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CENTRAL LIMIT THEOREMS FOR MIXING SEQUENCES OF RANDOM VARIABLES UNDER MINIMAL CONDITIONS

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Let $\{X_j, j \geq 1\}$ be a strictly stationary sequence of random variables with mean zero, finite variance, and satisfying a strong mixing condition. Denote by S_n the nth partial sum and suppose that $\operatorname{Var} S_n$ is regularly varying of order 1. We prove that if $S_n(\operatorname{Var} S_n)^{-1/2}$ does not converge to zero in L^1, then $\{X_j, j \geq 1\}$ is in the domain of partial attraction of a Gaussian law. If, however, no subsequence of $\{S_n(\operatorname{Var} S_n)^{-1/2}, n \geq 1\}$ converges to zero in L^1 and if $E|S_n|$ is regularly varying of order $\frac{1}{2}$, then $\{X_j, j \geq 1\}$ is in the domain of attraction to a Gaussian law. In each case the norming constant can be chosen as $E|S_n|$.

1. Introduction. Throughout this paper we assume that $\{X_j, j \geq 1\}$ is a strictly stationary sequence of random variables with nth partial sum S_n and satisfying

$$EX_1 = 0, \qquad EX_1^2 = 1, \tag{1.1}$$

$$\sigma(n)^2 = \sigma_n^2 := ES_n^2 = nL(n), \tag{1.2}$$

where L is a slowly varying function in the sense of Karamata. Let $\mathfrak{F}_a^b$ be the σ-field generated by $X_a, X_{a+1}, \dots, X_b$. The sequence $\{X_j, j \geq 1\}$ is said to satisfy a strong mixing condition if

$$\begin{aligned} \alpha(n) := \sup\big\{&|P(A \cap B) - P(A)P(B)|: \\ &A \in \mathfrak{F}_1^k, B \in \mathfrak{F}_{k+n}^\infty, k \geq 1\big\} \to 0. \end{aligned} \tag{1.3}$$

It is called uniformly (or φ-)mixing if

$$\varphi(n) := \sup\big\{|P(B|A) - P(B)|: B \in \mathfrak{F}_{k+n}^\infty, A \in \mathfrak{F}_1^k, k \geq 1\big\} \to 0. \tag{1.4}$$

In 1962 Ibragimov [4] proved that if $\{X_j, j \geq 1\}$ satisfies (1.1), (1.4), and $\sigma_n^2 \to \infty$, then (1.2) holds. Moreover, if

$$E|X_1|^{2+\delta} < \infty \quad \text{for some } \delta > 0, \tag{1.5}$$

then

$$\lim_{n\to\infty} \mathfrak{L}\left(\sigma_n^{-1} S_n\right) = \mathfrak{N}(0,1). \tag{1.6}$$

Later, without assuming (1.5), Ibragimov [5] proved that (1.1), $\sigma_n^2 \to \infty$ and (1.4) with φ satisfying $\sum \varphi^{1/2}(2^n) < \infty$ together imply (1.6). (See also Bradley [2, pages 586 and 587].) But the conjecture whether (1.1), (1.4), and $\sigma_n^2 \to \infty$ together imply (1.6) is still an unsolved problem. (See [6, page 393].)

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For strongly mixing stationary sequences it can be shown that the weak convergence in (1.6) is equivalent to the uniform integrability of the sequence $\{\sigma_n^{-2}S_n^2, n \geq 1\}$. Indeed from [6, Theorem 18.4.2 with a correction] and some routine calculations this equivalence follows easily.

But even in the uniformly mixing case (1.4) the uniform integrability of $\{\sigma_n^{-2}S_n^2, n \geq 1\}$ has not yet been established. (See, e.g., [4, Lemma 1.9] in case (1.5) holds.) Recently, Peligrad [7] showed that the uniform integrability of $\{\sigma_n^{-2}S_n^2, n \geq 1\}$ is equivalent to the Lindeberg condition

$$\lim_{n\to\infty} n\sigma_n^{-2}EX_1^2 1\{|X_1| \geq \varepsilon\sigma_n\} = 0 \quad \text{for all } \varepsilon > 0,$$

where $1\{\cdot\}$ denotes the indicator of the set $\{\cdot\}$. On the other hand, by a remark of Bradley [1, page 101] one might be able to construct a counterexample to Ibragimov's conjecture if one could construct a uniformly mixing sequence with $n^{-1}\sigma_n^2 \to 0$ and $\varphi(1) < \varepsilon$ $(\varepsilon > 0)$. Since the uniform integrability of $\{\sigma_n^{-2}S_n^2, n \geq 1\}$ appears to be intractable and since asymptotic normality still might hold if this integrability condition is not satisfied, we propose the use of different normalizing constants

$$\rho_n := (\pi/2)^{1/2}E|S_n| \tag{1.7}$$

instead of σ_n.

