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THE CENTRAL LIMIT THEOREM FOR EXCHANGEABLE RANDOM VARIABLES WITHOUT MOMENTS[1]

BY MICHAEL KLASS AND HENRY TEICHER

University of California, Berkeley and Rutgers University

If $\{X_n,\ n \geq 1\}$ is an exchangeable sequence with $(1/b_n(\Sigma_1^n X_i - a_n)) \to N(0,1)$ for some constants a_n and $0 < b_n \to \infty$ then b_n/n^α is slowly varying with $\alpha = 1$ or $\frac{1}{2}$ and necessary conditions (depending on α) which are also sufficient, are obtained. Three such examples are given, one with infinite mean, one with no positive moments, and the third with almost all conditional distributions belonging to no domain of attraction of any law.

1. Introduction. Exchangeable random variables $\{X_n,\ n = 1, 2, \dots\}$ have long been recognized as a natural generalization of i.i.d. random variables and the de Finetti representation of the corresponding probability measure [5] in the case of an infinite sequence has played a fundamental role in the subject. In particular, this has paved the way for a central limit theorem for exchangeable random variables with finite variance under the classical normalization $1/\sqrt{n}$. Specifically [1], if $EX_1 = 0$, $EX_1^2 = 1$, then $\mathscr{L}((1/\sqrt{n})\Sigma_1^n X_i) \to N(0,1)$ iff $\mathrm{Cov}(X_1, X_2) = 0 = \mathrm{Cov}(X_1^2, X_2^2)$. Contrary to the i.i.d. case, a central limit theorem may also obtain under the normalization $1/n$. Specifically [9], if $EX_1 = 0$ and $\mathrm{Cov}(X_1, X_2) = \mu > 0$, then $\mathscr{L}((1/n\mu)\Sigma_1^n X_i) \to N(0,1)$ iff $EX_1 X_2 \cdots X_k$ exists and coincides with the kth moment of a standard normal distribution for all $k \geq 1$.

Suppose, however, that nothing is stipulated about finiteness of moments. Under what conditions will there exist constants a_n and $0 < b_n \to \infty$ for which $\mathscr{L}((1/b_n)(\Sigma_{i=1}^n X_i - a_n)) \to N(0,1)$ and which normalizations b_n are permissible? It turns out that b_n/n^α must be slowly varying with $\alpha = \frac{1}{2}$ or 1 and Theorem 2 gives accompanying necessary and sufficient conditions.

In contradistinction to the i.i.d. case, a central limit theorem can be obtained for exchangeable random variables with infinite mean. An example is also given of exchangeable random variables obeying a central limit theorem for which almost all distributions of the conditionally i.i.d. sequences do not belong to the domain of attraction of any law.

2. Mainstream. Let $S_n = \Sigma_{j=1}^n X_j$, $n \geq 1$ where $\{X_n,\ n \geq 1\}$ is a sequence of exchangeable r.v.'s on the probability space $(\Omega, \mathscr{F}, P)$. Then $\{X_n,\ n \geq 1\}$ are conditionally i.i.d. given some σ-algebra $\mathscr{G} \subset \mathscr{F}$ and ([3], Corollary 7.3.5) there exists a regular conditional distribution P^ω given $\mathscr{G}$ such that for each $\omega \in \Omega$ the coordinate r.v.'s $\{\xi_n \equiv \xi_n^\omega,\ n \geq 1\}$ of the probability space $(R^\infty, B^\infty, P^\omega)$ are i.i.d.

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Theorem 1. *If there are constants* $0 < b_n \to \infty$ *and* a_n *for which*

$$\mathscr{L}\left(\frac{S_n - a_n}{b_n}\right) \to N(0,1), \tag{1}$$

then there exists a positive sequence $\varepsilon_n \downarrow 0$ *such that*

$$\mathscr{L}\left(\frac{S_{nn} - a_n}{b_n}\right) \to N(0,1), \tag{2}$$

where

$$S_{nn} = \sum_{j=1}^{n} X_{jn} \equiv \sum_{j=1}^{n} X_j I_{[|X_j| \le \varepsilon_n b_n]}. \tag{3}$$

The decisive portion of the proof is contained in

Lemma 1. *Under the hypothesis of* (1),

$$n \cdot P^{\omega}\{|\xi_1| > \varepsilon b_n\} \to_P 0, \qquad \varepsilon > 0.$$

Proof. Let $T_n^* = \sum_{j=1}^{n} \xi_j^*$ where $\{\xi_j^*,\ j \ge 1\}$ is a symmetrized version of $\{\xi_j,\ j \ge 1\}$. Then for any $\varepsilon > 0$ and $k > 0$, via (1)

$$\begin{aligned}
\frac{(2/\pi)^{1/2} e^{-(k\varepsilon)^2/2}}{k\varepsilon} &\ge \lim_{n\to\infty} P\{|S_n - a_n| > k\varepsilon b_n\} \\
&\ge \tfrac{1}{2} \limsup_{n\to\infty} \int_{\Omega} P^{\omega}\{|T_n^*| > 2k\varepsilon b_n\}\, dP \\
&\ge \tfrac{1}{2} \limsup_{n\to\infty} \int_{\Omega} P^{\omega}\left\{ \sum_{j=1}^{n} \xi_j^* I_{[|\xi_j^*| > \varepsilon b_n]} > 2k\varepsilon b_n \right\} dP.
\end{aligned} \tag{4}$$

Define $\tau = \inf\{1 \le h \le n: \sum_{j=1}^{h} I_{[|\xi_j^*| > \varepsilon b_n]} \ge 2k\}$ and ∞ if no such integer exists and note that

$$\begin{aligned}
P^{\omega}\left\{ \sum_{j=1}^{n} \xi_j^* I_{[|\xi_j^*| > \varepsilon b_n]} \ge 2k\varepsilon b_n \right\} &\ge \tfrac{1}{2} P^{\omega}\left\{ \sum_{j=1}^{\tau} \xi_j^* I_{[|\xi_j^*| > \varepsilon b_n]} > 2k\varepsilon b_n,\ \tau \le n \right\} \\
&\ge 2^{-(2k+1)} P^{\omega}\{\tau \le n\}.
\end{aligned}$$

