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ON THE NUMBER OF CROSSINGS OF EMPIRICAL DISTRIBUTION FUNCTIONS

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Let F and G be two continuous distribution functions that cross at a finite number of points $-\infty \leq t_1 < \cdots < t_k \leq \infty$. We study the limiting behavior of the number of times the empirical distribution function G_n crosses F and the number of times G_n crosses F_n. It is shown that these variables can be represented, as $n \to \infty$, as the sum of k independent geometric random variables whose distributions depend on F and G only through $F'(t_i)/G'(t_i)$, $i = 1, \ldots, k$. The technique involves approximating $F_n(t)$ and $G_n(t)$ locally by Poisson processes and using renewal-theoretic arguments. The implication of the results to an algorithm for determining stochastic dominance in finance is discussed.

1. Introduction. Consider two given continuous distribution functions F and G, and let F_n and G_n be the corresponding empirical distribution functions (edf's) based on n independent and identically distributed (i.i.d.) random variables (rv's). Throughout F_n and G_n are assumed to be independent. Let $K(n) =$ number of times $G_n(t)$ equals $F(t)$, and let $L(n) =$ number of intervals for which the graph of G_n equals the graph of F_n. The distributions of $K(n)$ and $L(n)$ have been studied when $F \equiv G$, and it is known [see Dwass (1961)] that

$$\lim_{n \to \infty} P\{K(n) < (2nt)^{1/2}\} = \lim_{n \to \infty} P\{L(n) < (4nt)^{1/2}\}$$
$$= 1 - e^{-t}.$$

We consider the situation where $F \not\equiv G$ but F crosses G at a finite number of points $-\infty \leq t_1 < \cdots t_k \leq \infty$. We were led to this problem while investigating certain types of stochastic dominance procedures that arise in finance.

We study the limiting behavior of $K(n)$, $L(n)$, and $L^*(n) =$ the number of times $G_n(t)$ strictly crosses $F_n(t)$ (see Section 2 for definition of strict crossing). It is shown that these variables can be represented, as $n \to \infty$, as the sum of k independent rv's each of which is the number of crossings in a local neighborhood around t_i, $i = 1, \ldots, k$. The distributions of these k rv's are related to geometric rv's whose distributions depend on F and G only through $F'(t_i)/G'(t_i)$, $i = 1, \ldots, k$. Furthermore, only a finite number of order statistics around t_i play a role in determining these crossings. The results are obtained under the assumption that F and G are continuously differentiable in a neighborhood of t_i with $F'(t_i) \neq G'(t_i)$, $i = 1, \ldots, k$.

The paper is organized as follows. The main results on the limiting distributions and an outline of the proofs are given in Section 2. Details of the proofs are

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deferred to Section 4 where we also derive a number of interesting lemmas concerning the local behavior and boundary crossing probabilities of empirical distributions. In Section 3 we discuss the implications of our results to an algorithm for determining stochastic dominance in finance. Our results suggest the need for using smoothed edf's in the algorithm.

Throughout the paper, we assume that both F and G are defined on $[0,1]$ and that $F(t)=t$. This presents no loss of generality since $K(n)$, $L(n)$, and $L^*(n)$ are invariant under monotone transformations of the underlying rv's that determine $F_n(t)$ and $G_n(t)$. To simplify matters we assume in addition that $0<G(t)<1$ for $0<t<1$ and that $G(0)=1-G(1)=0$.

2. Limiting distributions. Let $\{U_j\}_{j=1}^{\infty}$ denote a sequence of i.i.d. uniform $(0,1)$ rv's, i.e., $U_j \sim F(t)=t$, $0\le t\le 1$. Similarly let $\{V_j\}_{j=1}^{\infty}$ be i.i.d. rv's with $V_j\sim G(t)$, $0\le t\le 1$. $G(t)$ equals t at k points $0=t_1<t_2<\cdots<t_k=1$. Given $\delta>0$, consider the δ-neighborhood of t_i defined by $B(t_i;\delta)=(t_i-\delta, t_i+\delta)\cap[0,1]$. *We assume throughout that $G(t)$ is continuously differentiable in $B(t_i;\delta)$ for some $\delta>0$ with $G'(t_i)\neq 1$, $i=1,\dots,k$.*

F_n and G_n denote the edf's based respectively on $\{U_j\}_{j=1}^{n}$ and $\{V_j\}_{j=1}^{n}$. We assume that F_n and G_n are independent. $K(n)$ is the number of times $G_n(t)$ equals t. The points $t=0$ and $t=1$ are included in the computation of $K(n)$. $L(n)$ is the number of intervals for which the graph of G_n equals the graph of F_n. $L^*(n)$ is the number of times $G_n(t)$ strictly crosses $F_n(t)$. To define a strict crossing, let $Z_1<\cdots<Z_{2n}$ be the $2n$ ordered values of $\{U_j\}_{j=1}^{n}$ and $\{V_j\}_{j=1}^{n}$. A strict crossing occurs at Z_j if either (i) $F_n(Z_{j-2})<G_n(Z_{j-2})$ and $F_n(Z_j)>G_n(Z_j)$ or (ii) $G_n(Z_{j-2})<F_n(Z_{j-2})$ and $G_n(Z_j)>F_n(Z_j)$. Again for the sake of consistency, we include the points $t=0$ and $t=1$ in the computation of $L^*(n)$.