Our first result gives a condition which implies that a strongly stationary sequence $\{X_j, j \geq 1\}$ belongs to the domain of partial attraction of $\mathfrak{N}(0,1)$.

THEOREM 1. *Suppose that* $\{X_j, j \geq 1\}$ *satisfies* (1.1), (1.2), *and* (1.3). *Then unless*

$$\lim_{n\to\infty} \sigma_n^{-1}E|S_n| = 0 \tag{1.8}$$

there exists an infinite sequence $Q \subset \mathbb{N}$ *of integers such that*

$$\mathfrak{L}(\rho_n^{-1}S_n) \to \mathfrak{N}(0,1) \quad \text{as } n \to \infty,\ n \in Q. \tag{1.9}$$

The property (1.8) really can occur: Herrndorff [3] constructed a sequence $\{X_j, j \geq 1\}$ satisfying (1.1), (1.2), and (1.3) such that $b_n^{-1}S_n \to 0$ in probability as $n \to \infty$ for any sequence $b_n \to \infty$. On the other hand, there are also examples satisfying condition (1.9) such that $\mathfrak{L}(\rho_n^{-1}S_n)$ does not converge to $\mathfrak{N}(0,1)$ along $n \in \mathbb{N}$. Theorem 1 of Bradley [1] shows that there exists a sequence $\{X_j, j \geq 1\}$ satisfying (1.1), (1.2), and (1.3) such that for some subsequence $\{n(l), l \geq 1\}$ one has $\mathfrak{L}(\sigma_{n(l)}^{-1}S_{n(l)}) \to F$, where F is neither normal nor degenerate. In particular $\liminf_l \sigma_{n(l)}^{-1}E|S_{n(l)}| > 0$ and hence (by Theorem 1), $\rho_n S_n$ converges weakly to $\mathfrak{N}(0,1)$ along some subsequence, different from $\{n(l), l \geq 1\}$. Hence, such a sequence $\{X_j, j \geq 1\}$ does not belong to the domain of attraction of a normal law.

In view of Theorem 1 it seems desirable to look for additional conditions on $\{X_j, j > 1\}$ which together with

$$\liminf_{n\to\infty} \sigma_n^{-1}E|S_n| > 0 \tag{1.10}$$

are sufficient for (1.9) to hold with $Q = \mathbb{N}$.

In the following results we give additional conditions on $\{X_j,\ j \geq 1\}$ guaranteeing that it belongs to the domain of attraction of a normal law.

THEOREM 2. *Suppose* $\{X_j,\ j \geq 1\}$ *satisfies* (1.1), (1.2), (1.3), *and* (1.10). *If there exists a sequence* $\{a(n),\ n \geq 1\}$ *with* $a(n) \to \infty$ *or* $a(n) = \infty$ *such that*

$$\limsup_{n\to\infty} \rho_n^{-2} \int_{\{|S_n| \leq a(n)\sigma(n)\}} S_n^2\, dP \leq 1, \tag{1.11}$$

then

$$\lim_{n\to\infty} \mathfrak{L}\left(\rho_n^{-1} S_n\right) = \mathfrak{N}(0,1). \tag{1.12}$$

THEOREM 3. *Suppose that* $\{X_j,\ j \geq 1\}$ *satisfies* (1.1), (1.2), (1.3), *and* (1.10). *Moreover, assume that*

$$E|S_n| = n^{1/2} L_1(n), \tag{1.13}$$

where L_1 *is slowly varying on the integers. Then* (1.12) *holds.*

THEOREM 4. *Suppose that* $\{X_j,\ j \geq 1\}$ *satisfies* (1.1), (1.2), *and* (1.3). *Then the following two conditions are equivalent:*

$$\lim_{n\to\infty} \mathfrak{L}\left(\sigma_n^{-1} S_n\right) = \mathfrak{N}(0,1) \tag{1.14}$$

and

$$\limsup_{n\to\infty} \sigma_n \rho_n^{-1} \leq 1. \tag{1.15}$$

Note that condition (1.10) follows from (1.15) or from (1.11) if $a(n) \equiv \infty$. Also we would like to point out that (1.15) might have applications to statistical mechanics: If mixing rates are not computable, $\sigma_n \rho_n^{-1}$ still could be estimated.

In Section 2 we prove an approximation lemma. Its proof is of rather routine nature. Theorem 1 is proven in Section 3, the other theorems in Section 4.