Now if $Y_{n1}, \ldots, Y_{nn}$ are i.i.d. with

$$P^{\omega}\{Y_{n1} = 1\} = \delta/n = 1 - P^{\omega}\{Y_{n1} = 0\},$$

then on the set $A_n = A_n(\varepsilon, \delta) = \{\omega: nP^{\omega}\{|\xi_1^*| > \varepsilon b_n\} > \delta\}$

$$\begin{aligned}
P^{\omega}\{\tau \le n\} &\ge P^{\omega}\left\{ \sum_{j=1}^{n} I_{[|\xi_j^*| > \varepsilon b_n]} \ge 2k \right\} \\
&\ge P^{\omega}\{Y_{n1} + \cdots + Y_{nn} \ge 2k\} = \frac{\delta^{2k} e^{-\delta}}{(2k)!} + O\left(\frac{1}{n}\right)
\end{aligned}$$

via monotonicity and the Poisson approximation to the binomial [7] so that via (4)

$$\frac{1}{k\varepsilon}\left(\frac{2}{\pi}\right)^{1/2} e^{-(k\varepsilon)^2/2} \geq \limsup_{n\to\infty} 2^{-(2k+2)} \int_{A_n} P^\omega\{\tau \leq n\}\, dP$$
$$\geq \frac{2^{-2}(\delta/2)^{2k} e^{-\delta}}{(2k)!} \limsup_{n\to\infty} P\{A_n\},$$

implying as $k \to \infty$ that for all $\varepsilon > 0$, $\delta > 0$,

$$P\{\omega: nP^\omega\{|\xi_1^*| > \varepsilon b_n\} > \delta\} = P\{A_n\} = o(1),$$

as $n \to \infty$.

Next, if $m(\cdot)$ denotes a median and $D_n = \{\omega: |m(\xi_1)| \leq b_n\varepsilon/2\}$, clearly $P\{D_n\} \to 1$ as $n \to \infty$. Thus, for all $\varepsilon > 0$, $\delta > 0$,

$$\lim_{n\to\infty} P\left\{\omega: P^\omega\{|\xi_1| > \varepsilon b_n\} > \frac{\delta}{n}\right\} = \lim_{n\to\infty} P\left\{D_n \cap \left[P^\omega\{|\xi_1| > \varepsilon b_n\} > \frac{\delta}{n}\right]\right\}$$
$$\leq \lim_{n\to\infty} P\left\{\omega: P^\omega\left\{|\xi_1 - m(\xi_1)| > \frac{\varepsilon b_n}{2}\right\} > \frac{\delta}{n}\right\}$$
$$\leq \lim_{n\to\infty} P\left\{\omega: P^\omega\left\{|\xi_1^*| > \frac{\varepsilon b_n}{2}\right\} > \frac{\delta}{2n}\right\} = 0,$$

yielding the lemma. □

PROOF OF THEOREM 1. According to the lemma, $\lim_{n\to\infty} P\{B_n(\varepsilon, \delta)\} = 0$, all $\varepsilon > 0$, $\delta > 0$, where $B_n(\varepsilon, \delta) = \{\omega: nP^\omega\{|\xi_1| > \varepsilon b_n\} > \delta\}$. Hence, there exists a sequence $\varepsilon_n \downarrow 0$ for which $\lim_{n\to\infty} P\{B_n(\varepsilon_n, \delta)\} = 0$, all $\delta > 0$. Thus,

$$P\{S_n \neq S_{nn}\} \leq \int P^\omega\left\{\bigcup_{j=1}^{n} \left[|\xi_j| > \varepsilon_n b_n\right]\right\} dP \tag{5}$$
$$\leq P\{B_n(\varepsilon_n, \delta)\} + \delta \to_{n\to\infty} \delta \to_{\delta\to 0} 0,$$

so that $(S_n - S_{nn})/b_n \to_P 0$ whence (2) follows via (1). □

Define

$$T_{j,n,\omega} = \sum_{i=1}^{j} \xi_{i,n,\omega} \equiv \sum_{i=1}^{j} \xi_i^\omega I_{[|\xi_1^\omega| \leq \varepsilon_n b_n]}, \qquad 1 \leq j \leq n, \tag{6}$$

where ε_n is as in Theorem 1 and set

$$a_{n\omega} = E^\omega T_{nn\omega}, \qquad v_{n\omega}^2 = E^\omega(T_{nn\omega} - a_{n\omega})^2, \qquad n \geq 1. \tag{7}$$

As usual, Φ will denote the standard normal distribution function.

LEMMA 2. *Under the hypothesis of* (1),

$$\lim_{n\to\infty}\int_\Omega \Phi\left(v_{n\omega}^{-1}[b_n x-(a_{n\omega}-a_n)]\right)dP=\Phi(x),\quad \textit{all } x. \tag{8}$$

PROOF. If Z is a standard normal random variable independent of ω, (8) may be recast as

$$\lim_{n\to\infty}\int P^\omega\left\{\frac{v_{n\omega}}{b_n}Z+\frac{a_{n\omega}-a_n}{b_n}<x\right\}dP=\Phi(x),\quad \text{all } x. \tag{9}$$

Set

$$Z_{n\omega}=\frac{v_{n\omega}}{b_n}Z+\frac{a_{n\omega}-a_n}{b_n},\qquad U_{n\omega}=\frac{T_{nn\omega}-a_{n\omega}}{b_n}+\frac{a_{n\omega}-a_n}{b_n}.$$

Theorem 1 ensures that

$$\lim_{n\to\infty}\int_\Omega P^\omega\{U_{n\omega}<x\}\,dP=\Phi(x),\quad \text{all } x,$$

and so to verify (8), it suffices to prove that as $n\to\infty$

$$\int_\Omega|P^\omega\{Z_{n\omega}<x\}-P^\omega\{U_{n\omega}<x\}|\,dP=o(1). \tag{10}$$

Define $A_n=\{\omega\colon v_{n\omega}<\varepsilon_n^{1/2}b_n\}$. Now,

$$\begin{aligned}
&\int_{A_n}|P^\omega\{Z_{n\omega}<x\}-P^\omega\{U_{n\omega}<x\}|\,dP\\
&\le\int_{A_n}P^\omega\left\{\left|\frac{v_{n\omega}}{b_n}Z\right|\ge\varepsilon_n^{1/4}\right\}dP+\int_{A_n}P^\omega\left\{\left|\frac{T_{nn\omega}-a_{n\omega}}{b_n}\right|\ge\varepsilon_n^{1/4}\right\}dP\\
&\quad+\int_{A_n}P^\omega\left\{\left|\frac{a_{n\omega}-a_n}{b_n}-x\right|\le\varepsilon_n^{1/4}\right\}dP\\
&\le\int_{A_n}P^\omega\{|Z|>\varepsilon_n^{-1/4}\}\,dP+2\int_{A_n}P^\omega\left\{\left|\frac{T_{nn\omega}-a_{n\omega}}{b_n}\right|\ge\varepsilon_n^{1/4}\right\}dP\\
&\quad+\int_\Omega P^\omega\left\{\left|\frac{T_{nn\omega}-a_{n\omega}}{b_n}+\frac{a_{n\omega}-a_n}{b_n}-x\right|\le 2\varepsilon_n^{1/4}\right\}dP\\
&\le o(1)+2\int_{A_n}\varepsilon_n^{-1/2}v_{n\omega}^2/b_n^2\,dP+\Phi\left(x+2\varepsilon_n^{1/4}\right)-\Phi\left(x-2\varepsilon_n^{1/4}\right)\\
&\le o(1)+2\int_{A_n}\varepsilon_n^{1/2}\,dP=o(1).
\end{aligned}$$

