, page 205)) now ensures that

$$\frac{\{N_1 - E(N_1)\}}{\{\operatorname{var}(N_1)\}^{1/2}} \to N(0,1) \tag{4.61}$$

in distribution.

STEP (ii). We may deduce from Theorem 3.1 that

$$E(N_i) = \lambda\eta^{n-1}C_1\|\mathscr{B}_i\| + o(\lambda\eta^{n-1}), \tag{4.62}$$

where the finite constant C_1 depends only upon n and the distribution of S.

CASE (a). $\lambda\eta^{n-1} \to a < \infty$. When $i = 2$, $\|\mathscr{B}_i\| \le C_2/d$ for a constant C_2 not depending on d, and when $i = 3$, $\|\mathscr{B}_i\| \to 0$ as $\eta \to 0$. Therefore by (4.62),

$$\lim_{d\to\infty} \limsup_{\eta\to 0} E(N_i) = 0.$$

This result, together with (4.58) and (4.59), implies that $N(n) \equiv N_1 + N_2 + N_3$ is asymptotically Poisson-distributed with mean $a\mu(n)$, as had to be proved.

CASE (b). $\lambda\eta^{n-1} \to \infty$. Both N_2 and N_3 may be written in the form

$$N_i = \sum_{j=1}^{m_i} \sum_{l=1}^{n_{ij}} M_{ijl}, \tag{4.63}$$

where $m_i \le 2^k$, the M_{ijl}'s are the numbers of nth order clumps centred within respective disjoint regions $\mathscr{A}_{ijl}$ of dimension no more than $(cd\delta) \times \cdots \times (cd\delta)$, and for fixed i and j, the variables $M_{ij1}, \ldots, M_{ijn_{ij}}$ are mutually independent. The total number of terms in the double series (4.63) is $O(\delta^{-k})$ as $\delta \to 0$.

Independence implies that

$$\operatorname{var}(N_i) \le C_2 \sum_{j=1}^{m_i} \sum_{l=1}^{n_{ij}} \operatorname{var}(M_{ijl}) \le C_2 \sum_{j=1}^{m_i} \sum_{l=1}^{n_{ij}} E(M_{ijl}^2),$$

where C_2 depends only on k. Now,

$$E(M_{ijl}^2) \le E(M_{ijl}) + E\{M_{ijl}^2 I(M_{ijl} \ge 2)\}.$$

An argument like that leading to (4.52) shows that

$$\sup_{i,j,l} E\{M_{ijl}^2 I(M_{ijl} \ge 2)\} = O(\eta^{2n})$$

as $\eta \to 0$. Combining these estimates we conclude that for a constant $C_3 > 0$,

$$\operatorname{var}(N_i) \le C_2 \sum_{j=1}^{m_i} \sum_{l=1}^{n_{ij}} E(M_{ijl}) + C_3 \sum_{j=1}^{m_i} \sum_{l=1}^{n_{ij}} \eta^{2n}$$
$$= C_2 E(N_i) + O(\delta^{-k}\eta^{2n}).$$

Referring to formula (4.60) for $\operatorname{var}(N_1)$, and to formula (4.62) for $E(N_i)$, we conclude that

$$\frac{\operatorname{var}(N_i)}{\operatorname{var}(N_1)} \le \frac{C_1 C_2 \|\mathscr{B}_i\|}{\mu_1} + o(1)$$

for $i = 2$ and 3.

In the case $i = 2$ we have $\|\mathscr{B}_i\| \le C_4/d$, and when $i = 3$, $\|\mathscr{B}_i\| \to 0$ as $\eta \to 0$. Therefore

$$\lim_{d\to\infty} \limsup_{\eta\to 0} \frac{\operatorname{var}(N_i)}{\operatorname{var}(N_1)} = 0$$

for $i = 2$ and 3. This result, together with (4.59)–(4.61), implies that $N(n)$ is asymptotically normally distributed with asymptotic variance $\lambda\eta^{n-1}\mu(n)$.

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