2. An approximation lemma. Throughout this section we assume that (1.1) and (1.3) hold. We introduce some more notation. Let $p \in \mathbb{N}$ and let $g \geq 2$ satisfy

$$g \leq \min\left(\alpha^{-1/4}\left(\sigma_p^{1/4}\right), \sigma_p^{1/4}\right), \tag{2.1}$$

where we set $\alpha(x) = \alpha([x])$, $x \in \mathbb{R}$. Put

$$v^2 := \sigma_p^{-2} \int_{\{g^{1/2} < |S_p|/\sigma_p \leq g\}} S_p^2\, dP, \tag{2.2}$$

$$u^2 := \int_{\{|S_p| \leq g\sigma_p\}} S_p^2\, dP, \tag{2.3}$$

and

$$r := \left[g^2 c\right], \tag{2.4}$$

where c satisfies

$$\max(2g^{-1/2}, v^2) \le c < 1. \tag{2.5}$$

Finally, we set

$$n := r\left(p + \left[\sigma_p^{1/4}\right]\right) \tag{2.6}$$

and

$$\tau^2 := ru^2. \tag{2.7}$$

With this notation we have the following lemma.

LEMMA 1. *We have*

$$\begin{aligned} &|E\exp(it\tau^{-1}S_n) - \exp(-t^2/2)| \\ &\quad \le 2c + |t|u^{-1}\sigma_p c^{1/2} + |t|^3 u^{-1}\sigma_p g^{-1/4} \\ &\qquad + |t|^3 u^{-3}\sigma_p^3 v + 4\alpha^{1/2}\left(\sigma_p^{1/4}\right) + t^4 g^{-1} + t^2 u^{-2}\sigma_p^2 g^{-1}. \end{aligned}$$

PROOF. Note that $u \le \sigma_p$. Hence if $|t| > r^{1/2}$, then the term

$$|t|^3 u^{-1}\sigma_p g^{-1/4} \ge r^{3/2} g^{-1/4} \ge \left((g^2 c - 1)g^{-1/6}\right)^{3/2} \ge g^2$$

by (2.4) and (2.5). Hence we can assume from now on that $|t| \le r^{1/2}$.

We use the standard blocking argument. We decompose S_n into r blocks of length p each, separated by blocks of length $q := [\sigma_p^{1/4}]$ each, i.e.,

$$S_n = \sum_{j=1}^{r} Y_j + \sum_{j=1}^{r} Z_j = U_n + V_n, \tag{2.8}$$

where

$$Y_j := \sum_{i=(j-1)(p+q)+1}^{jp+(j-1)q} X_i, \qquad Z_j := \sum_{i=jp+(j-1)q+1}^{j(p+q)} X_i.$$

Since V_n is a sum of at most $r\sigma_p^{1/4}$ terms we have by Minkowski's inequality, by (2.1), (2.4), and (2.5)

$$EV_n^2 \le r^2\sigma_p^{1/2} \le g^4\sigma_p^{1/2} \le \sigma_p^{3/2}. \tag{2.9}$$

Hence by (2.4), (2.5), and (2.7)

$$\begin{aligned} &|E\exp(it\tau^{-1}S_n) - E\exp(it\tau^{-1}U_n)| \\ &\quad \le |E(\exp it\tau^{-1}V_n) - 1| \le t^2\tau^{-2}EV_n^2 \\ &\quad \le t^2\sigma_p^{3/2}u^{-2}r^{-1} \le t^2u^{-2}\sigma_p^2 g^{-1}. \end{aligned} \tag{2.10}$$

The blocks Y_j of U_n are separated by the blocks Z_j of V_n, having length $[\sigma_p^{1/4}]$ each. Thus by a well-known lemma (see, e.g., [6, Lemma 17.2.1]), stationarity, (2.1), (2.4), and (2.5)

$$\begin{aligned} |E\exp(it\tau^{-1}U_n) - \left(E\exp(it\tau^{-1}S_p)\right)^r| &\le 4r\alpha\left(\sigma_p^{1/4}\right) \\ &\le 4\alpha^{1/2}\left(\sigma_p^{1/4}\right). \end{aligned} \tag{2.11}$$

Next, we estimate $|E\exp(it\tau^{-1}S_p) - (1 - t^2/(2r))|$. By Chebyshev's inequality and (2.4)

$$\left|\int_{\{|S_p|>g\sigma_p\}} \exp\left(it\tau^{-1}S_p\right) dP\right| \le g^{-2} \le cr^{-1}. \tag{2.12}$$

