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AN EXTENSION OF SPITZER'S INTEGRAL REPRESENTATION THEOREM WITH AN APPLICATION

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Using a new approach, an extended version of Spitzer's integral representation for stationary measures of a discrete branching process is obtained. This result is used to provide a complete solution to a problem in damage models satisfying a generalized Rao–Rubin condition.

1. Introduction. Consider a modified discrete branching process $\{Z_n\colon n = 0, 1, \dots\}$ with one-step transition probabilities given by

$$(1.1)\quad p_{ij} = P(Z_{n+1} = j | Z_n = i) = \begin{cases} cp_j^{(i)}, & i = 0, 1, \dots,\ j = 1, 2, \dots, \\ 1 - c + cp_0^{(i)}, & i = 0, 1, \dots,\ j = 0, \end{cases}$$

where $0 < c \le 1$ and $\{p_j^{(i)}\colon\ j = 0, 1, \dots\}$ is the i-fold convolution of some probability distribution $\{p_j\}$ having $0 < p_0 < 1$ with itself for $i > 0$ and the degenerate distribution at zero for $i = 0$. (It is seen that the process reduces to a Bienaymé–Galton–Watson branching process when $c = 1$.) Define

$$(1.2)\qquad m = \sum_{j=1}^{\infty} jp_j, \qquad m^* = \sum_{j=1}^{\infty} (j \log j) p_j.$$

In this paper, we extend Spitzer's integral representation for stationary measures of a Bienaymé–Galton–Watson process with $m < 1$ and $m^* < \infty$ to a branching process of the type defined in (1.1). Such an extension could be arrived at by using arguments similar to those of Spitzer (1967) based on the potential theory of Markov chains. However, we give a new approach based on Bernstein's theorem on absolutely monotonic functions. (For the definition of an absolutely monotonic function and the relevant details concerning Bernstein's theorem, see Widder (1946; Chapter IV).) We use the extended version of Spitzer's theorem to obtain a complete identification of the solution to a certain functional equation in damage models considered earlier by Talwalker (1980) and Rao, Srivastava, Talwalker and Edgar (1980), and thus establish a link between branching processes and damage models.

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2. An extended version of Spitzer's integral representation. We establish the following theorems.

THEOREM 1. *The g.f. (generating function)* $U(s) = \Sigma \eta_j s^j$ *of any stationary measure* $\{\eta_j\}$ *of a modified branching process defined in* (1.1) *is analytic for* $|s| < q$, *where* q *is the smallest positive root of the equation* $s = f(s)$, *and (if normalized so that* $U(p_0) = 1$) *satisfies the equation*

$$cU(f(s)) = c + U(s), \tag{2.1}$$

where $f(s)$ *is the g.f. of* $\{p_j\}$. *Conversely, if* $U(s) = \Sigma \eta_j s^j$, $\eta_j \geq 0$, $|s| < q$ *satisfies* (2.1), *then* $\{\eta_j\}$ *is a stationary measure.*

PROOF. If $U(s)$ is analytic for $|s| < s_0$ for some $s_0 > 0$, then

$$\begin{aligned} U(s) &= \sum_{j=1}^{\infty} \eta_j s^j = \sum_{i=1}^{\infty} \eta_i \sum_{j=1}^{\infty} p_{ij} s^j \\ &= c \sum_{i=1}^{\infty} \eta_i \big([f(s)]^i - [f(0)]^i \big) \\ &= c\big(U(f(s)) - U(p_0)\big), \qquad |s| < s_0, \end{aligned}$$

which implies that (2.1) is valid at least for $|s| < s_0$. Then using the arguments exactly as in A.N. (Theorem 2, page 68) [where we use the abbreviation A.N. for the book by Athreya and Ney (1972)], we find that $U(s)$ is analytic for $|s| < q$. The converse assertion easily follows by equating the coefficients of s^j in (2.1). □

