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ON THE EXISTENCE OF THE ERGODIC HILBERT TRANSFORM

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Let u be a unitary operator acting in $\mathbb{L}_2(\Omega, F, p)$, where p is a probability measure. We prove that the limit $\lim_{n\to\infty}\Sigma_{0<|k|\le n}u^k f/k$ exists almost surely, for every $f \in \mathbb{L}_2(\Omega, F, p)$ if and only if the limit $\lim_{n\to\infty} n^{-1}\Sigma_{k=0}^{n-1}u^k f$ exists almost surely, for every $f \in \mathbb{L}_2(\Omega, F, p)$.

Let (Ω, F, p) be a probability space and let $\alpha: \Omega \to \Omega$ be a measure preserving automorphism of Ω. Cotlar [2] proved (as a part of a more general result) that the limit

$$(*) \qquad \lim_{n\to\infty} \sum_{0<|k|\le n} \frac{f(\alpha^k\omega)}{k}$$

exists with probability 1, for every function $f \in \mathbb{L}_1(\Omega, F, p)$. This limit is called the ergodic Hilbert transform of f. The most direct and elementary proof of the above result was given recently by Petersen [5] (cf. also [6]). In comparison with the (Garsia's) proof of the individual ergodic theorem, the known proofs of the existence of the ergodic Hilbert transform $(*)$ are much more complicated. That is why it seems to be interesting to clarify the connection between the ergodic Hilbert transform and the ergodic averages. In this note we compare the conditions for the existence of the limits

$$(1) \qquad \lim_{n\to\infty} \sum_{0<|k|\le n} \frac{u^k f}{k}$$

and

$$(2) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} u^k f,$$

where u is a unitary operator in $\mathbb{L}_2$ and $f \in \mathbb{L}_2$. Our main result is the following.

THEOREM 1. *The limit* (1) *exists for all* $f \in \mathbb{L}_2$ *almost surely if and only if the limit* (2) *exists for all* $f \in \mathbb{L}_2$ *almost surely.*

The proof of this theorem will be reduced to a characterization of the a.e. convergence of (1) in terms of the spectral measure of u. Let us recall first a result of Gaposhkin [4] of this type concerning the ergodic averages.

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THEOREM 2 [4]. *Let $f \in \mathbb{L}_2$. The limit* (2) *exists a.e. if and only if*

$$\lim_{n\to\infty} V(I_n)f = 0 \quad a.e., \tag{3}$$

where $I_n = \{t\colon 0 < |t| < 2^{-n}\}$ and V is the spectral measure of u in the representation

$$u = \int_{-\pi}^{\pi} e^{it}V(dt). \tag{4}$$

We shall prove the following analogue of this result.

THEOREM 3. *Let $f \in \mathbb{L}_2$. The limit* (1) *exists a.e. if and only if*

$$\lim_{n\to\infty} \left[V(I_n') - V(I_n'')\right] f = 0 \quad a.e., \tag{5}$$

where $I_n' = \{t\colon -2^{-n} < t < 0\}$ and $I_n'' = \{t\colon 0 < t < 2^{-n}\}$.

Theorem 1 follows immediately from the last two theorems just formulated. Indeed, put $P = V[-\pi, 0)$, $Q = V(0, \pi]$, $P_n = V(I_n')$, $Q_n = V(I_n'')$. Assume that, for all $f \in \mathbb{L}_2$, the limit (1) exists a.e. Fix some $f \in \mathbb{L}_2$. Then we have, by Theorem 3,

$$P_n f = P_n(Pf) = (P_n - Q_n)Pf \to 0 \quad \text{a.e.}$$

In the same way we show that $Q_n f \to 0$ a.e. Consequently, $V(I_n)f = (P_n + Q_n)f \to 0$ a.e., which implies (by Theorem 2) the existence of the limit (2) almost surely. In a similar way we show the converse implication.

PROOF OF THEOREM 3. Let us fix an arbitrary function $f \in \mathbb{L}_2$. Let

$$s_n(t) = \sum_{k=1}^{n} \frac{\sin kt}{k}, \qquad \sigma_n(t) = \sum_{k=n+1}^{\infty} \frac{\sin kt}{k}. \tag{6}$$

We have

$$s_n(t) + \sigma_n(t) = -t/2 + \pi/2(\operatorname{sgn} t) \quad \text{for } |t| \le \pi. \tag{7}$$

Put

$$\gamma_n = \sum_{0<|k|\le n} \frac{u^k f}{k} = \int_{-\pi}^{\pi} s_n(t)M(dt), \tag{8}$$

where $M(\cdot) = 2iV(\cdot)f$. Let

$$\Gamma_n = \int_{-\pi}^{\pi} \sigma_n(t)M(dt), \tag{9}$$

and let us remark that $\lim_{n\to\infty}\gamma_n$ exists a.e. if and only if

$$\lim_{n\to\infty} \Gamma_n = 0 \quad \text{a.e.} \tag{10}$$

Indeed, (7) implies that $\lim_n \gamma_n$ exists a.e. if and only if $\lim_n \Gamma_n$ exists a.e. Moreover, the sums (6) are uniformly bounded in the interval $[-\pi, \pi]$ (see, for example, [9], page 183). This implies that

$$\|\Gamma_n\|_2^2 = \int_{-\pi}^{\pi} \sigma_n(t)^2 \|M(dt)\|_2^2 \to 0 \quad \text{as } n \to \infty.$$

Consequently, if $\lim_n \Gamma_n$ exists a.e. then $\Gamma_n \to 0$ a.e.