It follows from the Glivenko–Cantelli theorem that, with probability $\to 1$ as $n\to\infty$, $G_n(t)$ does not cross t or $F_n(t)$ outside of $\bigcup_{i=1}^{k} B(t_i;\delta)$ for every sufficiently small $\delta>0$. Now let $\delta>0$ be small enough so that $B(t_i;\delta)$, $i=1,\dots,k$, are nonoverlapping. Let $K_i(n)$, $L_i(n)$, and $L_i^*(n)$ be the corresponding values of $K(n)$, $L(n)$, and $L^*(n)$ restricted to the interval $B(t_i;\delta)$. $M_\alpha(u)$ denotes a Poisson process with rate α, $\to_P$ denotes convergence in probability, and $\to_D$ denotes convergence in distribution. Let $\alpha_i=G'(t_i)$ and if $\alpha_i>1$, let $\theta(\alpha_i)<1$ be the solution of

$$\theta(\alpha_i)e^{-\theta(\alpha_i)}=\alpha_i e^{-\alpha_i}. \tag{2.1}$$

THEOREM 1. *As $n\to\infty$, $K_i(n)\to_D K_i$, $i=1,\dots,k$, where the K_i's are independent and*

$$P\{K_i=j\}$$

$$=\begin{cases}(1-\alpha_i)\alpha_i^{j-1}, & j=1,2,\dots \quad \textit{if } \alpha_i<1,\\ [1-\theta(\alpha_i)]\theta^{j-1}(\alpha_i), & j=1,2,\dots \quad \textit{if } \alpha_i>1,\ t_i=0 \textit{ or } 1,\\ [\alpha_i+\theta(\alpha_i)-2]/(\alpha_i-1), & j=0 \quad \textit{if } \alpha_i>1, 0<t_i<1,\\ [1-\theta(\alpha_i)]^2\theta^{j-1}(\alpha_i)/(\alpha_i-1), & j=1,2,\dots \quad \textit{if } \alpha_i>1, 0<t_i<1.\end{cases}$$

REMARK. We in fact show that $K_i(n) \to_P K_i$ in the case $\alpha_i < 1$ and in the case $\alpha_i > 1$, $t_i = 0$ or 1. When $\alpha_i > 1$ and $0 < t_i < 1$, we can show only that conditionally $\{K_i(n) | K_i(n) \geq 1\} \to_P \{K_i | K_i \geq 1\}$.

Let $\rho(\alpha_i) = \min(\alpha_i, 1/\alpha_i)$ and let

$$\pi(\alpha_i) = 1 - [1 - \rho(\alpha_i)]/[1 + \rho(\alpha_i)]. \tag{2.2}$$

THEOREM 2. *As $n \to \infty$, $L_i(n) \to_P L_i$, $i = 1, \dots, k$, where L_i's are independent and*

$$P\{L_i = j + 1\} = [1 - \pi(\alpha_i)] \pi^j(\alpha_i), \qquad j = 0, 1, \dots .$$

THEOREM 3. *As $n \to \infty$, $L_i^*(n) \to_P L_i^*$, $i = 1, \dots, k$, where the L_i^*'s are independent and*

$$P\{L_i^* = j + 1\} = [1 - \rho(\alpha_i)] \rho^j(\alpha_i), \qquad j = 0, 1, \dots .$$

As the proof is rather lengthy, we defer the details to Section 4 and give only an outline of the proof here. The following two sequences $\{\varepsilon_n\}$ and $\{\delta_n\}$ will be used extensively in the proofs in Section 4:

(i) $\{\varepsilon_n\}$ is any sequence such that as $n \to \infty$

$$\varepsilon_n \to 0 \quad \text{but } n^{1/2}\varepsilon_n \to \infty; \tag{2.3}$$

(ii) $\{\delta_n\}$ is any sequence such that as $n \to \infty$

$$n^{1/2}\delta_n \to 0 \quad \text{but } c_n \equiv n\delta_n \to \infty. \tag{2.4}$$

Since $[G_n(t) - G(t)] = o_p(\varepsilon_n)$ and $G(t_i) = t_i$, it can be seen that with probability $\to 1$ as $n \to \infty$, all the crossings occur inside the shrinking neighborhoods $B(t_i; \varepsilon_n)$. Within these shrinking neighborhoods, we show that all the crossings in fact occur in a δ_n-subinterval. Further, within these δ_n-subintervals, say $[T_{i,n}, T_{i,n} + \delta_n]$, we approximate $n[G_n(t) - G_n(T_{i,n})]$ and $n[F_n(t) - F_n(T_{i,n})]$ by Poisson processes $M_{\alpha_i}(\cdot)$ and $M_1(\cdot)$ with rates α_i and 1, respectively. Finding the distributions of $L_i(n)$ and $L_i^*(n)$ then reduces to finding the distributions of L_i = number of times $M_1(u)$ equals $M_{\alpha_i}(u)$ for all $u \geq 0$ and of L_i^* = number of times $M_1(u)$ strictly crosses $M_{\alpha_i}(u)$ for all $u \geq 0$. Similarly, if $\alpha_i < 1$ or if $\alpha_i > 1$ and $t_i = 0$ or 1, $K_i(n)$ reduces to K_i = number of times $M_{\alpha_i}(u)$ equals u for all $u \geq 0$. The case $\alpha_i > 1$ and $0 < t_i < 1$ is more complicated because in this case when $G_n(t)$ first crosses $F(t)$ from below, it overshoots $F(t)$. The distribution of $K_i(n)$ depends on the distribution of this random overshoot. However if $K_i(n) \geq 1$, i.e., $G_n(t)$ crosses $F(t)$ again from above, say at $T_{i,n}$, so that $G_n(T_{i,n}) = F(T_{i,n})$ we can approximate $n[G_n(t) - G_n(T_{i,n})]$ by $M_{\alpha_i}(\cdot)$ and use the same arguments as in the case $\alpha_i > 1$ and $t_i = 0$ to get the results. We use renewal-theoretic arguments and the application of Wiener–Hopf techniques to obtain the distributions of K_i, L_i, L_i^* and to calculate $\lim_{n\to\infty} P\{K_i(n) \geq 1\}$ when $\alpha_i > 1$ and $0 < t_i < 1$.

En route, we derive the following results on the boundary crossing probabilities of Poisson processes. We show that for $\alpha > 1$

$$P\{M_\alpha(u) = u \text{ for some } u > 0\} = \theta(\alpha)$$

and

$$\lim_{a \to \infty} P\{M_\alpha(u) = a + u \text{ for some } u > 0\} = [1 - \theta(\alpha)]/(\alpha - 1).$$

These two results may already be known in the literature. Even if this is the case, our method of proof based on Lemma 4.4 in Section 4 should be of independent interest.

3. A stochastic dominance problem. We were led to the rv's $K(n)$, $L(n)$, and $L^*(n)$ while investigating certain types of stochastic dominance procedures that arise in finance. In this section, we briefly describe the implication of the results in Section 2 to an algorithm proposed by Bawa, Lindenberg, and Rafsky (1979) for determining stochastic dominance. We start with some background on the problem.