THEOREM 2. (*An extended version of Spitzer's theorem.*) *If* $m < 1$ *and* $m^* < \infty$ *with* f *as in Theorem* 1, *then for every probability measure* ν *on* $[0, 1)$

$$U(s) = K \int_{[0,1)} U(s,t)\, d\nu(t) \tag{2.2}$$

is the g.f. of a stationary measure, where

$$U(s,t) = \sum_{n=-\infty}^{\infty} \big[\exp\{(\mathscr{B}(s) - 1)m^{n-t}\} - \exp\{-m^{n-t}\}\big] c^{n-t},$$

with $\mathscr{B}(s)$ *as the unique p.g.f. among those vanishing at* $s = 0$ *and satisfying the equation*

$$\mathscr{B}(f(s)) = m\mathscr{B}(s) + 1 - m, \tag{2.3}$$

and K the appropriate normalizing constant. Conversely, every stationary measure has the representation (2.2) *for some probability measure* ν *on* $[0, 1)$. (*For an interpretation of* $\mathscr{B}$ *in Bienaymé–Galton–Watson branching processes* (*see A.N., page* 17).)

PROOF. The first part of the theorem is easily verified. To prove the converse, it is sufficient to establish that the representation is valid for every

$s \in [0,1)$. Define then for every $s \in [0,1)$

$$U^*(s) = U(\mathscr{B}^{-1}(s)),$$

where $\mathscr{B}$ is as mentioned in the statement of Theorem 2. In view of Theorem 1 and Equation (2.3), we have

$$(2.4)\quad c^n U^*(m^n s + 1 - m^n) = U^*(s) + \xi_n, \qquad n = 1,2,\dots,\ 0 \le s < 1,$$

where

$$\xi_n = \begin{cases} c\left(\dfrac{1-c^n}{1-c}\right), & \text{if } c \ne 1, \\ n, & \text{if } c = 1. \end{cases}$$

We can write (2.4) also as

$$U^*(s) = c^n U(1 + m^n Q_n(\mathscr{B}^{-1}(s))) - \xi_n, \qquad n = 1,2,\dots,\ 0 \le s < 1,$$

where $Q_n(s) = [f_n(s) - 1]/m^n$ with $f_n(s)$ as the nth iterate of $f(s)$. Consequently

$$(2.5)\quad U^*(s) = \lim_{n\to\infty} \{c^n U(1 + m^n Q_n(0) - m^n Q_n(0)\mathscr{B}_n(\mathscr{B}^{-1}(s))) - \xi_n\}$$

for $0 \le s < 1$, where $\mathscr{B}_n(s) = [Q_n(0) - Q_n(s)]/Q_n(0)$. If $0 \le s_1 < s < s_2 < 1$, then noting in particular that $\mathscr{B}_n \to \mathscr{B}$ pointwise (see A.N., page 47), and $\mathscr{B}$ and $\mathscr{B}^{-1}$ are strictly increasing on $[0,1)$, we find that

$$(2.6)\quad \begin{aligned} -m^n Q_n(0)\mathscr{B}_n(\mathscr{B}^{-1}(s_1)) &< -m^n Q_n(0)s = -m^n Q_n(0)\mathscr{B}(\mathscr{B}^{-1}(s)) \\ &< -m^n Q_n(0)\mathscr{B}_n(\mathscr{B}^{-1}(s_2)) \end{aligned}$$

for large enough n. Since U is increasing on $[0,1)$, we obtain from (2.5) using (2.6),

$$(2.7)\quad \begin{aligned} U^*(s_1) &\le \liminf_{n\to\infty} \{c^n U(1 + m^n Q_n(0) - m^n Q_n(0)s) - \xi_n\} \\ &\le \limsup_{n\to\infty} \{c^n U(1 + m^n Q_n(0) - m^n Q_n(0)s) - \xi_n\} \le U^*(s_2). \end{aligned}$$

