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MEASURABILITY PROBLEMS FOR EMPIRICAL PROCESSES

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To a class $\mathscr{F}$ of bounded functions on a probability space we associate two classes $\mathscr{F}_r$ and $\mathscr{F}_s$. The class $\mathscr{F}$ is a Donsker class if and only if $\mathscr{F}_r$ and $\mathscr{F}_s$ are Donsker classes. The class $\mathscr{F}_r$ corresponds to a separable version of the empirical process. It is obtained by applying a special type of lifting to $\mathscr{F}$. The class $\mathscr{F}_s$ consists of positive functions that are zero almost surely. It concentrates the pathology of $\mathscr{F}$ with respect to measurability. We use this method to prove without any measurability assumption a general contraction principle for processes that satisfy the central limit theorem.

1. Introduction. Let T be an index set provided with a pseudometric d. Let X be a uniformly bounded process on T (complete definitions are given in the next section). In order to obtain a reasonable behavior of the process, the probabilist will consider a separable modification of X. It has been observed for a long time ([5], page 107) that liftings can be used to define such a separable modification, although this observation does not seem to have been used much. In some respects, liftings are a very orderly way to define a separable modification, since, for example, they preserve lattice operations.

Let us say that the process X satisfies the CLT if the natural map $\Omega \to Z = l^\infty(T)$ satisfies the CLT. The definition of the CLT involves the n-dimensional process $X^{(n)}$ on Ω^n given by

$$X_t^{(n)}(\omega_1, \dots, \omega_n) = \big(X_t(\omega_1), \dots, X_t(\omega_n)\big).$$

We shall describe a special class of liftings, called consistent liftings, which have the further property that for each n, $(\rho X)^{(n)}$ is a separable modification of $X^{(n)}$. We then show that X satisfies the CLT, if and only if, the two processes ρX and $\overline{X} = |X - \rho X|$ satisfy the CLT. The study of ρX is easier than the study of X, since ρX has much better measurability properties than X. For example, it satisfies the measurability hypothesis of [4]. Intuitively ρX is the "useful" part of X. The study of $\overline{X}$ is also easier, since $\overline{X} \geq 0$, and for each t, $\overline{X}_t = 0$ a.s., so there is no problem of finding the limit measure. Intuitively, $\overline{X}$ is the "singular" part of X. As an application of this technique, we prove the following comparison principle. If Y is another process on T such that

$$\forall s, t \in T, \forall \omega \in \Omega, \qquad |Y_s(\omega) - Y_t(\omega)| \leq |X_s(\omega) - X_t(\omega)|,$$

then Y satisfies the CLT.

Let us also mention that the regularization using liftings can also be used if the hypothesis that X is uniformly bounded is weakened to the hypothesis that for each ω, $\sup_t |X_t(\omega)| < \infty$. The standard technique is described in [7], where many other examples of regularization via liftings are given.

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2. Separable versions via liftings. We denote by (Ω, Σ, P) a complete probability space, and by T an index set. A n-dimensional process is a map $X: T \times \Omega \to \mathbb{R}^n$ such that for $t \in T$, the map $\omega \to X(t, \omega) = X_t(\omega)$ is measurable. A process X is called uniformly bounded if the collection of maps X_t is uniformly bounded. A process Y is called a *modification* of X if for each t, $Y_t = X_t$ a.s. (where the negligible set may depend on t). Suppose now T is provided with a pseudometric d. We say that a process X is *separable* if there exists a countable set D of T, that is dense in (T, d), and a null set N in Ω such that

$$\forall \omega \notin N, \forall t \in T, \forall \varepsilon > 0,$$
$$X_t(\omega) \text{ is a cluster point of } \{X_u(\omega),\ u \in D,\ d(t, u) \leq \varepsilon\}.$$

A lifting of $L^\infty = L^\infty(\Omega)$ is a map ρ: $L^\infty \to \mathscr{L}^\infty = \mathscr{L}^\infty(\Omega)$ that is linear, multiplicative, positive, with $\rho(1) = 1$, and $\rho(f) \in f$ for each $f \in L^\infty$ (note that $f \in L^\infty$ is a class in $\mathscr{L}^\infty$). For $f \in \mathscr{L}^\infty$, we write $\rho(f)$ instead of $\rho(\text{class } f)$.