Decision-making under uncertainty may be viewed as choices between alternative probability distributions of returns. In the stochastic dominance (SD) approach to this problem, one restricts an individual's utility function to a certain reasonable class and obtains the admissible set of alternatives for this case [see Whitmore and Findlay (1978) and references therein]. The first-order SD (SD_1) admissible set is valid for all decision makers with nondecreasing utility functions, the second-order (SD_2) admissible set for the subset of risk-averse individuals, and the third-order SD (SD_3) admissible set for further subset with decreasing absolute risk aversion.

The determination of the SD-admissible sets involves pairwise comparisons of probability distributions. For example, if F and G denote the distributions of two alternatives, it is known that F dominates G in the SD_2 sense if and only if $\int_{-\infty}^{t}[F(u) - G(u)]\,du \leq 0$ for all t with strict inequality for some t, and in the SD_3-sense if and only if $\int_{-\infty}^{t}\int_{-\infty}^{v}[F(u) - G(u)]\,du\,dv \leq 0$ for all t with strict inequality for some t (Bawa, 1975). In practice, the distributions F and G are unknown and are estimated by the edf's F_n and G_n based on past economic data. The SD-admissible sets are then determined by comparing the distributions F_n and G_n at the $2n$ jump points. However, there are typically a large number of such distributions and it can be rather time-consuming to determine the SD-admissible sets.

Recently Bawa, Lindenberg, and Rafsky (1979) introduced an efficient algorithm for determining SD-admissible sets. Among other things, the algorithm exploits a zero-crossing property which observes that the point t_0 where F_n crosses G_n is a point of either local minima or maxima for $H_n(t) = \int_{-\infty}^{t}[F_n(u) - G_n(u)]\,du$. To determine SD_2-dominance therefore it suffices to look at the $L^*(n)$ points where $F_n(t)$ crosses $G_n(t)$. A similar property is used for SD_3-comparisons.

At first glance one may speculate that if F crosses G at k points (and typically k will be small), $L^*(n)$ will be close to k for large n so that the number of computations does not grow with n. However, our results in Section 2 show that the value of $L^*(n)$ can be rather large even if k is small. For example, let $F(t) = \Phi(t/\sigma_1)$ and $G(t) = \Phi(t/\sigma_2)$ where $\Phi(t)$ is the standard normal df, and $\sigma_2 < \sigma_1$. Then F crosses G at $t = \pm\infty$ and at $t = 0$, and $G'(t)/F'(t) = 0$ at

$t = \pm\infty$ and $= \sigma_2/\sigma_1$ at $t = 0$. From Theorem 3 in Section 2 we note that if σ_2 is close to σ_1, $L^*(n)$ can take on large values. For example,

$$E(L^*) = \sigma_1/(\sigma_1 - \sigma_2),$$

which $\to \infty$ as $\sigma_2 \to \sigma_1$.

It is possible to construct smoothed edf's for which the number of crossings converges in probability to k as $n \to \infty$. The nearest-neighbor or kernel type approaches to density estimation can be used to obtain such smoothed edf's. This suggests that one may gain by smoothing the edf's before using them in the algorithm of Bawa et al. (1979) to determine SD-admissible sets. However, to be useful, these estimates need to be computationally simple.

4. Auxiliary results and proofs. Let $\{N_j\}_{j=1}^{\infty}$ be a sequence of Poisson rv's which are independent of the underlying rv's $\{U_j\}$ and $\{V_j\}$ with $EN_j = j$. Let $\{\varepsilon_n\}$ and $\{\delta_n\}$ be defined as in (2.3) and (2.4) in Section 2 with $c_n \equiv n\delta_n$. We use the notation $I(A)$ to denote the indicator function of a set A. We start with some results on the relationships between the edf's and Poisson processes. The following result is well known [see, for example, Kac (1949)].

LEMMA 4.1. *For $u \in [0, n]$, $\sum_{j=1}^{N_n} I(U_j \le u/n)$ is a Poisson process with rate* 1.

Next fix t_0 and α, where $G(t_0) = t_0$, $0 \le t_0 \le 1$, and $\alpha = G'(t_0) \ne 1$. Let $\{t_n\}$, $0 < t_n < 1$, be any sequence for which $t_n \to t_0$ as $n \to \infty$. For $t \in [t_n, t_n + \delta_n]$ we can approximate $n[G_n(t) - G_n(t_n)]$ by a Poisson process M_α.

LEMMA 4.2. *There exists a Poisson process M_α such that*

$$\lim_{n\to\infty} P\left\{ \sup_{t_n \le t \le t_n + \delta_n} \left| n[G_n(t) - G_n(t_n)] - M_\alpha(n(t - t_n)) \right| > 0 \right\} = 0.$$

PROOF. Let $W_j = G(V_j)$ where $V_j \sim G$. Consider $M_\alpha(u) = \sum_{j=1}^{N_n} I(t_n < W_j \le t_n + \alpha u/n)$. It follows from Lemma 4.1 that $M_\alpha(u)$ is in fact a Poisson process with rate α on $[0, c_n)$. So

$$\begin{aligned} P&\left\{ \sup_{t_n \le t \le t_n + \delta_n} \left| \sum_{j=1}^{n} I(t_n < V_j \le t) - \sum_{j=1}^{N_n} I(t_n < W_j \le t_n + \alpha(t - t_n)) \right| > 0 \right\} \\ &\le [G(t_n + \delta_n) - G(t_n)] E|N_n - n| \\ &\quad + P\left\{ \sup_{t_n \le t \le t_n + \delta_n} \left| \sum_{j=1}^{N_n} \left[I(G(t_n) < W_j \le G(t)) \right.\right.\right. \\ &\qquad\qquad \left.\left.\left. - I(t_n \le W_j \le t_n + \alpha(t - t_n)) \right] \right| > 0 \right\}. \end{aligned}$$