On the other hand, via the Berry–Esseen theorem, setting $x_{n\omega} = (b_n x - a_{n\omega} + a_n)/v_{n\omega}$,

$$\begin{aligned}
&\int_{A_n^c} |P^\omega\{Z_{n\omega} < x\} - P^\omega\{U_{n\omega} < x\}|\, dP \\
&= \int_{A_n^c} \left| P^\omega\{Z < x_{n\omega}\} - P^\omega\left\{\frac{T_{nn\omega} - a_{n\omega}}{v_{n\omega}} < x_{n\omega}\right\}\right| dP \\
&\le \int_{A_n^c} \frac{nE|\xi_{n1\omega} - E\xi_{n1\omega}|^3}{v_{n\omega}^3}\, dP \\
&\le \int_{A_n^c} \frac{2\varepsilon_n b_n}{v_{n\omega}}\, dP \le 2\varepsilon_n^{1/2} = o(1),
\end{aligned}$$

proving (10) and hence the lemma. □

COROLLARY 1. *Under* (1),

$$\lim_{n\to\infty} \int_\Omega E^\omega \exp\left\{ it\left(\frac{a_{n\omega} - a_n}{b_n}\right) - \frac{v_{n\omega}^2}{b_n^2}\frac{t^2}{2}\right\} dP = e^{-t^2/2}, \quad \textit{all real } t.$$

LEMMA 3. *Under the hypothesis of* (1), $\{v_{n\omega}/b_n,\ n \ge 1\}$ *and* $\{(a_{n\omega} - a_n)/b_n,\ n \ge 1\}$ *are tight sequences.*

PROOF. Apropos of tightness of $\{v_{n\omega}/b_n,\ n \ge 1\}$, the stronger result

(11) $$\lim_{n\to\infty} P\{\omega\colon v_{n\omega} \ge \alpha b_n\} = 0, \quad \text{for } \alpha > 1,$$

will be proved. To this end, note that on $B_n = \{\omega\colon v_{n\omega} \ge \alpha b_n\}$

$$\frac{E|\xi_{n1\omega} - E\xi_{n1\omega}|^3}{n^{1/2}\left(v_{n\omega}^2/n\right)^{3/2}} \le \frac{2\varepsilon_n b_n}{v_{n\omega}} \le \frac{2\varepsilon_n}{\alpha} = o(1).$$

Thus, if $A_n = B_n\{\omega\colon a_{n\omega} \ge a_n\}$, via Theorem 1 and the Berry–Esseen theorem, for $y > 0$

$$\begin{aligned}
1 - \Phi(y) &\ge \limsup_{n\to\infty} \int_{A_n} P^\omega\{T_{nn\omega} - a_{n\omega} > b_n y\}\, dP \\
&\ge \limsup_{n\to\infty} \int_{A_n} \left[1 - \Phi\left(\frac{y b_n}{v_{n\omega}}\right)\right] dP - \limsup_{n\to\infty} \int_{A_n} \frac{E|\xi_{n1\omega} - E\xi_{n1\omega}|^3}{n^{1/2}\left(v_{n\omega}^2/n\right)^{3/2}}\, dP \\
&\ge \limsup_{n\to\infty} \int_{A_n} [1 - \Phi(y/\alpha)]\, dP = [1 - \Phi(y/\alpha)] \limsup_{n\to\infty} P\{A_n\},
\end{aligned}$$

implying as $y \to \infty$ that $\lim_{n\to\infty} P\{A_n\} = 0$ for $\alpha > 1$. Analogously, commencing with $\Phi(-y)$, it follows that $P\{B_n[\omega\colon a_{n\omega} < a_n]\} = o(1)$ as $n \to \infty$ and (11) follows.

To verify the second portion of the theorem, let $D_n(C) = \{\omega: |a_{n\omega} - a_n| > Cb_n\}$ and suppose there were positive sequences $n_k \uparrow \infty$, $C_k \uparrow \infty$, and a positive number δ such that $P\{D_{n_k}(C_k)\} > \delta$, $k \geq 1$. If Z is as in Lemma 2, it follows therefrom for $x > 0$ that

$$\begin{aligned}
2\Phi(x) - 1 &= \lim_{n\to\infty} \int_\Omega P^\omega\{|v_{n\omega}Z + a_{n\omega} - a_n| < b_n x\}\, dP \\
&\leq 1 - \delta + \limsup_{k\to\infty} \int_{D_{n_k}} P^\omega\{|v_{n_k\omega}Z + a_{n_k\omega} - a_{n_k}| < b_{n_k}x\}\, dP \\
&\leq 1 - \delta + 2\limsup_{k\to\infty} \int_{D_{n_k}B^c_{n_k}} P^\omega\{\alpha Z < x - C_k\}\, dP \\
&\leq 1 - \delta + 2\limsup_{k\to\infty} \Phi\left(\frac{x - C_k}{\alpha}\right) = 1 - \delta
\end{aligned}$$

via (11) and yielding a contradiction as $x \to \infty$. Thus, $\{(a_{n\omega} - a_n)/b_n, n \geq 1\}$ is a tight sequence. □

Let $\{X_n^*, n \geq 1\}$ be the symmetrized version of the interchangeable process $\{X_n, n \geq 1\}$. That is, if $\{X_n^{(i)}, n \geq 1\}$ is an exchangeable process with the same finite-dimensional distributions as $\{X_n, n \geq 1\}$ and is defined on $\{\Omega_i, \mathscr{F}_i, P)$ where $(\Omega_i, \mathscr{F}_i, P)$, $i = 1, 2$, are copies of the original probability space, then $X_n^* = X_n^{(1)} - X_n^{(2)}$, $n \geq 1$, is defined on $(\Omega, \mathscr{F}, P_2)$ where $\Omega = \Omega_1 \times \Omega_2$, $\mathscr{F} = \mathscr{F}_1 \times \mathscr{F}_2$ is the product σ-algebra and $P_2 = P \times P$ is the product measure. Moreover, if $\mathscr{G}_i = \mathscr{G}$, $i = 1, 2$, and P^{ω_i} is the regular conditional distribution given $\mathscr{G}_i$ (such that for each $\omega_i \in \Omega_i$, the coordinate random variables $\xi_n^{\omega_i}$, $n \geq 1$, of $(R^\infty, B^\infty, P^{\omega_i})$ are i.i.d.), then $P_2^\omega = P^{\omega_1} \times P^{\omega_2}$ is a regular conditional distribution given $\mathscr{G} = \mathscr{G}_1 \times \mathscr{G}_2$ and $\xi_n^\omega = \xi_n^{\omega_1} - \xi_n^{\omega_2}$, $n \geq 1$, is the symmetrized version of the original i.i.d. random variables.