For the next estimate we use Taylor's theorem. We obtain

$$\begin{aligned}
&\left|\int_{\{|S_p|\le g\sigma_p\}} \exp\left(it\tau^{-1}S_p\right) dP - \left(1 - t^2/(2r)\right)\right| \\
&\quad\le \left|P\left(|S_p| \le g\sigma_p\right) + it\tau^{-1}\int_{\{|S_p|\le g\sigma_p\}} S_p\, dP\right. \\
&\qquad \left. - \tfrac{1}{2}t^2\tau^{-2}\int_{\{|S_p|\le g\sigma_p\}} S_p^2\, dP - \left(1 - t^2/(2r)\right)\right| \\
&\qquad + \tfrac{1}{6}|t|^3\tau^{-3}\int_{\{|S_p|\le g\sigma_p\}} |S_p|^3\, dP.
\end{aligned} \tag{2.13}$$

As in (2.12) we have

$$|1 - P\left(|S_p| \le g\sigma_p\right)| \le g^{-2} \le cr^{-1}. \tag{2.14}$$

Since $ES_p = 0$ we have, by (2.7),

$$\begin{aligned}
\left|\tau^{-1}\int_{\{|S_p \le g\sigma_p\}} S_p\, dP\right| &= \tau^{-1}\left|\int_{\{|S_p|>g\sigma_p\}} S_p\, dP\right| \\
&\le g^{-1}\sigma_p\tau^{-1} \le u^{-1}\sigma_p r^{-1}c^{1/2}.
\end{aligned} \tag{2.15}$$

The cubic term is estimated as follows. By (2.7), (2.4), (2.2), and (2.5)

$$\begin{aligned}
\tau^{-3}\int_{\{g^{1/2}\sigma_p<|S_p|\le g\sigma_p\}} |S_p|^3\, dP &\le \tau^{-3}g\sigma_p\sigma_p^2v^2 \\
&\le u^{-3}r^{-3/2}\sigma_p^3 2r^{1/2}c^{-1/2}v^2 \\
&\le 2u^{-3}\sigma_p^3 r^{-1}v
\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}
\tau^{-3}\int_{\{|S_p|\le g^{1/2}\sigma_p\}} |S_p|^3\, dP &\le \tau^{-3}g^{1/2}\sigma_p u^2 \\
&\le u^{-1}\sigma_p r^{-3/2}g^{1/2} \\
&\le u^{-1}\sigma_p r^{-1}g^{-1/4}.
\end{aligned} \tag{2.17}$$

By (2.3),

$$\tfrac{1}{2}t^2\tau^{-2}\int_{\{|S_p|\le g\sigma_p\}} S_p^2\, dP = t^2/(2r)$$

and hence substituting (2.14)–(2.17) into (2.13) we obtain, by (2.12),

$$|E\exp\big(it\tau^{-1}S_p\big) - \big(1 - t^2/(2r)\big)| \le r^{-1}\eta,$$

where

$$\eta := 2c + |t|u^{-1}\sigma_p c^{1/2} + \tfrac{1}{6}|t|^3 u^{-1}\sigma_p g^{-1/4} + \tfrac{1}{3}|t|^3 u^{-3}\sigma_p^3 v.$$

Hence, and since $|a^r - b^r| \le r|a - b|$ for $|a| \le 1$, $|b| \le 1$, we have for $|t| < r^{1/2}$

$$|E\exp\big(it\tau^{-1}U_n\big) - \big(1 - t^2/(2r)\big)^r| \le \eta + 4\alpha^{1/2}\big(\sigma_p^{1/4}\big) \tag{2.18}$$

by (2.11). Since $|e^x - (1 + x)| \le x^2$ for $|x| \le \frac{1}{2}$ we obtain by the above remark $|\exp(-\frac{1}{2}t^2) - (1 - t^2/(2r))^r| \le \frac{1}{4}t^4 r^{-1}$. The result follows now from (2.18), (2.10), (2.4), and (2.5). □

3. Proof of Theorem 1. If (1.8) does not hold then there exists an infinite subsequence $R \subset \mathbb{N}$ such that

$$\gamma := \inf\big\{\sigma_p^{-1}E|S_p|: p \in R\big\} > 0. \tag{3.1}$$

We apply Lemma 1 for each $p \in R$ to show the existence of an infinite sequence $Q \subset \mathbb{N}$ and of real numbers τ_n, $n \in Q$ satisfying

$$\mathfrak{L}\big(\tau_n^{-1}S_n\big) \to \mathfrak{N}(0,1) \qquad n \to \infty,\ n \in Q. \tag{3.2}$$