The first term $\to 0$ as $n \to \infty$ since $G(t_n + \delta_n) - G(t_n) \sim \delta_n G'(t_0)$ and $E|N_n - n| \le \sqrt{E(N_n - n)^2} = n^{1/2}$. To show that the second term $\to 0$, note that $\sup_{|t - t_n| \le \delta} |G(t) - G(t_n) - \alpha(t - t_n)| = o(\delta)$ as $\delta \to 0$. Bound this $o(\delta)$ term by $\delta \cdot \beta(\delta)$ where $\beta(\delta) \to 0$ monotonically as $\delta \to 0$. Now choose δ_n with $c_n \le \sqrt{1/\beta(n^{-1/2})}$. Then the second term is bounded by $\delta_n \beta(\delta_n) EN_n = c_n \beta(\delta_n) \le \beta(\delta_n)/\sqrt{\beta(n^{-1/2})} \le \sqrt{\beta(n^{-1/2})} \to 0$ as $n \to \infty$. □

Next we need some results on the boundary crossing probabilities of Poisson processes. It is known that for $\alpha < 1$

$$P\{M_\alpha(u) = u \text{ for some } u > 0\} = 1 - P\{M_\alpha(u) < u \text{ for all } u > 0\} = \alpha. \tag{4.1}$$

The second equality follows from Theorem 1 in Chapter 3 of Takacs (1967). To get the first equality, note from the strong law of large numbers that $M_\alpha(u)/u \to \alpha < 1$ as $u \to \infty$. Hence if $M_\alpha(u) \ge u$ for some u, it must cross the line u from above again, say at u^*, so that $M_\alpha(u^*) = u^*$. The following lemma gives the probability for $\alpha > 1$. Let $\theta(\alpha) < 1$ satisfy $\theta(\alpha)e^{-\theta(\alpha)} = \alpha e^{-\alpha}$.

LEMMA 4.3. *For* $\alpha > 1$,

$$P\{M_\alpha(u) = u \textit{ for some } u > 0\} = \theta(\alpha).$$

Before proving Lemma 4.3, we calculate

$$f(z) = P\{M_\alpha(u) + z > u \text{ for all } u > 0\}. \tag{4.2}$$

Let $\{X_j\}_{j=1}^{\infty}$ be i.i.d. standard exponential rv's, and let

$$P_n(\alpha, t) = P\{X_1 + \cdots + X_j \le \alpha(j-1) + t,\ j = 1, \dots, n\}.$$

It can be checked that

$$f(z) = P_\infty(\alpha, \alpha z). \tag{4.3}$$

LEMMA 4.4.

(i)
$$P_n(\alpha, t) = \begin{cases} 0 & \textit{if } \alpha(j-1) + t \le 0 \textit{ for some } j \le N, \\ 1 - \sum_{j=0}^{n-1} e^{-j\alpha - t} t(j\alpha + t)^{j-1}/j! & \textit{otherwise}. \end{cases}$$

(ii)
$$P_\infty(\alpha, t) = \begin{cases} 0 & \textit{if } \alpha \le 1, \\ 1 - e^{-t(1-\theta(\alpha)/\alpha)} & \textit{if } \alpha > 1. \end{cases}$$

PROOF. The easiest proof we know is by induction. Let $B_0 = 1$ and for $j \ge 1$,

$$B_j = \int_0^t dx_1 \int_0^{\alpha + t - x_1} dx_2 \cdots \int_0^{(j-1)\alpha + t - x_1 - \cdots - x_{j-1}} dx_j.$$

Then if $P_n \equiv P_n(\alpha, t)$,

$$P_n = \int_0^t dx_1 \int_0^{\alpha+t-x_1} dx_2 \cdots \int_0^{(n-1)\alpha+t-x_1-\cdots-x_{n-1}} dx_n\, e^{-x_1-\cdots-x_n}$$
$$= P_{n-1} - B_{n-1} e^{-(n-1)\alpha - t}.$$

We show that $B_n = t(n\alpha + t)^{n-1}/n!$ which immediately gives the expression for P_n. Integrating B_n over x_n, it can be shown that

$$\sum_{j=0}^{n} B_{n-j}[(j-1)\alpha]^j/j! = [(n-1)\alpha + t]^n/n!.$$

Now Abel's identity (Riordan, 1968) gives

$$x^{-1}(x + y + n\beta)^n = \sum_{k=0}^{n} \binom{n}{k}(x + k\beta)^{k-1}[y + (n-k)\beta]^{n-k}. \tag{4.4}$$

Setting $x = t$, $y = -\alpha$, $\beta = \alpha$, and $k = n - j$ in (4.4) we get

$$\sum_{j=0}^{n} \frac{t}{(n-j)!}[t + (n-j)\alpha]^{n-j-1} \frac{[(j-1)\alpha]^j}{j!} = \frac{[t + (n-1)\alpha]^n}{n!},$$

i.e., $B_{n-j} = t[(n-j)\alpha + t]^{n-j-1}/(n-j)!$ as claimed. Now as $n \to \infty$, $P_n \to P_\infty$ where

$$P_\infty = 1 - e^{-t} \sum_{j=0}^{\infty} (t/\alpha)(j + t/\alpha)^{j-1}(\alpha e^{-\alpha})^j/j!.$$

To show P_∞ is of the form claimed in (ii), let $y = -n[1 - 1/\theta(\alpha)]$, $\beta = 1$ in (4.4) and let $n \to \infty$. We get

$$e^{x\theta(\alpha)} = x \sum_{k=0}^{\infty} [\theta(\alpha)]^{-k}(x + k)^{k-1} e^{-\theta(\alpha)}/k!.$$

Setting $x = t/\alpha$ gives the desired result. □

PROOF OF LEMMA 4.3. Let $T =$ first $u > 0$: $M_\alpha(u) > u$. Since $\alpha > 1$, $P\{T < \infty\} = 1$ by the law of large numbers. Let $Z = M_\alpha(T) - T$, the amount of overshoot at T, and let $M_\alpha'(u)$ be a Poisson process independent of Z. Then

$$\begin{aligned} P\{M_\alpha(u) \neq u \text{ for all } u > 0\} &= P\{M_\alpha(u) > u \text{ for all } u > T\} \\ &= P\{M_\alpha'(u) + Z > u \text{ for all } u > 0\} \\ &= Ef(Z) \end{aligned}$$

from the definition of $f(z)$ in (4.2). The distribution of Z can be obtained from the Wiener–Hopf results in Feller (1971). Let $Y_i = (X_i/\alpha) - 1$ where the X_i's are i.i.d. standard exponential. Then Z can be represented as the ladder height variable for the process $S_n = Y_1 + \cdots + Y_n$, $S_0 = 0$. The results on page 405 of Feller (1971) show, after some algebraic manipulations, that Z has density

$$h(z) = \alpha e^{-[\alpha - \theta(\alpha)](1-z)}, \qquad 0 < z < 1. \tag{4.5}$$