Defining $S_n^* = \sum_{j=1}^n X_j^*$, (1) ensures

$$\mathscr{L}(S_n^*/b_n) \to N(0, 2). \tag{12}$$

Moreover, setting

$$X_{jn}^* = X_j^{(1)} I_{[|X_j^{(1)}| \leq \varepsilon_n b_n]} - X_j^{(2)} I_{[|X_j^{(2)}| \leq \varepsilon_n b_n]},$$

$$S_{kn}^* = \sum_{j=1}^k X_{jn}^*, \qquad 1 \leq k \leq n,$$

where ε_n is as previously defined, it follows via (5) that

$$\begin{aligned}
P_2\{S_n^* \neq S_{nn}^*\} &\leq P_2\left\{ \bigcup_{j=1}^n \left[|X_j^{(1)}| > \varepsilon_n b_n\right] \cup \bigcup_{j=1}^n \left[|X_j^{(2)}| > \varepsilon_n b_n\right] \right\} \\
&\leq 2P\left\{ \bigcup_{j=1}^n \left[|X_j^{(1)}| > \varepsilon_n b_n\right] \right\} = o(1),
\end{aligned}$$

which, in conjunction with (12) guarantees

$$\mathscr{L}(S^*_{n,n}/b_n) \to N(0,2). \tag{13}$$

For $i = 1,2$ define

$$T_{k,n,\omega_i} = \sum_{h=1}^{k} \xi_h^{\omega_i} I_{[|\xi_h^{\omega_i}| \le \varepsilon_n b_n]},$$

$$a_{k,n,\omega_i} = ET_{k,n,\omega_i}, \qquad v^2_{k,n,\omega_i} = E\left[T_{k,n,\omega_i} - a_{k,n,\omega_i}\right]^2, \tag{14}$$

$$a_{n,\omega_i} = a_{n,n,\omega_i}, \qquad v^2_{n,\omega_i} = v^2_{n,n,\omega_i}.$$

Since, for each ω_i, $\{\xi_h^{\omega_i} I_{[|\xi_h^{\omega_i}| \le \varepsilon_n b_n]},\ h \ge 1\}$ are i.i.d., for any $\alpha > 0$

$$\frac{v^2_{[\alpha n],n,\omega_i}}{v^2_{n,\omega_i}} = \frac{[\alpha n]}{n} \to \alpha, \qquad \frac{a_{[\alpha n],n,\omega_i}}{a_{n,\omega_i}} = \frac{[\alpha n]}{n} \to \alpha. \tag{15}$$

Furthermore, for any distinct, positive numbers $\alpha_1, \alpha_2, \alpha_3$ and any sequence N_k of the positive integers, there is a further subsequence $n_k \to \infty$ for which

$$b_{[\alpha_j n_k]}/b_{n_k} \to q_j \alpha_j^{1/2}, \qquad j = 1,2,3, \tag{16}$$

as $k \to \infty$ where $q_j \in [0, \infty]$, $j = 1,2,3$.

LEMMA 4. *Under* (1), *the random vectors*

$$\left\{\left(\frac{a_{n\omega_1} - a_{n\omega_2}}{b_n}, \frac{\left(v^2_{n\omega_1} + v^2_{n\omega_2}\right)^{1/2}}{b_n}\right) n \ge 1\right\}$$

are tight relative to the product measure P_2.

PROOF. Via Lemma 3, for any $\varepsilon > 0$, $C > C_\varepsilon$ and all $n \ge 1$,

$$\begin{aligned} &P_2\left\{\left(a_{n\omega_1} - a_{n\omega_2}\right)^2 + \left(v^2_{n\omega_1} + v^2_{n\omega_2}\right) > 8C^2 b_n^2\right\} \\ &\quad \le P_2\left\{\left(a_{n\omega_1} - a_{n\omega_2}\right)^2 > 4C^2 b_n^2\right\} + P_2\left\{\left(v^2_{n\omega_1} + v^2_{n\omega_2}\right) > 4C^2 b_n^2\right\} \\ &\quad \le 2P\left\{\left(a_{n\omega_1} - a_n\right)^2 > C^2 b_n^2\right\} + 2P\left\{v^2_{n\omega_1} > 2C^2 b_n^2\right\} < \varepsilon. \end{aligned}$$

□

Just as (2) of Theorem 1 implied Lemma 2, so its counterpart (13), or more precisely,

$$\mathscr{L}\left(S^*_{[\alpha n],n}/b_{[\alpha n]}\right) \to N(0,2), \tag{13'}$$

guarantees for any $\alpha > 0$ that

$$\lim_{n\to\infty} \int_{\Omega_1 \times \Omega_2} \Phi\left(\frac{b_{[\alpha n]}x - \left(a_{[\alpha n],\omega_1} - a_{[\alpha n],\omega_2}\right)}{\left(v^2_{[\alpha n],\omega_1} + v^2_{[\alpha n],\omega_2}\right)^{1/2}}\right) dP_2 = \Phi\left(\frac{x}{\sqrt{2}}\right) \tag{17}$$

or equivalently

$$(17')\qquad \lim_{n\to\infty}\int_{\Omega_1\times\Omega_2} P^{\omega}\left\{ \frac{\left(v^2_{[\alpha n],\omega_1} + v^2_{[\alpha n],\omega_2}\right)^{1/2}}{b_{[\alpha n]}} Z + \frac{a_{[\alpha n],\omega_1} - a_{[\alpha n],\omega_2}}{b_{[\alpha n]}} < x \right\} dP_2 = \Phi\left(\frac{x}{\sqrt{2}}\right),$$

where Z is a standard normal random variable independent of ω.