It is seen that $U^*(s)$ is continuous and $U^*(0) = 0$, and hence from (2.7), in particular, we have

$$(2.8)\quad \begin{aligned} U^*(s) &= \lim_{n\to\infty} \{c^n U(1 + m^n Q_n(0) - m^n Q_n(0)s) - \xi_n\} \\ &= \lim_{n\to\infty} \{c^n[U(1 + m^n Q_n(0) - m^n Q_n(0)s) - U(1 + m^n Q_n(0))]\} \end{aligned}$$

for $s \in [0,1)$, with the limits well defined. Since for each $n \ge 1$, the expression within the second limit in (2.8) can be expressed as the g.f. of some nonnegative sequence for $s \in [0,1)$, the extended continuity theorem given in Feller (1971, page 433) implies that U^* is the g.f. of a nonnegative sequence. Define now a function $\hat{U}$ on $(-\infty,1)$ such that its restriction to $[0,1)$ is indeed U^* and for $s \in (-\infty,0)$,

$$\hat{U}(s) = c^n U^*(m^n s + 1 - m^n) - \xi_n,$$

where n is the smallest integer for which $m^n s + 1 - m^n \in [0,1)$. It is easily seen that

$$\text{(2.9)}\quad c^n\hat{U}(m^n s + 1 - m^n) = \hat{U}(s) + \xi_n, \qquad n = 1,2,\dots, s \in (-\infty, 1),$$

which, since $\hat{U} = U^*$ on $[0,1)$, implies that $\hat{U}'$ (i.e., $d\hat{U}(s)/ds$, $s \in (-\infty,1)$) exists and is absolutely monotonic on $(-\infty,1)$ with $\lim \hat{U}'(s) = 0$ as $s \to -\infty$. Since $\hat{U}(0) = 0$, we conclude from Bernstein's theorem that for some measure μ on $(0,\infty)$

$$\text{(2.10)}\qquad \hat{U}(s) = \int_{(0,\infty)} (e^{sx} - 1)\, d\mu(x), \qquad s \in (-\infty, 1).$$

From (2.9), we find that the measure μ is such that

$$\text{(2.11)}\qquad c^n \int_{(0,\infty)} \left(e^{sxm^n + (1-m^n)x} - 1\right) d\mu(x) = \int_{(0,\infty)} (e^{sx} - 1)\, d\mu(x) + \xi_n,$$
$$n = 1,2,\dots, s \in (-\infty,1).$$

If $s_0 \in (-\infty,1)$ and $s \in (-\infty,0]$, then subtracting the identity (2.11) from the corresponding identity with s replaced by $s + s_0$, we obtain

$$\text{(2.12)}\quad c^n \int_{(0,\infty)} e^{sxm^n}(e^{s_0xm^n} - 1)e^{(1-m^n)x}\, d\mu(x) = \int_{(0,\infty)} e^{sx}(e^{s_0x} - 1)\, d\mu(x),$$

which is valid for all $s \in (-\infty,0]$. Then, in view of the uniqueness theorem for Laplace–Stieltjes transforms, (2.12) implies

$$\text{(2.13)}\quad \int_{[m^n, m^{n-1})} (e^{s_0x} - 1)\, d\mu(x) = c^n \int_{[1, m^{-1})} (e^{s_0xm^n} - 1)e^{(1-m^n)x}\, d\mu(x)$$

for $n = 0, \pm1, \pm2, \dots$. Since s_0 in (2.13) is arbitrary, we conclude using (2.10) that

$$\text{(2.14)}\quad \begin{aligned} U^*(s) = \hat{U}(s) &= \sum_{n=-\infty}^{\infty} \int_{[m^n, m^{n-1})} (e^{sx} - 1)\, d\mu(x) \\ &= \int_{[1,m^{-1})} \sum_{n=-\infty}^{\infty} c^n(e^{-(1-s)xm^n} - e^{-xm^n})e^x\, d\mu(x) \\ &= K\int_{[0,1)} \left\{ \sum_{n=-\infty}^{\infty} c^{n-t}\left(e^{-m^{n-t}(1-s)} - e^{-m^{n-t}}\right)\right\} d\nu(t) \end{aligned}$$

for $s \in [0,1)$, where K is a positive constant and ν is a probability measure on $[0,1)$ such that for every Borel subset A of $[0,1)$ we have

$$\text{(2.15)}\qquad \nu(A) = K^{-1} \int_{S_A} e^x c^{-\log x/\log m}\, d\mu(x),$$

where $S_A = \{x\colon (-\log x/\log m) \in A\}$. [The operation of interchanging the order of summation and integration in (2.14) is justified by either Fubini's theorem or the monotone convergence theorem.] The required result now follows on observing that $U(s) = U^*(\mathscr{B}(s))$, $s \in [0,1)$. □

REMARK 1. In Definition 2 on page 432 of Feller (1971), $\overline{0, \infty}$ should be changed to $\overline{|0, \infty}$ to make the extended continuity theorem given on page 433 nonambiguous. [We used this theorem in the proof of our Theorem 2.]