For this purpose we show that there exist a sequence $\{g(p), p \in R\}$ and a monotone sequence $\{c(p), p \in R\}$ with the following properties:

$$\lim_{p\in R} g(p) = \infty, \qquad \lim_{p\in R} c(p) = 0, \tag{3.3}$$

$$g(p) \le \min\big(\alpha^{-1/4}\big(\sigma_p^{1/4}\big), \sigma_p^{1/4}\big), \tag{3.4}$$

$$v^2(p) := \sigma_p^{-2}\int_{\{g(p)^{1/2} < |S_p|/\sigma_p \le g(p)\}} S_p^2\, dP \to 0, \qquad p \to \infty,\ p \in R, \tag{3.5}$$

and

$$1 > c(p) \ge \max\big(2g(p)^{-1/2}, v^2(p)\big). \tag{3.6}$$

We first choose a sequence $\{z(p),\ p \in R\}$ with

$$\lim_{p\in R} z(p) = \infty, \qquad z(p) \le \min\big(\alpha^{-1/4}\big(\sigma_p^{1/4}\big), \sigma_p^{1/4}\big). \tag{3.7}$$

Next, we choose a sequence $\{i(p),\ p \in R\}$ such that

$$\lim_{p\in R} i(p) = \infty, \qquad \lim_{p\in R} 2^{-i(p)}\log z(p) = \infty. \tag{3.8}$$

Now, fix $p \in R$. Since the intervals $I_i(p) :=]z(p)^{2^{-i-1}}, z(p)^{2^{-i}}]$, $0 \le i < i(p)$, are disjoint there exists an integer $k = k(p)$ with $0 \le k < i(p)$ such that

$$\sigma_p^{-2}\int_{\{|S_p|/\sigma_p \in I_k(p)\}} S_p^2\, dP \le i(p)^{-1}. \tag{3.9}$$

$$g(p) := z(p)^{2^{-k(p)}}. \tag{3.10}$$

Then $g(p) \to \infty$ by (3.8) and because of (3.7)–(3.9) conditions (3.4) and (3.5) are satisfied. Since $\max(2g(p)^{-1/2}, v(p)^2) \to 0$, $p \in R$, we now can choose $\{c(p), p \in R\}$ with $c(p) \searrow 0$ and satisfying (3.6).

With these choices of $\{g(p), p \in R\}$ and $\{c(p), p \in R\}$ we define $u(p)$ by (2.3), $r(p)$ by (2.4), and $n(p)$ by (2.6). We put $Q := \{n(p), p \in R\}$ and define τ_n^2, $n \in Q$ by (2.7). Since by Hölder's inequality

$$(3.11)\qquad \begin{aligned}\sigma_p^{-1}E|S_p| &= \sigma_p^{-1}\int_{\{|S_p|/\sigma_p \le g(p)\}} |S_p|\,dP + \sigma_p^{-1}\int_{\{|S_p|/\sigma_p > g(p)\}} |S_p|\,dP \\ &\le \sigma_p^{-1}u(p) + g(p)^{-1}\end{aligned}$$

we have, by (3.1), for sufficiently large $p \in R$

$$(3.12)\qquad \sigma_p^{-1}u(p) \ge \tfrac{1}{2}\gamma > 0.$$

Lemma 1 now implies (3.2). It remains to show that

$$(3.13)\qquad \lim_{n\in Q} \tau_n/\rho_n = 1.$$

To see this we choose a sequence $\{\varkappa(m), m \in \mathbb{N}\}$ with

$$(3.14)\qquad \begin{aligned}&\lim_{m\to\infty} \varkappa(m) = \infty, \qquad \text{and}\\ &\lim_{m\to\infty} (\sup\{|(L(mt)/L(m)) - 1| : 1 \le t \le \varkappa(m)\} = 0.\end{aligned}$$

This is possible. Indeed, by the Karamata theorem there exists an increasing sequence $\{m_k, k \ge 2\}$ such that

$$\left|\sup_{1\le t\le k} L(tm)/L(m) - 1\right| \le 1/k, \qquad m \ge m_k.$$

Then $\varkappa(\cdot)$ defined by $\varkappa(m) = k$ for $m_k < m \le m_{k+1}$, has the desired properties. Of course, we can assume that $\{z(p), p \in R\}$ was chosen so that in addition to (3.7) we have $z(p) \le \frac{1}{2}\varkappa(p)^{1/2}$. Then we have for all sufficiently large $p \in R$, by (2.6) and since by (1.1) $\sigma_p^2 \le p^2$,

$$\begin{aligned}\frac{\sigma^2(n(p))}{r(p)\sigma_p^2} &= \frac{r(p)\big(p + [\sigma_p^{1/4}]\big)L\big(r(p)\big(p + [\sigma_p^{1/4}]\big)\big)}{r(p)pL(p)} \\ &= \big(1 + O(p^{-1/2})\big)\frac{L\big(r(p)\big(p + [\sigma_p^{1/4}]\big)\big)}{L(p)} = 1 + o(1)\end{aligned}$$

by (3.14). Thus, by (2.7) and (3.11), we have for sufficiently large $p \in R$,

$$(3.15)\qquad E\big(\tau_{n(p)}^{-1}S_{n(p)}\big)^2 = \tau_{n(p)}^{-2}\sigma_{n(p)}^2 \le 2u(p)^{-2}\sigma_p^2 \le 8\gamma^{-2}.$$