In view of Lemma 4, there is a subsequence such that the random vectors

$$(18)\qquad \left(\frac{\left(v^2_{n_k,\omega_1} + v^2_{n_k,\omega_2}\right)^{1/2}}{b_{n_k}}, \frac{a_{n_k,\omega_1} - a_{n_k,\omega_2}}{b_{n_k}} \right) \to_{\mathscr{L}} \left(R_{\omega_1,\omega_2}, C_{\omega_1,\omega_2}\right).$$

Then as already noted, there is a further subsequence (also denoted by n_k) such that (16) and (15) hold for $\alpha_1, \alpha_2, \alpha_3$. Hence, via (17′), (18), (16) and (15)

$$(19)\qquad \int_{\Omega} P^{\omega}\left\{R_{\omega_1,\omega_2}Z + \alpha_j^{1/2}C_{\omega_1,\omega_2} < q_j x\right\} dP_2 = \Phi\left(\frac{x}{\sqrt{2}}\right),$$

for $j = 1, 2, 3$ and all real x where (R, C) is a fictitious random vector, independent of the standard normal variable Z.

Moreover, in view of (11) of Lemma 3, the distribution, say F, of R cannot assign positive mass to $\{R > 2\}$. Consequently, expressing (19) in terms of moment generating functions,

$$(20)\qquad \exp\left[q_j^2 t^2/\alpha_j\right] = \int_0^2 \exp\left[t^2 r^2/2\alpha_j\right] M_r(t)\, dF(r),$$

where $M_r(t) = E\{e^{tC} | R = r\}$.

Let $1 = \alpha_1 < \alpha_2 < \alpha_3$. Then $\lambda/\alpha_1 + (1-\lambda)/\alpha_3 = 1/\alpha_2$ for some λ in $(0,1)$ and as $\alpha_3 \to \infty$, so $\lambda \to \alpha_1/\alpha_2$. Hence, via (20), denoting the moment generating function of C by $M_C(t)$,

$$\begin{aligned} &\exp\left[q_2^2 t^2/\alpha_2\right] \\ &\quad = \left[\int_0^2 \exp\left[t^2 r^2/2\alpha_1\right] M_r(t)\, dF(r)\right]^{\lambda} \left[\int_0^2 \exp\left[t^2 r^2/2\alpha_3\right] M_r(t)\, dF(r)\right]^{1-\lambda} \\ &\quad \to_{\alpha_3\to\infty} \left[\int_0^2 \exp\left[t^2 r^2/2\alpha_1\right] M_r(t)\, dF(r)\right]^{\alpha_1/\alpha_2} \left[M_C(t)\right]^{1-\alpha_1/\alpha_2} \\ &\quad = \exp\left[t^2/\alpha_2\right]\left(M_C(t)\right)^{(\alpha_2-1)/\alpha_2}, \end{aligned}$$

whence, since $q = q(\alpha)$,

$$(21)\qquad M_C(t) = \exp\left[t^2\left[\frac{q^2(\alpha) - 1}{\alpha - 1}\right]\right], \quad \text{for } \alpha > 1.$$

Hence, for some constant σ_0, depending on $\{n_k\}$, ω_1 and ω_2, necessarily

$q^2(\alpha) = 1 + (\sigma_0^2/2)(\alpha - 1)$ for $\alpha \geq 1$ implying

$$M_C(t) = \exp[t^2\sigma_0^2/2] \tag{22}$$

and

$$\int_0^2 \exp[t^2r^2/2\alpha]\,M_r(t)\,dF(r) = \exp\left[t^2\left(\frac{\sigma_0^2}{2} + \frac{1-\sigma_0^2/2}{\alpha}\right)\right], \qquad \alpha \geq 1. \tag{23}$$

Differentiating (23) with respect to α, cancelling factors of t^2/α^2, and setting $t = 0$ for $k = 1$ or differentiating and cancelling again (before setting $t = 0$) for the case $k = 2$,

$$\int_0^2 r^{2k}\,dF(r) = (2 - \sigma_0^2)^k, \qquad k = 1, 2,$$

so that the distribution of R^2 is degenerate at $(2 - \sigma_0^2)$ for some σ_0^2 in $[0, 2]$ while according to (22), the distribution of C is $N(0, \sigma_0^2)$.

Thus, setting

$$R_n^2 = R_{n,\omega}^2 = \frac{v_{n\omega_1}^2 + v_{n\omega_2}^2}{b_n^2}, \qquad C_n = C_{n,\omega} = \frac{a_{n\omega_1} - a_{n\omega_2}}{b_n}, \tag{24}$$

every subsequence $\{N_k\}$ of positive integers has a further subsequence $\{n_k\}$ such that for some σ_0^2 in $[0, 2]$

$$\mathscr{L}(R_{n_k}^2) \to \delta_{2-\sigma_0^2}, \qquad \mathscr{L}(C_{n_k}) \to N(0, \sigma_0^2), \tag{25}$$

and so in view of Lemma 3, for some subsequence n_k

$$\mathscr{L}(v_{n_k,\omega_1}^2/b_{n_k}^2) \to \delta_{1-\sigma_0^2/2}, \qquad \mathscr{L}\left(\frac{a_{n_k,\omega_1} - a_{n_k}}{b_{n_k}}\right) \to N\left(0, \frac{\sigma_0^2}{2}\right). \tag{26}$$

Next, it will be shown that σ_0 is almost surely independent of ω_1, ω_2, and of the subsequence.

Suppose that $R_{n_k} \to_{P_2} r_1$. According to (25), $0 \leq r_1 \leq \sqrt{2}$ and moreover

$$R_{[\alpha n_k]} \sim \frac{R_{n_k}}{q(\alpha)} \to_{P_2} \frac{r_1}{[\alpha(1 - r_1^2/2) + r_1^2/2]^{1/2}} = r_\alpha \quad \text{(say)}. \tag{27}$$

If $r_1 = \sqrt{2}$, then $r_\alpha \equiv \sqrt{2}$ whence $R_n \to_{P_2} \sqrt{2}$. On the other hand, if $r_1 < \sqrt{2}$, it will be shown that $r_1 = 0$ and so is independent of the subsequence. Suppose rather that $0 < r_1 < \sqrt{2}$. Since $R_{[\alpha n_k]} \to_{P_2} r_\alpha$ and $r_\alpha \to 0$ as $\alpha \to \infty$, we may define integers $k(j)$ such that

$$P_2\{|R_{j\cdot k(j)} - r_j| > 1/j\} < 1/j, \qquad j \geq 1,$$

whence, setting $m_j = jk(j)$, necessarily $R_{m_j} \to_{P_2} 0$.