From this we get

$$Ef(Z) = \int_0^1 P_\infty(\alpha, \alpha z)h(z)\,dz$$
$$= 1 - \theta(\alpha). \qquad \square$$

LEMMA 4.5. *For* $\alpha > 1$,

$$\lim_{a\to\infty} P\{M_\alpha(u) = a + u \text{ for some } u > 0\} = [1 - \theta(\alpha)]/(\alpha - 1).$$

PROOF. For fixed $a > 0$, let T_a = first $u > 0$: $M_a(u) > a + u$, and let $W_a = M_\alpha(T_a) - (a + T_a)$. Then by the same arguments as in the proof of Lemma 4.3,

$$P\{M_\alpha(u) \neq a + u \text{ for all } u > 0\} = Ef(W_a). \tag{4.6}$$

To obtain the limiting distribution of W_a as $a \to \infty$, once again let $Y_i = (X_i/\alpha) - 1$, X_i's i.i.d. standard exponential. Note that W_a has the same distribution as the amount of overshoot for the process $S_n = Y_1 = \cdots + Y_n$, with $S_0 = 0$, when it first exceeds a. Consider now the related renewal process $S_n' = Z_1 + \cdots + Z_n$ with $S_0' = 0$ where the Z_i's are the ladder height variables that we considered in the proof of Lemma 4.3 for the process S_n. W_a also has the same distribution as the amount of overshoot for the S_n' process when it first exceeds a. Since the Z_i's are nonnegative, we can apply the results in Feller (1971, page 369) to get the limiting distribution of W_a as $a \to \infty$ as

$$\lim_{a\to\infty} P\{W_a \le z\} = \mu^{-1}\int_0^z [1 - H(x)]\,dx \tag{4.7}$$

where $H(x) = \int_0^x h(y)\,dy$, h is given by (4.5), and $\mu = \int_0^1[1 - H(x)]\,dx$. From this we see that the limiting distribution has density

$$w(z) = [\alpha/(\alpha - 1)](1 - e^{-[\alpha - \theta(\alpha)](1-z)}), \qquad 0 < z < 1. \tag{4.8}$$

Since $f(z) = P_\infty(\alpha, \alpha z)$ is bounded and continuous, we get

$$\lim_{a\to\infty} Ef(W_a) = \int_0^1 P_\infty(\alpha, \alpha z)w(z)\,dz$$
$$= 1 - [1 - \theta(\alpha)]/(\alpha - 1).$$

The result now follows from (4.6). $\square$

We study the crossings of F_n and G_n in several steps. We start with the crossings of $F_n(t)$ and $t_0 + \alpha(t - t_0)$, the line through t_0 with slope α. Here t_0 and α are fixed with $0 \le t_0 \le 1$ and $0 \le \alpha \neq 1$. Let $\tau_n^f(t_0)$ be the first time $F_n(t)$ crosses $t_0 + \alpha(t - t_0)$, i.e.,

$$\tau_n^f(t_0) = \begin{cases} 0 & \text{if } t_0 = 0, \\ \text{first } t > 0\text{: } F_n(t) \le t_0 + \alpha(t - t_0) & \text{if } t_0 > 0,\ \alpha > 1, \\ \text{first } t > 0\text{: } F_n(t) > t_0 + \alpha(t - t_0) & \text{if } t_0 > 0,\ \alpha < 1. \end{cases} \tag{4.9}$$

Similarly define $\tau_n^l(t_0)$ as the last time $F_n(t)$ crosses $t_0 + \alpha(t - t_0)$ with $\tau_n^l(1) = 1$.

From the fact that $1 - F_n(1 - t)$ has the same distribution as $F_n(t)$, it follows that

$$P\{\tau_n^l(t_0) \le t_0 + x\} = P\{\tau_n^f(1 - t_0) \ge 1 - t_0 - x\}. \tag{4.10}$$

This observation will prove useful later. Consider the shrinking neighborhood $B(t_0; \varepsilon_n)$. Since $F_n(t) = F(t) + O_p(n^{-1/2})$, it can be shown using Chebychev's inequality that for $\alpha > 1$

$$\lim_{n\to\infty} P\{F_n(t) \le t_0 + \alpha(t - t_0) \text{ for all } t \in [t_0 + \varepsilon_n, 1]\} = 1 \tag{4.11}$$

and

$$\lim_{n\to\infty} P\{F_n(t) \ge t_0 + \alpha(t - t_0) \text{ for all } t \in [0, t_0 - \varepsilon_n]\} = 1.$$

For $\alpha < 1$, (4.11) holds with the inequalities reversed. Therefore

$$\lim_{n\to\infty} P\{[\tau_n^f(t_0), \tau_n^l(t_0)] \subset B(t_0; \varepsilon_n)\} = 1. \tag{4.12}$$

We now show that

$$\lim_{n\to\infty} P\{\tau_n^l(t_0) - \tau_n^f(t_0) \le \delta_n\} = 1, \tag{4.13}$$

i.e., all the crossings occur in a δ_n-interval. This means that at most c_n order statistics around t_t are involved in the crossings. Since this is true for any $c_n = n\delta_n \to \infty$ with $n^{1/2}\delta_n \to 0$, there is only a finite number of them.

LEMMA 4.6.

(i) $\lim_{n\to\infty} P\{F_n(t) < t_0 + \alpha(t - t_0)$ *for all* $t \ge \tau_n^f(t_0) + \delta_n\} = 1$ *for* $\alpha > 1$.

(ii) $\lim_{n\to\infty} P\{F_n(t) > t_0 + \alpha(t - t_0)$ *for all* $t \ge \tau_n^f(t_0) + \delta_n\} = 1$ *for* $\alpha < 1$.