If $X_t(\omega) = (X_t^1(\omega), \ldots, X_t^n(\omega))$, we set

$$\rho X_t(\omega) = \big(\rho X_t^1(\omega), \ldots, \rho X_t^n(\omega)\big), \tag{1}$$

where for simplicity we write

$$\rho X_t^i(\omega) = \rho\big(X_t^i(\cdot)\big)(\omega).$$

The following result is proved in [5]. For completeness we give the short proof in our setting.

PROPOSITION 1. *If (T, d) has a countable basis of open sets, for each multidimensional process X, ρX is a separable modification of X.*

PROOF. Let U be an open set of (T, d) and B be a closed ball in R^n. For $t \in U$, let $A_t = \{\omega; \rho X_t(\omega) \in B\}$. Let D be a countable subset of U such that $\bigcup_{u \in D} A_u$ has the largest probability among all choices of D. Then, for $t \in U$, we have $P(A_t \setminus \bigcup_{u \in D} A_u) = 0$ so we have $\rho(A_t) \subset \rho(\bigcup_{u \in D} A_u)$. Let

$$N = \rho\Big(\bigcup_{u \in D} A_u\Big) \Big\backslash \bigcup_{u \in D} A_u.$$

For $t \in U$, we have $A_t \subset \rho(A_t)$ since B is closed (see [5], page 52, Remark 3). So, for $\omega \notin N$, we have $\omega \in A_t \Rightarrow \omega \in \bigcup_{u \in D} A_u$. In other words

$$\rho X_t(\omega) \in B \Rightarrow \exists u \in D, \qquad \rho X_u(\omega) \in B.$$

The result is now clear, since (T, d) and R^n have countable basis of open sets. □

As we will show in the next section, it is useful to consider the process $X^{(k)}$ valued in $(R^n)^k$, with basic probability space Ω^k where $X^{(k)}$ is given by

$$X_t^{(k)}(\omega_1, \ldots, \omega_k) = \big(X_t(\omega_1), \ldots, X_t(\omega_k)\big).$$

When ρ is a lifting, there is no reason why the process $(\rho X)^{(k)}$ given by

$$(\rho X)_t^{(k)}(\omega_1, \ldots, \omega_k) = \big(\rho X_t(\omega_1), \ldots, \rho X_t(\omega_k)\big) \tag{2}$$

should be separable (actually the example of [8] shows that this is not the case in general). Suppose, however, that the lifting ρ has the property that there exists a lifting ρ^k on $L^\infty(\Omega^k, \Sigma^k, P^k)$ with the property that for each function g of $L^\infty(\Omega)$, and each $1 \le i \le k$, we have

$$\rho^k(g^i)(\omega_1, \dots, \omega_k) = \rho(g)(\omega_i), \tag{3}$$

where $g^i(\omega_1, \dots, \omega_k) = g(\omega_i)$. Then, if we set $Y_t^i(\omega_1, \dots, \omega_k) = X_t(\omega_i)$, we have

$$\rho^k(Y_t^i(\cdot, \dots, \cdot))(\omega_1, \dots, \omega_k) = \rho X_t(\omega_i).$$

In other words, (2) shows that $(\rho X)^{(k)} = \rho^k X^{(k)}$. It then follows from Proposition 1 that $(\rho X)^{(k)}$ is a separable modification of $X^{(k)}$.

It is now natural to state:

DEFINITION 2. A lifting ρ of $L^\infty(\Omega)$ is called *consistent* if for each k there exists a lifting ρ^k of $L^\infty(\Omega^k, \Sigma^k, P^k)$ that satisfies (3).

We have proved:

THEOREM 3. *If (T, d) is separable, and if ρ is a consistent lifting, for each k, $(\rho X_1^{(k)})$ is a separable modification of $X^{(k)}$.*

It has been shown in [6] that every complete probability space admits a consistent lifting.