In view of (27), $\limsup R_n > r_1/2$ with probability one, implying

$$f_j = \inf\{n \geq m_j\colon R_n \geq \tfrac{1}{2}r_1\}$$

is a bonafide random variable, $j \geq 1$. As earlier, the subsequence $\{[\frac{1}{2}f_j]\}$ has a further subsequence $[\frac{1}{2}f_{i_j}]$ with

$$R_{[\frac{1}{2}f_{i_j}]} \to_{P_2} r^*.$$

Since $f_j/m_j \to \infty$ and $R_n \le \frac{1}{2}r_1$ for $m_{i_j} \le n < f_{i_j}$, necessarily $r^* \le \frac{1}{2}r_1$. Consequently,

$$R_{2[\frac{1}{2}f_{i_j}]} \to_{P_2} \frac{r^*}{[2(1 - r^{*2}/2) + r^{*2}/2]^{1/2}} < r^* \le \tfrac{1}{2}r_1,$$

contradicting the definition of f_j. Hence, $r_1 = 0$ and $R_n \to_{P_2} 0$. It follows that the only permissible values of σ_0^2 in (25), (26) are $\sigma_0^2 = 2$ and 0.

Note via (27) that when $r_1 = 0$, $q(\alpha) = \sqrt{\alpha}$ implying $b_{[\alpha n]} \sim q_\alpha \sqrt{\alpha}\, b_n = \alpha b_n$ so that b_n/n is slowly varying. Alternatively, if $r_1 = \sqrt{2}$, $q(\alpha) = 1$, implying $b_{[\alpha n]} \sim \sqrt{\alpha}\, b_n$, whence $b_n/\sqrt{n}$ is slowly varying.

Thus the value of σ_0 is independent (a.s.) of (ω_1, ω_2) and we have established the necessity portion of

THEOREM 2. *If $\{X_n, n \ge 1\}$ is a sequence of exchangeable random variables with*

$$\mathscr{L}\left(\frac{1}{b_n}\left(\sum_{i=1}^{n} X_i - a_n\right)\right) \to N(0,1), \tag{28}$$

for some constants a_n, b_n where $0 < b_n \to \infty$, then there exists a positive sequence $\varepsilon_n \downarrow 0$ such that

$$nP^{\omega}\{|\xi_1| > \varepsilon_n b_n\} \to_P 0. \tag{29}$$

Moreover, either $b_n/n^{1/2}$ is slowly varying with

$$v_{n,\omega}/b_n \to_P 1, \qquad \frac{a_{n,\omega} - a_n}{b_n} \to_P 0 \tag{30}$$

or b_n/n is slowly varying and

$$v_{n,\omega}/b_n \to_P 0, \qquad \mathscr{L}\left(\frac{a_{n,\omega} - a_n}{b_n}\right) \to N(0,1). \tag{31}$$

Conversely, if there exist sequences a_n, $0 < b_n \to \infty$, $\varepsilon_n \downarrow 0$, such that (29) *and either* (30) *or* (31) *hold, then* (28) *obtains.*

PROOF. It suffices to verify sufficiency and in view of (29) it is enough to establish (28) with $\Sigma_1^n X_i$ replaced by S_{nn} as defined in (3).

If (31) holds, recalling the notation of (6), (7) and setting

$$E_n = [\omega\colon P^{\omega}\{|T_{n,n,\omega} - a_{n,\omega}| > \varepsilon b_n\} > \delta], \qquad \delta > 0,$$

$$P\{E_n\} \le P\{v_{n,\omega}^2 > \delta\varepsilon^2 b_n^2\} = o(1),$$

as $n \to \infty$, whence

$$\begin{aligned}
P\left\{\frac{S_{nn}-a_n}{b_n}<x\right\} &= \int P^\omega\left\{\frac{T_{n,n,\omega}-a_n}{b_n}<x\right\}dP \\
&\le \int P^\omega\left\{\frac{T_{n,n,\omega}-a_n}{b_n}<x, \left|\frac{T_{n,n,\omega}-a_{n,\omega}}{b_n}\right|\le \varepsilon\right\}dP \\
&\quad + \int P^\omega\left\{\left|\frac{T_{n,n,\omega}-a_{n,\omega}}{b_n}\right|>\varepsilon\right\}dP \\
&\le \int P^\omega\left\{\frac{a_{n\omega}-a_n}{b_n}<x+\varepsilon\right\}dP \\
&\quad + \int_{E_n} P^\omega\left\{\left|\frac{T_{n,n,\omega}-a_{n,\omega}}{b_n}\right|>\varepsilon\right\}dP+\delta \\
&\le P\left\{\frac{a_{n\omega}-a_n}{b_n}<x+\varepsilon\right\}+P\{E_n\}+\delta,
\end{aligned}$$

implying

$$\limsup_{n\to\infty} P\left\{\frac{S_{nn}-a_n}{b_n}<x\right\}\le \Phi(x+\varepsilon)+\delta.$$

In similar fashion

$$\limsup_{n\to\infty} P\left\{\frac{S_{nn}-a_n}{b_n}\ge x\right\}\le 1-\Phi(x-\varepsilon)+\delta,$$

yielding

$$\liminf_{n\to\infty} P\left\{\frac{S_{nn}-a_n}{b_n}<x\right\}\ge \Phi(x-\varepsilon)-\delta.$$

The desired conclusion now follows from continuity of Φ and the arbitrariness of ε and δ.

Alternatively, if (30) holds, setting

$$D_n=\{\omega\colon (1-\delta)b_n\le v_{n,\omega}\le b_n(1+\delta), |a_{n,\omega}-a_n|\le \delta b_n\}, \qquad \delta>0,$$

clearly, $P\{D_n^c\}=o(1)$ as $n\to\infty$. Now

$$\begin{aligned}
P\left\{\frac{S_{nn}-a_n}{b_n}<x\right\} &= \int_\Omega P^\omega\left\{\frac{T_{n,n,\omega}-a_{n\omega}}{v_{n\omega}}<\frac{b_n}{v_{n\omega}}\left[x-\frac{(a_{n\omega}-a_n)}{b_n}\right]\right\}dP \\
&\le \int_{D_n} P^\omega\left\{\frac{T_{n,n,\omega}-a_n}{v_{n\omega}}<\frac{x+\delta}{1-\delta}\right\}dP+P\{D_n^c\} \\
&\le \int_\Omega P^\omega\left\{\frac{T_{n,n,\omega}-a_n}{v_{n\omega}}<\frac{x+\delta}{1-\delta}\right\}dP+P\{D_n^c\},
\end{aligned}$$

implying

$$\limsup_{n\to\infty} P\left\{\frac{S_{nn}-a_n}{b_n} < x\right\} \le \Phi\left(\frac{x+\delta}{1-\delta}\right).$$

Analogously,

$$\liminf_{n\to\infty} P\left\{\frac{S_{nn}-a_n}{b_n} < x\right\} \ge \Phi\left(\frac{x-\delta}{1+\delta}\right)$$

and the conclusion follows as $\delta \to 0$. □

Note that sufficiency of (29), (30) may be recast as

$$\begin{aligned} &nP^{\omega}\{|\xi_1| > \varepsilon_n b_n\} \to_P 0,\\ (32)\qquad &\frac{1}{b_n}\left[nE^{\omega}\xi_1 I_{[|\xi_1|\le \varepsilon_n b_n]} - a_n\right] \to_P 0,\\ &\frac{n}{b_n^2}\operatorname{Var}^{\omega}\left(\xi_1 I_{[|\xi_1|\le\varepsilon_n b_n]}\right) \to_P 1. \end{aligned}$$

In contradistinction to the i.i.d. case, (28) does not require a finite mean.