We shall show that (10) is equivalent to

$$\lim_{n\to\infty} \Gamma_{2^n} = 0 \quad \text{a.e.} \tag{11}$$

In order to do this let us notice that the asymptotics of the kernel $\sigma_n(t)$ are the same as those of $(1/n)\ \sum_{k=0}^{n-1} e^{ikt}$, which have been studied by Gaposhkin [4]. Namely, we have

$$|\sigma_n(t)| \le \frac{C}{n|t|} \quad \text{for } 0 < |t| \le \pi,\ n = 1,2,\dots, \tag{12}$$

and, for $n > m$ and all t, we have

$$|\sigma_n(t) - \sigma_m(t)| = |s_n(t) - s_m(t)| \le \frac{n-m}{n}, \tag{13}$$

$$|\sigma_n(t) - \sigma_m(t)| \le C(n-m)|t| \tag{14}$$

(as a rule, the constants C are different in different formulae).

Indeed, there exists a constant $C > 0$ such that $|\sum_{k=1}^{n} \sin kt| \le C/|t|$ for all positive integers n and $0 < |t| \le \pi$. Using the Abel transformation of $\sum_{k=n}^{m} \sin kt/k$, we easily obtain (12). Put

$$\delta_n = \max_{1 \le k < 2^n} |\Gamma_{2^n+k} - \Gamma_{2^n}|. \tag{15}$$

The formulae (12)–(14) and the spectral decomposition (9) make it possible to use the dyadic expansion method well-known in the theory of orthogonal series ([1], [7]) to show that

$$\sum_{n=1}^{\infty} \|\delta_n\|_2^2 < \infty \tag{16}$$

holds. The proof of the last formula can be obtained (mutatis mutandis) from the estimations provided by Gaposhkin in [4] (compare Gaposhkin's reasoning in the proof of Theorem 1 leading to the inequality $\sum \|\delta_n^*\|^2 < \infty$). Formula (16) implies $\delta_n \to 0$ a.e., which gives the equivalence of (10) and (11). Now, let us put

$$\begin{aligned}\alpha_n = &\int_{\{0<|t|<2^{-n}\}} (\sigma_{2^n}(t) - \pi/2(\operatorname{sgn} t))M(dt) \\ &+ \int_{\{2^{-n} \le |t| \le \pi\}} \sigma_{2^n}(t) M(dt)\end{aligned} \tag{17}$$

and let $\mu(\cdot) = \|M(\cdot)\|_2^2$. Then, we have

$$\sum_{n=1}^{\infty} \|\alpha_n\|_2^2 \le C\left[\sum_{n=1}^{\infty}\left(2^{2n}\int_{\{0<|t|<2^{-n}\}}|t|^2\mu(dt) + 2^{-2n}\int_{\{2^{-n}\le|t|\le\pi\}}|t|^{-2}\mu(dt)\right)\right] < \infty. \tag{18}$$

Indeed, it is enough to use (12) and the inequality

$$\begin{aligned}|\sigma_n(t) - \pi/2(\operatorname{sgn} t)| &\le |\sigma_n(t) - (\pi/2(\operatorname{sgn} t) - t/2)| + |t/2| \\ &= |s_n(t)| + |t/2| \le Cn|t|.\end{aligned}$$

Consequently, $\alpha_n \to 0$ a.e. Thus, $\Gamma_{2^n} \to 0$ a.e. if and only if

$$\int_{\{0<|t|<2^{-n}\}} (\operatorname{sgn} t) M(dt) \to 0 \quad \text{a.e.,} \tag{19}$$

which is equivalent to (5). Since $\delta_n \to 0$ a.e., the proof is complete. □

COROLLARY. *For a unitary operator in* $\mathbb{L}_2(\Omega, F, p)$, *each of the following conditions is sufficient for the existence of the ergodic Hilbert transform*

$$\sum_{k=-\infty}^{\infty}{}' \frac{u^k f}{k} = \lim_{n\to\infty} \sum_{0<|k|\le n} \frac{u^k f}{k} \quad a.e.,$$

for every $f \in \mathbb{L}_2$:

(a) $uf \ge 0$ *a.e. for* $f \ge 0$ *a.e.*

(b) *There is a nonnegative function* $g \in \mathbb{L}_1(\Omega, F, p)$ *such that*

$$\int_\Omega |u^{-k} f|\, dp \le \int_\Omega g|f|\, dp, \quad \textit{for all } f \in \mathbb{L}_\infty(\Omega, F, p) \textit{ and } k \ge 1.$$

This follows immediately, by Theorem 1, from the well-known ergodic theorems of Stein ([8], page 87) and Duncan [3].

REMARK. A similar argument can be used to establish the following continuous parameter version of Theorem 1.

For a one-parameter weakly measurable unitary group $(u_t, -\infty < t < \infty)$ on $\mathbb{L}_2(\Omega, F, p)$, the limit

$$\lim_{\varepsilon\searrow 0} \int_{\{\varepsilon\le|t|<1/\varepsilon\}} \frac{u_t f}{t}\, dt$$

exists almost surely, for every $f \in \mathbb{L}_2$, if and only if the limit

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t u_t f\, dt$$

exists almost surely, for every $f \in \mathbb{L}_2$.

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