REMARK 2. If we define $G(i, j)$ to be the Green function corresponding to the substochastic matrix (p_{ij}), $i, j = 1, 2, \dots,$ then an argument essentially as in A.N. (page 70) implies that for every subsequence of positive integers $\{k_i\}$ such that the fractional part of $(-\log k_i)/\log m \to t$ with $t \in [0, 1)$ we have

$$\frac{\sum_{j=1}^{\infty} G(k_i, j) s^j}{G(k_i, 1)} \to U^*(s, t) = \frac{\tilde{U}(s, t)}{\tilde{U}'(0, t)},$$

where $\tilde{U}(s, t)$ is the $U(s, t)$ of Theorem 2 with m^{n-t} replaced by $(-Q(0)m^{n-t})$, $Q(0)$ being as defined in A.N. (page 40), and

$$\tilde{U}'(0, t) = \left.\frac{\partial \tilde{U}(s, t)}{\partial s}\right|_{s=0}.$$

We note that the proof in A.N. (page 69) remains valid in the present case as well with $U(s, t)$ changed to $U^*(s, t)$. Consequently, the Poisson–Martin integral representation for a stationary measure analogous to the one given in Seneta (1973, page 151) for a super regular vector yields the validity of our Theorem 2 with $U(s, t)$ replaced by $\tilde{U}(s, t)$. Now, it is easy to verify that Theorem 2 remains valid even when $Q(0)$ is not necessarily equal to -1 and hence we have an alternative proof of Theorem 2.

REMARK 3. The version of Theorem 2 with $U(s, t)$ replaced by $\tilde{U}(s, t)$ of Remark 2 yields Spitzer's theorem when $c = 1$.

COROLLARY. *If $f(s) = 1 - m + ms$, then every sequence $\{\eta_j\}$ is a stationary measure iff it is of the form*

$$\eta_j = K \sum_{n=-\infty}^{\infty} \int_{[0,1)} c^{n-t} e^{-m^{n-t}} \frac{m^{(n-t)j}}{j!} \, d\nu(t), \qquad j = 1, 2, \dots, \tag{2.16}$$

where ν is a probability measure on $[0, 1)$, and K is a positive constant as in Theorem 2.

The result (2.16) is obvious from Theorem 2 in view of the fact that $\mathscr{B}(s) = s$.

3. An application to damage models. Let (X, Y) and (X', Y') be two-vector random variables with nonnegative integer-valued components such that X and X' have the same marginal distribution $\{g_n\}$ with $g_0 < 1$, and Y and Y' are such that for each n with $g_n > 0$,

$$P(Y = r | X = n) = \binom{n}{r} \pi^r (1 - \pi)^{n-r}, \qquad r = 0, 1, \dots, n,$$

$$P(Y' = r | X' = n) = \binom{n}{r} \pi'^r (1 - \pi')^{n-r}, \qquad r = 0, 1, \dots, n,$$

where $0 < \pi, \pi' < 1$ are fixed. Talwalker (1980) and Rao, Srivastava, Talwalker and Edgar (1980) considered the problem of characterizing the distribution of X or X' by the equation

$$P(Y = r) = P(Y' = r|X' = Y'), \qquad r = 0, 1, \dots, \tag{3.1}$$

which is an extended version of the Rao–Rubin condition (Rao and Rubin (1964)). It is seen that (3.1) is equivalent to

$$G(1 - \pi + \pi s) = G(\pi' s)/G(\pi'), \qquad |s| \le 1 \tag{3.2}$$

where G is the g.f. of $\{g_n\}$.