EXAMPLE 1. Let $\{Y, Y_n, n \ge 1\}$ be i.i.d. random variables with Y as in Theorem 4 of [6]. Then $E|Y| = \infty$ and there exist slowly varying, positive constants $b_n/n \uparrow \infty$ for which $1/b_n \Sigma_{i=1}^n Y_i \to_P -1$. Thus, if $X_n = \omega Y_n$, $n \ge 1$ where ω is a standard normal variable independent of $\{Y_n, n \ge 1\}$, clearly $\{X_n, n \ge 1\}$ is exchangeable with $((1/b_n)\Sigma_{i=1}^x X_i) \to N(0,1)$ but $E\{|x||\omega\} = \infty$ for all $\omega \ne 0$.

By slightly altering an example suggested by the referee, it is even possible for X to lack all moments despite the fact that its partial sums can be normalized to converge to a standard normal, as we now illustrate:

EXAMPLE 2. Let $\{Y_i\}$ be exchangeable random variables obtained by a 2^{-m} weighting on the distribution F_m which are symmetric and put mass $\frac{1}{2}$ at the points m^m and $-m^m$. Then

$$\begin{aligned} \lim_{n\to\infty} P\left(n^{-1/2}\sum_{i=1}^n Y_i \le x\right) &= \lim_{n\to\infty}\sum_{m=1}^{\infty} 2^{-m} P_{F_m}\left(n^{-1/2}\sum_{i=1}^n Y_i \le x\right)\\ &= \sum_{m=1}^{\infty} 2^{-m}\Phi(xm^{-m}), \end{aligned}$$

a *mixture* of normals. Now let $Z_1, Z_2, \dots,$ be i.i.d. symmetric random variables, independent of $\{Y_i\}$, such that for some b_n with $n^{-1/2}b_n \to \infty$, $\Sigma_{j=1}^n Z_j/b_n \to_L N(0,1)$. Finally, put $X_j = Y_j + Z_j$. Then $\{X_j\}$ are exchangeable, $E|X_j|^{\alpha} = \infty$ for all $\alpha > 0$, and $\Sigma_{j=1}^n X_j/b_n - \Sigma_{j=1}^n Z_j/b_n \to_{P_r} 0$ so that $\Sigma_{j=1}^n X_j/b_n \to_L N(0,1)$.

It is possible for a sequence of exchangeable random variables to belong to the domain of attraction of a normal distribution yet conditionally on ω (for almost all ω) not belong to the domain of attraction of any law:

EXAMPLE 3. Let $\{a_n\}, \{q_n\}$ be positive sequences such that

$$a_{n+1}/n^{1/2}a_n \to \infty, \qquad \sum_{n=1}^{\infty} a_n^{-2} < 1,$$

$$q_n \le 1, \qquad \sum_{n=1}^{\infty} q_n = \infty, \qquad \sum_{j=n}^{n^2} q_j \to 0, \qquad \sum_{n=1}^{\infty} q_n^2 < \infty$$

and define

$$\xi_i^{\omega} = Y_i \sum_{j=1}^{\infty} \left(j^{1/2} I_{[\omega_j = 1]} + I_{[\omega_j = 0]} \right) I_{[|Y_i| = a_j]},$$

where $\{Y_i\}$ are i.i.d. random variables with

$$P\{Y_i = \pm a_j\} = \tfrac{1}{2} a_j^2, \qquad j \ge 1, \qquad P\{Y_i = 0\} = 1 - \sum_{j=1}^{\infty} 1/a_j^2,$$

while $\{\omega_j\}$ are independent random variables, independent of $\{Y_i\}$, satisfying

$$P\{\omega_j = 1\} = q_j = 1 - P\{\omega_j = 0\}, \qquad j \ge 1.$$

If X_n, $n \ge 1$, is an exchangeable sequence which, conditionally on $\omega = (\omega_1, \omega_2, \dots,)$, is distributed as ξ_n^{ω}, $n \ge 1$, then setting

$$b_n = \sup\{b \colon nE(Y_1^2 \wedge b^2) \ge b^2\},$$

$(b_n^{-1}\Sigma_{i=1}^n X_i) \to N(0,1)$ whereas the distribution of ξ^{ω} is not in the domain of attraction of any law for almost all ω. Of course, the only possible candidate is a stable law.

To verify this, note at the outset that for all $k \ge 1$

$$\sum_{j=n}^{n^{2^k}-1} q_j = \sum_{i=1}^{k} \left(\sum_{j=n^{2^{i-1}}}^{n^{2^i}-1} q_j \right) = o(1),$$

whence there is a sequence $k_n \to \infty$ with e^{k_n} slowly varying and

$$\sum_{j=n}^{n^{2^{k_n}}-1} q_j = o(1) \quad \text{or} \quad \sum_{j=[n^{1/k_n}]}^{n-1} q_j = o(1).$$

Define

$$B_n = \bigcap_{j=[n^{1/k_n}]}^{n-1} \{\omega_j = 0\}, \qquad A_n = \{\omega_n = 1\}.$$

PROPOSITION 1. $P\{A_n B_n, \text{ i.o.}\} = 1$.