Hence $\{\tau_n^{-1}S_n, n \in Q\}$ is uniformly integrable and thus, by (3.2),

$$\lim_{n\in Q} \tau_n^{-1}E|S_n| = (2\pi)^{-1/2}\int_{-\infty}^{\infty} |x|\exp\big(-\tfrac{1}{2}x^2\big)\,dx = (2/\pi)^{1/2}.$$

In view of (1.7) this proves (3.13) and thus Theorem 1. □

4. Proofs of Theorems 2, 3, and 4.

PROOF OF THEOREM 2. Since no subsequence of $\{\sigma_n^{-1}S_n, n \geq 1\}$ converges to zero in L^1 we can repeat the construction of Section 3 with $R = \mathbb{N}$. Suppose we knew that $r(p+1) - r(p) = O(p^{-1})$. Then because of (2.6) we could choose $Q = \mathbb{N}$ and the conclusion of Theorem 2 would follow at once. But $k(p)$ as chosen in (3.9) could oscillate wildly and so could $r(p)$.

From the construction of Section 3 we obtain a subsequence

$$Q = \left\{n(p) \colon n(p) = r(p)\left(p + \left[\sigma_p^{1/4}\right]\right), p \in \mathbb{N}\right\}$$

with

$$\lim_{n \in Q} \mathfrak{L}\left(\rho_n^{-1}S_n\right) = \mathfrak{N}(0,1). \tag{4.1}$$

We assume that this construction was carried out with a sequence $\{z(p), p \in \mathbb{N}\}$ with $z(p) \nearrow \infty$ and satisfying

$$z(p) \leq \min\left(\alpha\left(\sigma_p^{1/4}\right)^{-1/16}, \sigma_p^{1/16}, p^{1/4}, a(p)^{1/4}, \tfrac{1}{2}\varkappa(p)^{1/8}\right) \tag{4.2}$$

and

$$z(p) \leq z(q) \leq z(p)^{3/2}, \qquad p \leq q \leq p^2, \tag{4.3}$$

where $\{\varkappa(m), m \in \mathbb{N}\}$ was the sequence chosen in (3.14). Such a sequence can be constructed as follows: First choose an increasing sequence $y(p)$ satisfying (4.2). By induction on k define $z(p) = \min(y(p_k), z(p_{k-1})^{3/2})$ for $p = p_k = 2^{2^k}$, $p_k + 1, \ldots, p_{k+1} - 1$. Let $\{h(n), n \in Q\}$ and $\{j(n), n \in Q\}$ be two arbitrary sequences of real numbers tending to infinity and with $h(n) < j(n) < a(n)$, $n \in Q$. Since $\mathfrak{L}(\rho_n^{-1}S_n) \to \mathfrak{L}(N) := \mathfrak{N}(0,1)$, $n \to \infty$, $n \in Q$, and since $\rho_n \leq \sigma_n(\pi/2)^{1/2} \leq 2\sigma_n$ we have for each $\alpha > 0$

$$\begin{aligned}\int_{\{|N| \leq \alpha\}} N^2\,dP &= \lim_{n \in Q} \int_{\{|S_n|/\rho_n \leq \alpha\}} \rho_n^{-2}S_n^2\,dP \\ &\leq \liminf_{n \in Q} \rho_n^{-2} \int_{\{|S_n|/\sigma_n \leq h(n)\}} S_n^2\,dP.\end{aligned}$$

Hence, letting $\alpha \to \infty$ we obtain, by (1.11),

$$\begin{aligned}1 &\leq \liminf_{n \in Q} \rho_n^{-2} \int_{\{|S_n|/\sigma_n \leq h(n)\}} S_n^2\,dP \\ &\leq \limsup_{n \in Q} \rho_n^{-2} \int_{\{|S_n|/\sigma_n < a(n)\}} S_n^2\,dP \leq 1.\end{aligned} \tag{4.4}$$

Since $E|S_n| \leq \sigma_n$, by (1.7) and since no subsequence of $\{\sigma_n^{-1}S_n, n \geq 1\}$ converges to 0 in L^1 we obtain

$$(2/\pi)^{1/2} \leq \limsup_{n \to \infty} \sigma_n \rho_n^{-1} < \infty.$$

Hence, (4.4) implies

$$\lim_{n\in Q} \sigma_n^{-2} \int_{\{h(n)<|S_n|/\sigma_n \le j(n)\}} S_n^2 \, dP = 0. \tag{4.5}$$