3. Processes that satisfy the CLT. Consider the map $\phi\colon \Omega \to Z = l^\infty(T)$ given by $\phi(\omega) = (X_t(\omega) - EX_t)_{t \in T}$. We say that the process X satisfies the central limit theorem (CLT) if ϕ satisfies the CLT as in [9]. This means the following: On the space $(\Omega^\infty, \Sigma^\infty, P^\infty)$ product of countably many copies of (Ω, Σ, P), define $S_n(\omega) = \sum_{i \le n} \phi(\omega_i)$. Then there is a (Radon) measure μ on $(Z, \|\cdot\|)$ such that for each bounded norm continuous function g on Z, we have

$$\lim_n \int^* g(S_n/\sqrt{n})\, dP^\infty = \int g\, d\mu. \tag{4}$$

If $\mathscr{F}$ is a class of (uniformly bounded) measurable functions on Ω, we say it is a Donsker class if the process $X_h(\omega) = h(\omega)$ indexed by $\mathscr{F}$ satisfies the CLT. Conversely, the process X satisfies the CLT if and only if the class of functions

$$\{X_t;\ t \in T\}$$

is a Donsker class. That this is equivalent to the usual definition of Donsker classes is proved in [3], Theorem 5.2. We now obtain the following decomposition.

THEOREM 4. *Let ρ be a consistent lifting. Then the process X satisfies the CLT if and only if the process ρX and the process $\overline{X}$ given by $\overline{X} = |\rho X - X|$ both satisfy the CLT.*

It follows from Theorem 3 that ρX has excellent measurability properties, and can be studied by the methods of [4]. An important point in the result is the absolute value in the definition of $\overline{X}$; this will make the use of comparison much easier.

The first step towards Theorem 4 is the following:

PROPOSITION 5. *Let Y be a uniformly bounded process on T. Assume that for each $t \in T$, $Y_t = 0$ a.s. Then Y satisfies the CLT if and only if $|Y|$ satisfies the CLT.*

PROOF. Assume that Y satisfies the CLT. Taking for g in (4) a function depending only on one coordinate, we see that μ can be only the Dirac measure at the origin. Taking for g in (4) the function $g(x) = \inf(1, \|x\|/\varepsilon)$, one sees that for each $\varepsilon > 0$, there is m such that for $n \geq m$ we have

$$(P^n)^*\left(\left\{\left\|\frac{S_n}{\sqrt{n}}\right\| \geq \varepsilon\right\}\right) \leq \varepsilon.$$

Conversely, this condition is easily seen to imply that Y satisfies the CLT.

Let $U \subset \Omega^n$ be a set of measure $\geq 1 - \varepsilon$ such that

$$\forall(\omega_1, \dots, \omega_n) \in U, \qquad \operatorname*{Sup}_t \frac{1}{\sqrt{n}}\left|\sum_{i \leq n} Y_t(\omega_i)\right| \leq \varepsilon.$$

If we fix a subset I of $\{1, \dots, n\}$, and denote by J its complement, we can identify the space Ω^n with $\Omega^I \times \Omega^J$. Let $p = \operatorname{card} I$, $q = \operatorname{card} J$. We suppose $p, q \geq 1$. The set

$$V_I = \{\alpha \in \Omega^I; P^q(\{\beta \in \Omega^J; (\alpha, \beta) \in U\}) = 0\}$$

is such that $P^n(U \cap (V_I \times \Omega^J)) = 0$, by Fubini's theorem. Let $V = \bigcup V_I \times \Omega^J$, where the union is taken over all choices of I with $1 \leq \operatorname{card} I < n$. Let $U' = U \setminus V$. We have $P^n(U') = P^n(U)$.

Fix now $\omega = (\omega_1, \dots, \omega_n) \in U'$ and $t \in T$. Let $I \subset \{1, \dots, n\}$ with $1 \leq \operatorname{card} I \leq n$. Let $N = \{Y_t \neq 0\}$. Write $\omega = (\alpha, \beta)$, $\alpha \in \Omega^I$, $\beta \in \Omega^J$. Since $\alpha \notin V_I$, we have

$$P^q(\{\beta \in \Omega^J; (\alpha, \beta) \in U\}) > 0,$$

so there exists β' with $(\alpha, \beta') \in U$, such that no component of β' belongs to N. If $\omega' = (\alpha, \beta')$, write $\omega' = (\omega_i')$. We have $Y_t(\omega_i') = 0$ for $i \notin I$, and $Y_t(\omega_i') = Y_t(\omega_i)$ for $i \in I$. It follows that

$$\frac{1}{\sqrt{n}}\left|\sum_{i \in I} Y_t(\omega_i)\right| \leq \varepsilon.$$