PROOF. Consider first the case $\alpha > 1$ for which $F_n(\tau_n^f(t_0)) \equiv t_0 + \alpha(\tau_n^f(t_0) - t_0)$. So

$$\begin{aligned} &P\{F_n(t) > t_0 + \alpha(t - t_0) \text{ for some } t > \tau_n^f(t_0) + \delta_n\} \\ &\quad \le P\{F_n(t) - F_n(\tau_n^f(t_0) + \delta_n) > \alpha(t - \tau_n^f(t_0) - \delta_n) \\ &\qquad\qquad + \delta_n(\alpha - 1)/2 \text{ for some } t > \tau_n^f(t_0) + \delta_n\} \\ &\quad + P\{F_n(\tau_n^f(t_0) + \delta_n) - F_n(\tau_n^f(t_0)) > \delta_n(\alpha + 1)/2\}. \end{aligned} \tag{4.14}$$

It follows from the strong Markov property of F_n that $F_n(\tau_n^f(t_0) + \delta_n) - F_n(\tau_n^f(t_0))$ has the same distribution as $F_n(\delta_n)$. So by Chebychev's inequality, the second term in (4.14) is less than $4\delta_n(1 - \delta_n)/(n\delta_n^2(\alpha - 1)^2)$ which $\to 0$ as $n \to \infty$. To show that the first term in (4.14) $\to 0$, consider the Poisson variables

$\{N_n\}$. Since $P\{N_n \geq n\} \to \frac{1}{2}$ as $n \to \infty$,

$$\lim_{n\to\infty} P\{F_n(t) - F_n(\tau_n^f(t_0) + \delta_n) > \alpha(t - \tau_n^f(t_0) - \delta_n) + \delta_n(\alpha - 1)/2 \text{ for some } t \geq \tau_n^f(t_0) + \delta_n\}$$

$$\leq \lim_{n\to\infty} 2P\left\{\sum_{j=1}^{N_n} I(\tau_n^f(t_0) + \delta_n < U_j \leq t) - n\alpha(t - \tau_n^f(t_0) - \delta_n) > c_n(\alpha - 1)/2 \text{ for some } t > \tau_n^f(t_0) + \delta_n\right\}$$

$$\leq \lim_{n\to\infty} 2P\left\{\sup_{0<u<\infty} M_1(u) - \alpha u > c_n(\alpha - 1)/2\right\},$$

which equals 0 from the following argument. Let

$$g_\alpha(a) = P\left\{\sup_{0<u<\infty} M_1(u) - \alpha u > a\right\}.$$

If $a' < a$,

$$g_\alpha(a) = g_\alpha(a')Eg_\alpha(a - a' - Z), \tag{4.15}$$

where Z is the amount by which $M_1(u)$ overshoots the line $a' + \alpha u$ at the point of first crossing. Let $g_\alpha = \lim_{a\to\infty} g_\alpha(a)$. By letting $a \to \infty$ and then $a' \to \infty$ in (4.15), we get $g_\alpha = g_\alpha^2$, i.e., $g_\alpha = 0$ or 1. But $g_\alpha(a)$ is monotone decreasing in a and from (4.1) we see that $g_\alpha(0) = 1/\alpha < 1$. Hence $g_\alpha = 0$.

When $\alpha < 1$, F_n overshoots the line $t_0 + \alpha(t - t_0)$ at the time of crossing $\tau_n^f(t_0)$. However, this overshoot is less than $1/n$ and so

$$\begin{aligned} P\{F_n(t) &< t_0 + \alpha(t - t_0) \text{ for some } t \geq \tau_n^f(t_0) + \delta_n\} \\ &\leq P\{F_n(t) - F_n(\tau_n^f(t_0) + \delta_n) < \alpha(t - \tau_n^f(t_0) - \delta_n) + \delta_n(\alpha - 1)/2 \text{ for some } t \geq \tau_n^f(t_0) + \delta_n\} \\ &\quad + P\{F_n(\tau_n^f(t_0) + \delta_n) - F_n(\tau_n^f(t_0)) < \delta_n(\alpha + 1)/2 - 1/n\}. \end{aligned} \tag{4.16}$$

The second term in (4.16) $\to$ 0 by Chebychev's inequality. The first term $\to$ 0 by an argument analogous to that for $\alpha > 1$ but now based on Lemma 4.3. □

Next consider the crossings of $F_n(t)$ and $h(t)$ in $B(t_0; \delta)$ when $h(t_0) = t_0$ and $h(t)$ is continuously differentiable in $B(t_0; \delta)$ with $0 \leq h'(t_0) \neq 1$. Let $\nu_n^f(t_0)$ and $\nu_n^l(t_0)$ respectively be the first and last crossings defined as in (4.9). It can once again be shown that

$$\lim_{n\to\infty} P\{[\nu_n^f(t_0), \nu_n^l(t_0)] \subset B(t_0; \varepsilon_n)\} = 1.$$

Similar arguments as in Lemma 4.6 can be used to obtain the following.

LEMMA 4.7.

$$\lim_{n\to\infty} P\{\nu_n^l(t_0) - \nu_n^f(t_0) \leq \delta_n\} = 1.$$

PROOF. We will consider just the case $h'(t_0) > 1$ since the other case is similar. There exists α with $1 < \alpha < h'(t_0)$ such that $h(t) > h(\nu_n^f(t_0)) + \alpha(t - \nu_n^f(t_0))$ for $\nu_n^f(t_0) \le t \le t_0 + \delta$. Since $F_n(\nu_n^f(t_0)) = h(\nu_n^f(t_0))$,

$$\lim_{n \to \infty} P\{F_n(t) < h(t) \text{ for all } t \ge \nu_n^f(t_0) + \delta_n\}$$
$$\ge \lim_{n \to \infty} P\{F_n(t) - F_n(\nu_n^f(t_0)) < \alpha(t - \nu_n^f(t_0)) \text{ for all } t \ge \nu_n^f(t_0) + \delta_n\},$$

which can be shown to equal one by using arguments similar to those in Lemma 4.6. □

Next consider the edf $G_n(t)$ with $G(t_0) = t_0$ and $G(t)$ continuously differentiable in $B(t_0; \delta)$. Let $\mu_n^f(t_0)$ and $\mu_n^l(t_0)$ be respectively the first and last crossings of $G_n(t)$ and the line $t_0 + \beta(t - t_0)$, $\beta \ne G'(t_0)$, in $B(t_0; \delta)$. Again we can show that

$$\lim_{n \to \infty} P\{[\mu_n^f(t_0), \mu_n^l(t_0)] \subset B(t_0; \varepsilon_n)\} = 1.$$