We shall apply Lemma 1 once more. To prepare for it we set

$$h(n) := h(n(p)) := z(n(p)) \quad \text{if } n = n(p),\ p \in \mathbb{N} \tag{4.6}$$

and

$$j(n) := j(n(p)) := \min\left(\alpha^{-1/4}\left(\sigma_n^{1/4}\right), \sigma_n^{1/4}, a(n), \tfrac{1}{2}\varkappa(n)^{1/2}\right) \quad \text{if } n = n(p),\ p \in \mathbb{N}. \tag{4.7}$$

For sufficiently large $p \in \mathbb{N}$ we have by (2.4), (2.6), (3.10), (4.2), and since $c(p) \to 0$,

$$n(p) \le g^2(p)c(p)(p + p^{1/4}) \le z^2(p)p \le p^{1/2}p < p^2. \tag{4.8}$$

By (4.2), (4.7), and (4.6)

$$j(n) \ge z^4(n) = h^4(n) > h(n), \qquad n \in Q. \tag{4.9}$$

Since, by (4.5),

$$w^2(n) := \sigma_n^{-2} \int_{\{h^{1/2}(n)<|S_n|/\sigma_n \le j(n)\}} S_n^2 \, dP \to 0, \qquad n \in Q, \tag{4.10}$$

we can choose a nonincreasing $\{d(n), n \in Q\}$ such that

$$\lim_{n\in Q} d(n) = 0 \quad \text{and} \quad d(n) \ge \min\left(2h(n)^{-1/2}, w^2(n)\right), \qquad n \in Q. \tag{4.11}$$

Let $Q := \{n_k, k \ge 1\}$ be arranged in increasing order and let J_k be the interval

$$J_k := \left[n_k h^2(n_k)d(n_k), n_k j^2(n_k)d(n_k)\right].$$

We show that there exists a k_0 such that

$$J_k \cap J_{k+1} \ne \varnothing, \qquad k \ge k_0. \tag{4.12}$$

Since $n_k \in R = \mathbb{N}$ we have $n(n_k) = r(n_k)(n_k + [\sigma(n_k)^{1/4}]) \in Q$. As n_{k+1} is the smallest member of Q bigger than n_k we must have for sufficiently large k

$$n_{k+1} \le n(n_k) \le n_k z^2(n_k) < n_k^2$$

by (4.8). Hence, by (4.6) and (4.3), the left endpoint of J_{k+1} does not exceed

$$n_{k+1}h^2(n_{k+1})d(n_{k+1}) \le n_k z^2(n_k) z^2(n_{k+1}) \le n_k z^2(n_k) z^2(n_k^2) \le n_k z^5(n_k)$$

for sufficiently large k. By (4.11) and (4.9) the right endpoint of J_k is bigger than

$$n_k j^2(n_k)d(n_k) \ge n_k j^2(n_k)h^{-1/2}(n_k) \ge n_k z(n_k)^{15/2}$$

for sufficiently large k. Since $z(n_k) \to \infty$ we obtain (4.12). Let $m \ge \min\{l: l \in J_{k_0}\}$. Then there is a $k \ge k_0$ such that $m \in J_k$. Thus we have for some

$g \in [h(n_k), j(n_k)]$ and some $|\theta| \le 2$

$$m = g^2 d(n_k) n_k = [g^2 d(n_k)](n_k + [\sigma^{1/4}(n_k)]) + \theta n_k \tag{4.13}$$
$$= M_k + \theta n_k, \quad \text{say.}$$

Now, by (2.4), M_k is of the form (2.6) and hence we can apply Lemma 1. We set $p := n_k$ and $c = d(n_k)$. Since $\gamma = \inf \sigma_n^{-1} E|S_n| > 0$ we have, by (3.1) and (3.12), $u(n_k)/\sigma(n_k) \ge \frac{1}{2}\gamma > 0$. Now $g \ge h(n_k) \to \infty$ and $\alpha(\sigma^{1/4}(n_k)) \to 0$. Finally, by (4.10) and since $h(n_k) \le g \le j(n_k)$, we have

$$v^2(n_k) := \sigma^{-2}(n_k) \int_{\{g^{1/2} < |S_{n_k}|/\sigma(n_k) \le g\}} S_{n_k}^2 \, dP \le w^2(n_k) \to 0.$$

Hence, by Lemma 1

$$\mathfrak{L}(\tau^{-1}(M_k) S_{M_k}) \to \mathfrak{N}(0,1). \tag{4.14}$$

Since $|\theta| \le 2$ we have by (3.14) for all k sufficiently large

$$\frac{ES_{\theta n_k}^2}{\sigma^2(n_k)} \le \frac{|\theta| n_k L(|\theta| n_k)}{n_k L(n_k)} \le 4.$$