Since this is true for all I, we have $\sum_{i \leq n}|Y_t(\omega_i)|/\sqrt{n} \leq 2\varepsilon$. So, for $\omega \in U'$, we have $\sum_{i \leq n}|Y|_t(\omega_i)/\sqrt{n} \leq 2\varepsilon$. This shows that $|Y|$ satisfies the CLT. The converse implication is straightforward. □

Before we start the proof of Theorem 4, we recall the following criteria, due to Dudley [2] in the case $d = d_0$. As stated below, it follows by essentially the same argument. The details have been carried out in [1]. We denote by d_0 the pseudometric on T given by

$$(5)\qquad d_0(s,t) = \left[E(X_s - X_t)^2 - (EX_s - EX_t)^2\right]^{1/2}.$$

PROPOSITION 6. *For a pseudometric d on T, consider the condition*

$$\forall \varepsilon > 0, \exists \alpha > 0, \exists n_0, \forall n \geq n_0,$$

$$(6)\qquad (P^n)^*\left(\left\{(\omega_1,\dots,\omega_n) \in \Omega^n;\ \operatorname*{Sup}_{d(s,t)<\alpha} \frac{1}{\sqrt{n}}\left|\sum_{i\leq n} \left(X_s(\omega_i) - EX_s - X_t(\omega_i) + EY_t\right)\right| \geq \varepsilon\right\}\right) \leq \varepsilon.$$

If X satisfies the CLT, condition (6) *holds with* $d = d_0$, *and* (T, d_0) *is a totally bounded. Conversely, if condition* (6) *holds for some* $d \geq d_0$, *and if* (T, d) *is totally bounded, X satisfies the CLT.*

PROOF OF THEOREM 4. Assume first that X satisfies the CLT. Consider the pseudometric d_0 given by (5). It is also the pseudometric associated to ρX. Theorem 3 shows that $(\rho X)^{(k)}$ is a separable modification of $X^{(k)}$. It follows that there is a countable set D of T and for each n a null set N_n of Ω^n such that

$$(\omega_1,\dots,\omega_n) \notin N_n \Rightarrow \operatorname*{Sup}_{d_0(s,t)<\alpha} \frac{1}{\sqrt{n}}\left|\sum_{i\leq n} \rho X_s(\omega_i) - \rho X_t(\omega_i) - EX_s + EX_t\right|$$
$$\leq \operatorname*{Sup}_{\substack{d_0(s,t)<\alpha\\ s,t\in D}} \frac{1}{\sqrt{n}}\left|\sum_{i\leq n} \rho X_s(\omega_i) - \rho X_t(\omega_i) - E\rho X_s + E\rho X_t\right|.$$

Let

$$N_n' = N_n \cup \left\{\omega \in \Omega^n, \exists i \leq n, \exists s \in D, \rho X_s(\omega_i) \neq X_s(\omega_i)\right\}.$$

Then $P^n(N_n') = 0$, and

$$(7)\qquad (\omega_1,\dots,\omega_n) \notin N_n' \Rightarrow \operatorname*{Sup}_{d_0(s,t)<\alpha} \frac{1}{\sqrt{n}}\left|\sum_{i\leq n} \rho X_s(\omega_i) - \rho X_t(\omega_i) - E\rho X_s + E\rho X_t\right|$$
$$\leq \operatorname*{Sup}_{\substack{d_0(s,t)<\alpha\\ s,t\in D}} \frac{1}{\sqrt{n}}\left|\sum_{i\leq n} X_s(\omega_i) - X_t(\omega_i) - EX_s + EX_t\right|.$$

It then follows from Proposition 6 that ρX satisfies the CLT. Let $X_s' = X_s - \rho X_s$. Then (7) shows that if $(\omega_1,\dots,\omega_n) \notin N_n'$ we have

$$(8)\qquad \operatorname*{Sup}_{d_0(s,t)<\alpha} \frac{1}{\sqrt{n}}\left|\sum_{i\leq n} X_s'(\omega_i) - X_t'(\omega_i)\right|$$
$$\leq 2 \operatorname*{Sup}_{d_0(s,t)<\alpha} \frac{1}{\sqrt{n}}\left|\sum_{i\leq n} X_s(\omega_i) - X_t(\omega_i) - EX_s + EX_t\right|.$$

Moreover, we know that for fixed t, $|\sum_{i\leq n} X_t'(\omega_i)|/\sqrt{n} = 0$ a.s.