LEMMA 4.8.

$$\lim_{n \to \infty} P\{\mu_n^l(t_0) - \mu_n^f(t_0) \le \delta_n\} = 1.$$

PROOF. Let $\nu_n^f(t_0)$ and $\nu_n^l(t_0)$ be the first and last times $F_n(t)$ crosses $t_0 + \beta(G^{-1}(t) - t_0)$ in $B(t_0; \delta)$. Since $G_n(t)$ has the same distribution as $F_n(G(t))$, $(\nu_n^f(t_0), \nu_n^l(t_0))$ has the same distribution as $(G(\mu_n^f(t_0)), G(\mu_n^l(t_0)))$. The result now follows from Lemma 4.7 and the fact that $G(\mu_n^l(t_0)) - G(\mu_n^f(t_0))$ is of the same order as $(\mu_n^l(t_0) - \mu_n^f(t_0))$. □

We can now complete the proofs of the results in Section 2.

PROOF OF THEOREM 1. We consider three separate cases.

CASE (i) $\alpha_i = G'(t_i) < 1$. Let $\mu_n^f(t_i)$ be the first time $G_n(t)$ equals t in $B(t_i; \delta)$. Note that $\mu_n^f(t_i)$ exists with probability $\to 1$ as $n \to \infty$. By Lemma 4.8, we can restrict attention to the number of times $G_n(t)$ equals t in $[\mu_n^f(t_i), \mu_n^f(t_i) + \delta_n]$. In this interval, we can use Lemma 4.2 to approximate $G_n(t) - G_n(\mu_n^f(t_i)) \equiv G_n(t) - \mu_n^f(t_i)$ by a Poisson process M_{α_i}. Therefore, with probability $\to 1$ as $n \to \infty$, $K_i(n)$ is the same as the number of times $M_{\alpha_i}(u)$ equals u in $[0, c_n]$. Since $c_n \to \infty$ as $n \to \infty$ and $\alpha_i < 1$, it can be shown that the probability $M_{\alpha_i}(u)$ equals u for some $u \in (c_n, \infty)$ tends to 0 as $n \to \infty$. So we may as well consider K_i = number of times $M_{\alpha_i}(u)$ equals u for all $u \ge 0$. Note that $K_i \ge 1$ since $M_{\alpha_i}(0) = 0$. From (4.1),

$$P\{K_i = 1\} = P\{M_{\alpha_i}(u) \ne u \text{ for all } u > 0\}$$
$$= 1 - \alpha_i.$$

From the strong Markov property and the stationarity of the increments of

$M_{\alpha_\iota}(u)$, it can be seen that

$$P\{K_i = j+1\} = (1-\alpha_\iota)\alpha_\iota^j, \qquad j = 0, 1, \dots .$$

CASE (ii): $\alpha_\iota = G'(t_\iota) > 1$, $t_\iota = 0$ or 1. It suffices to consider the case $t_\iota = 0$ since the case $t_\iota = 1$ can be obtained from this by considering the process $1 - G_n(1-t)$. By Lemma 4.8, we restrict attention to the crossings in $[0, \delta_n]$. By Lemma 4.2, with probability $\to 1$ as $n \to \infty$, this is the same as the number of times $M_{\alpha_\iota}(u)$ equals u in $[0, c_n]$. Again since $c_n \to \infty$ we may as well consider K_ι = number of times $M_{\alpha_\iota}(u)$ equals u for all $u \geq 0$. By Lemma 4.3,

$$P\{K_\iota = j+1\} = \left[1 - \theta(\alpha_i)\right]\theta^j(\alpha_i), \qquad j = 0, 1, \dots .$$

CASE (iii): $\alpha_\iota = G'(t_i) > 1$, $0 < t_i < 1$. Let $\mu_n^f(t_\iota)$ be the first $t \in B(t_\iota; \delta)$ such that $G_n(t)$. By Lemma 4.8, we can restrict attention to the number of crossings in $[\mu_n^f(t_i), \mu_n^f(t_i) + \delta_n]$. The calculation in this case is a bit more involved because $G_n(t)$ overshoots t at $\mu_n^f(t_\iota)$. If we condition on $\{K_i(n) \geq 1\}$ so that there is a first $t^* \in B(t_i; \delta)$ at which $G_n(t^*) = t^*$, we can use the same arguments as in case (ii) to get

$$P\{K_\iota = j+1 | K_\iota \geq 1\} = \left[1 - \theta(\alpha_i)\right]\theta^j(\alpha_\iota), \qquad j = 0, 1, \dots .$$

We now compute $\lim_{n\to\infty} P\{K_\iota(n) \geq 1\}$. Let

$$R_n = n\left[G_n\left(\mu_n^f(t_i)\right) - \mu_n^f(t_i)\right].$$

By approximating $n[G_n(t) - G_n(\mu_n^f(t_i))]$ for $t \in [\mu_n^f(t_\iota), \mu_n^f(t_i) + \delta_n]$ by the Poisson process M_{α_ι}, we see that

$$\begin{aligned}\lim_{n\to\infty} P\{K_\iota(n) = 0\} &= \lim_{n\to\infty} P\left\{M_{\alpha_\iota}(u) + R_n > u \text{ for all } u > 0\right\}\\ &= \lim_{n\to\infty} Ef(R_n),\end{aligned}$$

where $f(\cdot)$ is given by (4.2). Let $V_{1:n} < \cdots < V_{n:n}$ denote the order-statistics corresponding to $\{V_j\}_{j=1}^n$ where $V_j \sim G$. Let J_n = first $j \geq n(t_i - \varepsilon_n)$: $V_{j:n} \geq j/n$. Then $R_n \equiv J_n - nV_{J_n:n}$. We know that $V_{j:n}$ has the same distribution as $G^{-1}(U_{j:n})$ where $U_{j:n}$ is the order statistic from the uniform distribution. Recall also that $\mu_n^f(t_\iota) \in B(t_i; \varepsilon_n)$ and that in this interval $|G(t) - t_i + \alpha_i(t - t_\iota)| = o(\varepsilon_n)$. By using these facts, it can be shown that the limiting distribution, as $n \to \infty$, of R_n is the same as the limiting distribution, as $a \to \infty$, of $W_a = M_{\alpha_\iota}(T_a) - (a + T_a)$ where T_a = first $u > 0$: $M_{\alpha_\iota}(u) > a + u$. It then follows from (4.6) and Lemma 4.5 that

$$\begin{aligned}\lim_{n\to\infty} P\{K_\iota(n) \geq 1\} &= \lim_{a\to\infty} P\left\{M_{\alpha_\iota}(u) = a + u \text{ for some } u > 0\right\}\\ &= \left[1 - \theta(\alpha_\iota)\right]/(\alpha_\iota - 1). \qquad \square\end{aligned}$$