PROOF. Since $\Sigma_{n=1}^{\infty} P\{A_n\} = \Sigma_{n=1}^{\infty} q_n = \infty$, $P\{A_n, \text{ i.o.}\} = 1$ and so $P\{\bigcup_{j=n}^{j_n} A_j\} > 1 - 1/n$ for some integer $j_n > n$, $n \ge 1$. In view of $P\{B_n\} =$

$\prod_{j=n^{1/k_n}}^{n-1}(1 - q_j) \sim \exp\{-\sum_{n^{1/k_n}}^{n-1} q_j\} \to 1$, setting $\tau_n = \sup\{j\colon\ \omega_j = 1$ for $n \le j \le j_n\}$,

$$P\left\{\bigcup_{j=n}^{j_n} A_j B_j\right\} \ge P\{A_{\tau_n} B_{\tau_n}\} = \sum_{j=n}^{j_n} P\{\tau_n = j,\, B_j\} = \sum_{j=n}^{j_n} P\{\tau_n = j\} P\{B_j\}$$

$$\ge (1 + o(1)) P\left\{\bigcup_{n}^{j_n} A_j\right\} = 1 + o(1)$$

so that $P\{A_n B_n,\ \text{i.o.}\} = 1$. □

PROPOSITION 2. *For almost all* ω, *the distribution of* ξ^ω *is not in the domain of attraction of any (stable) law whereas the distribution of* Y *is in this domain.*

PROOF. For fixed ω (in a set of probability one), there is an increasing random sequence j_n such that $A_{j_n} B_{j_n}$ occurs, $n \ge 1$. Since $\omega_{j_n} = 1$ and $\omega_i = 0$, $j_n^{1/k_{j_n}} \le i < j_n$,

$$E_\omega \xi^2 I_{[\xi^2 \le (j_n - 1/n) a_{j_n}^2]} = E_\omega \xi^2 I_{[\xi^2 \le a_{j_n}^2 - 1]}$$

$$\le E_\omega \left[\sum_{i=1}^{j_n^{1/k_{j_n}}} i a_i^2 I_{[|Y| = a_i]} + \sum_{j_n^{1/k} j_n}^{j_n - 1} a_i^2 I_{[|Y| = a_i]}\right]$$

$$= \sum_{i=1}^{j_n^{1/k_{j_n}}} i + \sum_{j_n^{1/k_{j_n}}}^{j_n} 1 \sim j_n,$$

$$\frac{(j_n - 1/n) a_{j_n}^2 P^\omega\{\xi^2 > (j_n - 1/n) a_{j_n}^2\}}{E_\omega \xi^2 I_{[\xi^2 \le (j_n - 1/n) a_{j_n}^2]}} \ge \frac{(j_n - 1) a_{j_n}^2 P\{|Y| = a_{j_n}\}}{j_n} \to 1,$$

whereas

$$\frac{(j_n + 1/n) a_{j_n}^2 P^\omega\{\xi^2 > (j_n + 1/n) a_{j_n}^2\}}{E_\omega \xi^2 I_{[\xi^2 \le (j_n + 1/n) a_{j_n}^2]}} \sim \frac{j_n a_{j_n}^2 / a_{j_n+1}^2}{2 j_n} \to 0,$$

implying the initial portion of the proposition. Apropos of the latter part, for $a_{n-1} \le t < a_n$

$$\frac{t^2 P\{|Y| > t\}}{E Y^2 I_{[|Y| \le t]}} \le \frac{2t^2 / a_n^2}{n - 1} \le \frac{2}{n - 1} \to 0.$$ □

Define an increasing sequence of integers $m_n \to \infty$ so that

$$a_n \le b_{m_n} < a_{n+1}.$$

PROPOSITION 3.

$$b_{m_n}^{-1} \sum_{i=1}^{m_n} X_i I_{[X_i^2 \le n^{1/k_n} a_{[n^{1/k_n}]}^2]} \to_P 0$$

and

$$b_{m_n}^{-1} \sum_{i=1}^{m_n} Y_i I_{[|Y_i| \le a_{[n^{1/k_n}]}]} \to_P 0.$$

Proof. Denoting the former by U_n and the latter by V_n,

$$EU_n^2 = Eb_{m_n}^{-2} m_n E_\omega(\xi^\omega)^2 I_{[|Y| \le a_{[n^{1/k_n}]}]} \le b_{m_n}^{-2} m_n \sum_{i=1}^{n^{1/k_n}} i$$

$$\le \frac{n^{2/k_n}}{E\left(Y^2 \wedge b_{m_n}^2\right)} \le \frac{n^{2/k_n}}{EY_1^2 I_{[|Y| \le a_n]}} = \frac{n^{2/k_n}}{n} = o(1)$$

and

$$EV_n^2 = b_{m_n}^{-2} m_n EY^2 I_{[|Y| \le a_{[n^{1/k_n}]}]} \le \frac{EY^2 I_{[|Y| \le a_{[n^{1/k_n}]}]}}{EY^2 I_{[|Y| \le a_n]}} = \frac{[n^{1/k_n}]}{n} = o(1). \qquad \square$$

Proposition 4. *$\mathscr{L}(b_{m_n}^{-1}\Sigma_{i=1}^{m_n} Y_i) \to N(0,1)$ and there is a sequence $\delta_n \downarrow 0$ for which $m_n P\{|Y| > \delta_n b_{m_n}\} = o(1)$ as $n \to \infty$.*

Proof. The first assertion is immediate via Proposition 2 while $P\{|Y| > t\} \le 2t^{-2}$ for all large t ensures

$$m_n P\{|Y| > \delta b_{m_n}\} \le \frac{2m_n}{\delta^2 b_{m_n}^2} = \frac{2}{\delta^2 E\left(Y^2 \wedge b_{m_n}\right)^2} \le \frac{2}{\delta^2 EY^2 I_{[|Y| \le b_{m_n}]}} \to 0,$$

yielding the second. $\square$

Proposition 5. $\mathscr{L}(b_n^{-1}\Sigma_{i=1}^n X_i) \to N(0,1)$.

Proof.

$$P\left\{\bigcup_{i=1}^{m_n}\left[X_i I_{[|X_i| \ge a_{[n^{1/k_n}]}]} \ne Y_i I_{[|Y_i| \ge a_{[n^{1/k_n}]}]}\right]\right\}$$

$$\le P\left\{B_n^c \cup \bigcup_{i=1}^{m_n}\left[|Y_i| > \delta_n b_{m_n}\right]\right\} \le P\{B_n^c\} + m_n P\{|Y| > \delta_n b_{m_n}\}$$

$$= o(1),$$

whence via Propositions 3 and 4

$$\mathscr{L}\left(b_{m_n}^{-1}\sum_{i=1}^{m_n} X_i\right) \to N(0,1).$$

However, for every subsequence $\{n'\}$ of the positive integers there exists a further subsequence m_n' such that if $a_k \le b_{m_n'} < a_{k+1}$ then $b_{m_{n+1}'} \ge a_{k+1}$. Hence, from the above

$$\mathscr{L}\left(b_{m_n'}^{-1}\sum_{i=1}^{m_n'}X_i\right)\to N(0,1).$$

Consequently,

$$\mathscr{L}\left(b_n^{-1}\sum_{i=1}^{n}X_i\right)\to N(0,1). \qquad \square$$

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DEPARTMENT OF STATISTICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720

DEPARTMENT OF STATISTICS
RUTGERS UNIVERSITY
NEW BRUNSWICK, NEW JERSEY 08903