From (6) and (8) it follows that

$$\forall \varepsilon > 0, \exists n_0, \forall n \geq n_0,$$

$$(P^n)^*\left(\left\{(\omega_1, \ldots, \omega_n);\ \operatorname{Sup}_s \frac{1}{\sqrt{n}}\left|\sum_{i \leq n} X_s'(\omega_i)\right| \geq \varepsilon\right\}\right) \leq \varepsilon.$$

This shows that X' satisfies the CLT. Proposition 5 then shows that $\overline{X} = |X'|$ satisfies the CLT. This completes the proof of half of Theorem 4. The proof that X satisfies the CLT when ρX and $\overline{X}$ do is routine and is left to the reader. □

A comment might be in order. What happens if one applies ρ to a process that is already nice? As implicitly shown in the above proof, for each n, there is a null set N_n' such that if $(\omega_1, \ldots, \omega_n) \in N_n'$, the set

$$\{(\rho X_t(\omega_1), \ldots, \rho X_t(\omega_n)) \in \mathbb{R}^n;\ t \in T\}$$

is contained in the closure of the set

$$\{(X_t(\omega_1), \ldots, X_t(\omega)) \in \mathbb{R}^n;\ t \in T\}.$$

So, for example, the process ρX will satisfy random entropy conditions that are at least as good as the random entropy conditions satisfied by X.

4. A comparison principle. The following theorem is an easy consequence of a new inequality of Fernique when enough measurability is assumed. The point here is that we prove it without any measurability assumption, as an application of Theorem 4.

THEOREM 7. *Let X, Y be two uniformly bounded processes. Assume that X satisfies the CLT and that*

$$\forall s, t \in T, \forall \omega \in \Omega, \qquad |Y_s(\omega) - Y_t(\omega)| \leq |X_s(\omega) - X_t(\omega)|.$$

Then Y satisfies the CLT.

In the measurable case, the proof will use the following criteria. In the case $d = d_0$, it follows from [4], Theorem 2.14d, e. As stated below, it is simple adaptation. We denote by (Ω', Ξ, Q) another probability space and by h a standard normal r.v. on Ω'.

PROPOSITION 8. *Let X be a uniformly bounded process satisfying the CLT such that $X^{(n)}$ is separable for each n. If $d_0(s,t) = (E(X_s - X_t)^2)^{1/2}$, then*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} E\left(\operatorname*{Sup}_{\substack{s, t \in T \\ d_0(s,t) < \delta}} \left|\frac{1}{\sqrt{n}} \sum_{i \leq n} (X_s(\omega_i) - X_t(\omega_i)) h(\omega_i')\right|\right) = 0.$$

Moreover, if for some pseudometric $d \geq d_0$ for which (T, d) is totally bounded,

the following condition holds:
$\forall \varepsilon > 0, \exists \delta > 0, \exists n_0, \forall n \geq n_0,$

$$(P^n \times Q^n)\left\{(\omega_1, \ldots, \omega_n, \omega'_1, \ldots, \omega'_n);\right.$$
$$\left.\operatorname*{Sup}_{d(s,t)<\delta}\left|\frac{1}{\sqrt{n}}\sum_{i\leq n}(X_s(\omega_i) - X_t(\omega_i))h(\omega'_i)\right| > \varepsilon\right\} < \varepsilon,$$

then X satisfies the CLT.

We now prove Theorem 7.

FIRST STEP. We first assume that for each n, the processes $X^{(n)}$ and $Y^{(n)}$ are separable. Let

$$d(s,t) = \left(E(X_s - X_t)^2\right)^{1/2},$$
$$d_0(s,t) = \left(E(Y_s - Y_t)^2\right)^{1/2},$$

so $d_0 \leq d$. Also (T, d) is totally bounded. Actually, if $M(\delta)$ denotes the smallest number of d balls of radius δ necessary to cover T, we have

$$\lim_{\delta\to 0}\delta^2 \log M(\delta) = 0.$$

(See [4], Theorem 2.16.)