Consequently

$$E(S_m - S_{M_k})^2 = ES_{\theta n_k}^2 \le 4\sigma^2(n_k). \tag{4.15}$$

Denoting $r^*(n_k) := [g^2 d(n_k)]$ we obtain by (3.14) for all sufficiently large k

$$\frac{\sigma^2(M_k)}{r^*(n_k)\sigma^2(n_k)} \ge \frac{r^*(n_k) n_k L(r^*(n_k)(n_k + [\sigma^{1/4}(n_k)]))}{r^*(n_k) n_k L(n_k)} \ge \frac{1}{2},$$

since $r^*(n_k) \le g^2 d(n_k) \le j^2(n_k) d(n_k) < \frac{1}{2}\varkappa(n_k)$ by (4.7). Hence, by (4.15) and as $r^*(n_k) \to \infty$, we have

$$\sigma^{-2}(M_k) E(S_m - S_{M_k})^2 \to 0. \tag{4.16}$$

In the same way as (3.15) one can prove

$$\sigma^2(M_k)/\tau^2(M_k) \le 8\gamma^{-2}. \tag{4.17}$$

We set

$$\tau_m := \tau(M_k) \quad \text{if } m \text{ and } M_k \text{ are as in (4.13).}$$

Then, by (4.14), (4.16), and (4.17),

$$\lim_{m \to \infty} \mathfrak{L}(\tau_m^{-1} S_m) = \mathfrak{N}(0,1). \tag{4.18}$$

Since by (4.16) and (4.17) the sequence $\{\tau_m^{-1} S_m, m \ge 1\}$ is uniformly integrable, we obtain Theorem 2 from (4.18) using the argument at the end of Section 3. □

Proof of Theorem 4. For the proof of Theorem 4 we note that (1.14) implies

$$\liminf \sigma_n^{-1} E|S_n| \ge (2\pi)^{-1/2} \int_{-\alpha}^{\alpha} |x| \exp\left(-\frac{x^2}{2}\right) dx \quad \text{for any } \alpha > 0;$$

hence, $\limsup_{n\to\infty}\sigma_n/\rho_n \le 1$. Conversely, if (1.15) holds then no subsequence of $\{\sigma_n^{-1}S_n, n \ge 1\} \to 0$ in L^1. Also (1.15) implies (1.11) with $a(n) \equiv \infty$. Hence, by Theorem 2,

$$\lim_{n\to\infty} \mathfrak{L}(\rho_n^{-1}S_n) = \mathfrak{N}(0,1). \tag{4.19}$$

Now the norming constants ρ_n can be replaced by σ_n, since by (4.19) and (1.15) we have for every $\alpha > 0$ and a random variable N with $\mathfrak{L}(N) = \mathfrak{N}(0,1)$,

$$\int_{\{|N|\le\alpha\}} N^2\,dP = \lim_{n\to\infty} \rho_n^{-2}\int_{\{|S|/\sigma_n\le\alpha\}} S_n^2\,dP \le \limsup_{n\to\infty} \sigma_n^2/\rho_n^2 \le 1.$$

We let $\alpha \to \infty$ and obtain $\sigma_n/\rho_n \to 1$. (1.14) follows now from (4.19). □

Proof of Theorem 3. We first note that without loss of generality we can assume that the sequence $\{\varkappa(m), m \in \mathbb{N}\}$ satisfies the following condition in addition to (3.14):

$$\lim_{m\to\infty} \max\{L_1(mt)/L_1(m)\colon t = 1,2,\dots,[\varkappa(m)]\} = 1. \tag{4.20}$$

In view of (1.10) we can repeat the construction of Section 3 with $R = \mathbb{N}$. We obtain sequences $Q = \{n(p)\colon\ n(p) = r(p)(p + [\sigma_p^{1/4}]), p \ge 1\}$ and $\{g(p), p \ge 1\}$ satisfying the conditions spelled out above. Thus, by (2.7),

$$\tau^2(n(p)) = r(p)u^2(p) = r(p)\int_{\{|S_p|\le g(p)\sigma_p\}} S_p^2\,dP. \tag{4.21}$$

By (1.13), (1.7), (3.13), and since $r(p) \le \varkappa(p)$ by (4.2) we have

$$\lim_{p\to\infty} \frac{\tau^2(n(p))}{r(p)\rho_p^2} = 1.$$

Hence, we obtain from (4.21)

$$\limsup_{p\to\infty} \rho_p^{-2}\int_{\{|S_p|\le g(p)\sigma_p\}} S_p^2\,dP \le 1.$$

Thus (1.11) is satisfied with $a(p) = g(p)$ and Theorem 3 follows from Theorem 2. □

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