PROOFS OF THEOREMS 2 AND 3. Let $\alpha_i = G(t_i)$ and let $\mu_n^f(t_i)$ be the first $t \in B(t_i;\delta)$ at which $G_n(t)$ equals $F_n(t)$. For $\alpha_i > 1$,

$$\begin{aligned} P\{G_n(t) > F_n(t) \text{ for all } t \in (\mu_n^f(t_i) + \delta_n, t_i + \delta]\} \\ \geq P\{G_n(t) - G_n(\mu_n^f(t_i)) > (t - \mu_n^f(t_i))(1 + \alpha_i)/2 \\ \text{for all } t \in (\mu_n^f(t_i) + \delta_n, t_i + \delta]\} \\ \times P\{F_n(t) - F_n(\mu_n^f(t_i)) < (t - \mu_n^f(t_i))(1 + \alpha_i)/2 \\ \text{for all } t \in (\mu_n^f(t_i) + \delta_n, t_i + \delta]\} \end{aligned}$$

$\to 1$ as $n \to \infty$ by Corollary 2. Similarly for $\alpha_i < 1$,

$$\lim_{n \to \infty} P\{G_n(t) < F_n(t) \text{ for all } t \in (\mu_n^f(t_i) + \delta_n, t_i + \delta]\} = 1.$$

Therefore we need to consider only the number of crossings in $[\mu_n^f(t_i), \mu_n^f(t_i) + \delta_n]$. In this interval we approximate $G_n(t) - G_n(\mu_n^f(t_i))$ and $F_n(t) - F_n(\mu_n^f(t_i))$ by M_{α_i} and M_1.

Therefore, with probability $\to 1$ as $n \to \infty$, $L_i(n)$ is the same as L_i = the number of intervals for which the graph of $M_{\alpha_i}(u)$ equals the graph of $M_1(u)$ for all $u \geq 0$. Again we note that $L_i \geq 1$ because $M_{\alpha_i}(0) = M_1(0) = 0$. To compute the distribution of L_i, let T = first $u > 0$: $M_{\alpha_i}(u) \neq M_1(u)$, and let

$$\pi(\alpha_i) = P\{M_{\alpha_i}(u) = M_1(u) \text{ for some } u > T\}.$$

Then by the strong Markov property and stationarity of the increments of $M_{\alpha_i}(u) - M_1(u)$,

$$P\{L_i = j + 1\} = [1 - \pi(\alpha_i)]\pi^j(\alpha_i), \qquad j = 0, 1, \dots .$$

To calculate $\pi(\alpha_i)$ first suppose $\alpha_i < 1$. Then $1 - \pi(\alpha_i) = P\{M_{\alpha_i}(u) < M_1(u)$ for all $u \geq T\}$. Let $\{X_i\}_{i=1}^\infty$ and $\{Y_i\}_{i=1}^\infty$ be mutually independent i.i.d. exponential rv's with means 1 and $1/\alpha_i$ respectively, and let $S_n = \sum_{j=1}^n (X_j - Y_j)$. Then

$$\begin{aligned} P\{M_{\alpha_i}(u) < M_1(u) \text{ for all } u \geq T\} &= P\{S_1 < 0\} \\ &\quad \times P\{M_{\alpha_i}(u) \leq M_1(u) \text{ for all } u \geq 0\} \\ &= P\{S_1 < 0\} P\{\max_n S_n \leq 0\}. \end{aligned}$$

If H denotes the distribution of S_1, from Eq. (5.9) in Feller (1971, page 410) we get

$$P\{\max_n S_n \leq 0\} = \alpha \cdot \kappa(\alpha), \tag{4.17}$$

where $\kappa(\alpha)$ solves the equation $\int_{-\infty}^{\infty} e^{\kappa(\alpha)y}\, dH(y) = 1$. This gives $\kappa(\alpha) = \alpha^{-1} - 1$. Also

$$P(S_1 < 0) = 1/(1 + \alpha). \tag{4.18}$$

Combining these results, we get

$$1 - \pi(\alpha_i) = (1 - \alpha)/(1 + \alpha).$$

The result for $\alpha_i > 1$ can be obtained from the fact the number of intervals for which the graph of $M_{\alpha_i}(u)$ equals that of $M_1(u)$ for all $u \geq 0$ has the same distribution as the number of intervals for which the graph of $M_1(u)$ equals that of $M_{1/\alpha_i}(u)$ for all $u \geq 0$. Therefore

$$1 - \pi(\alpha_i) = [1 - \rho(\alpha_i)]/[1 + \rho(\alpha_i)],$$

where

$$\rho(\alpha_i) = \min(\alpha_i, 1/\alpha_i).$$

To prove Theorem 3, observe that with probability $\to 1$ as $n \to \infty$, $L_i^*(n)$ is the same as L_i^* = the number of times $M_{\alpha_i}(u)$ crosses $M_1(u)$ for all $u \geq 0$. Note from (4.17) that for $\alpha_i < 1$

$$\begin{aligned} P\{M_{\alpha_i}(u) \leq M_1(u) \text{ for all } u \geq 0\} &= P\{\max_n S_n \leq 0\} \\ &= 1 - \alpha_i. \end{aligned}$$

Similarly for $\alpha_i > 1$,

$$P\{M_{\alpha_i}(u) \geq M_1(u) \text{ for all } u \geq 0\} = 1 - 1/\alpha_i.$$

Using this, it can be shown that

$$P\{L_i^* = j + 1\} = [1 - \rho(\alpha_i)]\,\rho^j(\alpha_i). \qquad \square$$

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