Let $\varepsilon > 0$. We fix $\delta > 0$ such that the following hold:

$$26\delta(\log M(\delta))^{1/2} \leq \varepsilon^2;$$

$\exists n_0, \forall n \geq n_0,$

$$E\left(\operatorname*{Sup}_{\substack{s,t\in T\\ d(s,t)<\delta}}\left|\frac{1}{\sqrt{n}}\sum_{i\leq n}(X_s(\omega_i) - X_t(\omega_i))h(\omega'_i)\right|\right) < \varepsilon^3. \tag{9}$$

It is obvious from the definition of the CLT that we have, for each $\alpha > 0$,

$$\lim_n (P^n)^*\left(\left\{\operatorname*{Sup}_{t\in T}\left|\frac{1}{n}\sum_{i\leq n}X_t(\omega_i)\right| \geq \alpha\right\}\right) = 0.$$

It then follows from [9], Proposition 24, that we have

$$\lim_n (P^n)^*\left(\left\{\operatorname*{Sup}_{s,t\in T}\left|\frac{1}{n}\sum_{i\leq n}(X_s(\omega_i) - X_t(\omega_i))^2 - d^2(s,t)\right| \geq \delta^2\right\}\right) = 0.$$

Let $n_1 \geq n_0$, such that for $n \geq n_1$, we have $P^n(A_n) \geq 1 - \varepsilon$, where

$$A_n = \left\{(\omega_1, \ldots, \omega_n) \in \Omega^n;\ \forall s, t \in T,\right.$$
$$\left.\frac{1}{n}\sum_{i\leq n}(X_s(\omega_i) - X_t(\omega_i))^2 \leq d^2(s,t) + \delta^2\right\}.$$

Let us denote by E_Q the conditional expectation at $\omega_1, \dots, \omega_n$ fixed. Let

$$B_n = \left\{ (\omega_1, \dots, \omega_n) \in \Omega^n, \right.$$

$$\left. E_Q\left(\operatorname*{Sup}_{\substack{s,t \in T \\ d(s,t) < \delta}} \left| \frac{1}{\sqrt{n}} \sum_{i \le n} (X_s(\omega_i) - X_t(\omega_i)) h(\omega_i') \right| \right) \le \varepsilon^2 \right\}.$$

It follows from (9) that $P^n(B_n) \ge 1 - \varepsilon$. Let $C_n = A_n \cap B_n$. We now fix $(\omega_1, \dots, \omega_n) \in C_n$. Consider the Gaussian process Θ on T, with basic probability space Q^n given by

$$\Theta_t(\omega_1', \dots, \omega_n') = \frac{1}{\sqrt{n}} \sum_{i \le n} X_t(\omega_i) h(\omega_i').$$

The pseudometric on T associated to Θ is given by

$$d_1(s,t) = \left(\frac{1}{n} \sum_{i \le n} (X_s(\omega_i) - X_t(\omega_i))^2 \right)^{1/2}.$$

We note that since $(\omega_1, \dots, \omega_n) \in A_n$, we have

$$d(s,t) < \delta \Rightarrow d_1(s,t) < 2\delta.$$

We now estimate

$$a = E_Q\left(\operatorname*{Sup}_{d(s,t) < \delta} \left| \frac{1}{\sqrt{n}} \sum_{i \le n} (Y_s(\omega_i) - Y_t(\omega_i)) h(\omega_i') \right| \right).$$

We have

$$a \le E_Q\left(\operatorname*{Sup}_{d_1(s,t) < 2\delta} \left| \frac{1}{\sqrt{n}} \sum_{i \le n} (Y_s(\omega_i) - Y_t(\omega_i)) h(\omega_i') \right| \right),$$

so Fernique's comparison theorem as in [2], (2.29), shows that

$$a \le E_Q\left(\operatorname*{Sup}_{d_1(s,t) < 2\delta} \left| \frac{1}{\sqrt{n}} \sum_{i \le n} (X_s(\omega_i) - X_t(\omega_i)) h(\omega_i') \right| \right) + 26\delta M(\delta)^{1/2} \le 2\varepsilon^2$$

since $(\omega_1, \dots, \omega_n) \in B_n$ and since T can be covered by $M(\delta)$ d_1 balls of radius 2δ. It follows that

$$Q^n\left(\left\{ \operatorname*{Sup}_{d(s,t) < \delta} \left| \frac{1}{\sqrt{n}} \sum_{i \le n} (Y_s(\omega_i) - Y_t(\omega_i)) h(\omega_i') \right| \ge \varepsilon \right\} \right) \le 2\varepsilon.$$

Using Fubini's theorem and $P^n(C_n) \ge 1 - 2\varepsilon$, we get

$$P^n \times Q^n\left(\left\{ \operatorname*{Sup}_{d(s,t) < \delta} \left| \frac{1}{\sqrt{n}} \sum_{i \le n} (Y_s(\omega_i) - Y_t(\omega_i)) h(\omega_i') \right| > \varepsilon \right\} \right) \le 4\varepsilon,$$

so Proposition 8 shows that Y satisfies the CLT.

SECOND STEP. We turn to the general case. We fix a consistent lifting ρ. Since liftings preserve lattice operations, we have

$$\forall s, t \in T, \forall \omega \in \Omega, \qquad |\rho Y_s(\omega) - \rho Y_t(\omega)| \le |\rho X_s(\omega) - \rho Y_t(\omega)|,$$

so Theorem 3 and the first step show that ρY satisfies the CLT. Let

$$Y' = Y - \rho Y, \qquad \overline{Y} = |Y'|, \qquad \overline{X} = |X - \rho X|.$$

For $s, t \in T$, $\omega \in \Omega$, we get

$$|Y_s'(\omega) - Y_t'(\omega)| \le 2|\rho X_s(\omega) - \rho X_t(\omega)| + \overline{X}_s(\omega) + \overline{X}_t(\omega).$$

Proposition 1 shows that there is a null set N_1 and a countable subset D of T such that

$$\forall \omega \notin N_1, \forall t \in T, \qquad \rho X_t(\omega) \text{ is a cluster point of } \{\rho X_s(\omega);\ s \in D\}.$$

Let N_2 be a null set containing N_1, such that for $\omega \notin N_2$, $s \in D$ we have $Y_s'(\omega) = 0$, $\overline{X}_s(\omega) = 0$.

We have

$$\forall \omega \notin N_2, \forall t \in T, \forall s \in D, \qquad |Y_t'(\omega)| \le \overline{X}_t(\omega) + 2|\rho X_s(\omega) - \rho X_t(\omega)|.$$

Taking the infimum for $s \in D$ gives

$$\forall \omega \notin N_2, \forall t \in T, \qquad \overline{Y}_t(\omega) \le \overline{X}_t(\omega).$$

Since $\overline{Y}_t(\omega) \ge 0$, this makes it obvious that $\overline{Y}$ satisfies the CLT. The proof is complete. □

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REFERENCES

[1] ANDERSEN, N. T. and DOBRIĆ, V. (1987). The central limit theorem for stochastic processes. *Ann. Probab.* **15** 164–177.

[2] DUDLEY, R. M. (1984). A course on empirical processes. *Ecole d'Eté de Probabilités de Saint-Flour XII—1982. Lecture Notes in Math.* **1097**. Springer, Berlin.

[3] DUDLEY, R. M. (1985). An extended Wichura theorem, definitions of Donsker class, and weighted empirical distributions. *Probability in Banach Spaces V. Lecture Notes in Math.* **1153** 141–178. Springer, Berlin.

[4] GINÉ, E. and ZINN, J. (1984). Some limit theorems for empirical processes. *Ann. Probab.* **12** 929–989.

[5] IONESCU TULCEA, A. and IONESCU TULCEA, C. (1969). *Topics in the Theory of Lifting*. Springer, Berlin.

[6] TALAGRAND, M. (1982). Closed convex hull of set of measurable functions, Riemann-measurable functions and measurability of translations. *Ann. Inst. Fourier (Grenoble)* **32** (1) 39–69.

[7] TALAGRAND, M. (1984). Pettis integral and measure theory. *Mem. Amer. Math. Soc.* **51** (307).

[8] TALAGRAND, M. (1986). On liftings and the regularization of stochastic processes. Unpublished manuscript.

[9] TALAGRAND, M. (1987). The Glivenko–Cantelli problem. *Ann. Probab.* **15**. To appear.

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