When $\pi = \pi'$, (3.1) is the Rao–Rubin condition, and it is shown by Rao and Rubin (1964) using Bernstein's theorem and in a simpler way by Shanbhag (1974), that in this case G is the p.g.f. of a Poisson distribution. When $\pi > \pi'$, it is easily seen that G is the p.g.f. of a binomial distribution with an arbitrary index and success probability $(\pi - \pi')/\pi(1 - \pi')$.

When $\pi < \pi'$, the picture is totally different, and the family of distributions $\{g_n\}$ for which (3.1) holds is somewhat curious and fairly large. Talwalker (1980) and Rao, Srivastava, Talwalker and Edgar (1980) identified the family as a mixture of Poisson distributions with the mixing measure itself satisfying a further functional equation. The following theorem gives a complete identification of the solution and provides a more satisfactory answer and a rigorous proof to the characterization problem than that given earlier.

THEOREM 3. *Let (X, Y) and (X', Y') be two-vector random variables as considered above with $0 < \pi < \pi' < 1$. Then (3.1) is valid iff*

$$g_j = K \sum_{n=-\infty}^{\infty} \int_{[0,1)} c^{n-t} e^{-(\pi/\pi')^{n-t}} \frac{(\pi/\pi')^{(n-t)j}}{j!} \left[\frac{\pi' - \pi}{\pi'(1 - \pi)}\right]^j d\nu(t), \qquad j = 0, 1, \dots,$$

where ν is a probability measure on $[0, 1)$ and c is a real number lying in $(0, 1)$ and K is a normalizing constant.

The result follows from the corollary to Theorem 2 (see (2.16)) since (3.1) is equivalent to (3.2), which can be written as

$$cU(1 - m + ms) = U(s) + cU(1 - m)$$

with

$$g_0 = \frac{c}{1 - c} U(1 - m), \qquad m = \frac{\pi}{\pi'},$$

$$U(s) = G\left(\frac{(1 - \pi)\pi'}{\pi' - \pi} s\right) - g_0.$$

(Observe that (3.2) implies that the p.g.f. $G(s)$ is defined for all s such that $|s| < \pi'(1 - \pi)/(\pi' - \pi)$ and it satisfies the equation in (3.2) for every

$$s \in \left(-\frac{\pi'(1 - \pi)}{\pi' - \pi}, \frac{\pi'(1 - \pi)}{\pi' - \pi}\right).)$$

REMARK 4. If the measure ν in Theorem 3 is taken as the Lebesgue measure on $[0,1)$, then the distribution $\{g_n\}$ in question reduces to a negative binomial distribution.

REMARK 5. It is interesting to note that if a p.g.f. G satisfies (3.2) simultaneously for two pairs (π_i, π_i'), $i = 1,2$, where $0 < \pi_i < \pi_i' < 1$ and $(\log \pi_1 - \log \pi_1')/(\log \pi_2 - \log \pi_2')$ is irrational, then G is the p.g.f. of a negative binomial distribution of the form

$$G(s) = \frac{\{[\pi_1'(1-\pi_1)/(\pi_1'-\pi_1)] - 1\}^{\alpha}}{\{[\pi_1'(1-\pi_1)/(\pi_1'-\pi_1)] - s\}^{\alpha}}$$

for some $\alpha > 0$. Since the condition implies G to be well defined also on $(1, s_0)$ where $s_0 = \pi_1'(1-\pi_1)/(\pi_1' - \pi_1)$, the result in question follows as a corollary to the result of Marsaglia and Tubilla (1975) by noting in particular that $f(x) = G(s_0 - s_0 e^{-x})/G(0)$, $x \ge 0$ is well defined and satisfies the equation $f(t_i + x) = f(t_i)f(x)$, $x \ge 0$, $i = 1,2$ with $t_i = \log(\pi_i'/\pi_i)$, $i = 1,2$. The same result was established in Rao, Srivastava, Talwalker and Edgar (1980) by a different and slightly more involved method.

REFERENCES

ATHREYA, K. B. and NEY, P. E. (1972). *Branching Processes.* Springer, Berlin.

FELLER, W. (1971). *An Introduction to Probability and Its Applications* **2**, 2nd ed. Wiley, New York.

MARSAGLIA, G. and TUBILLA, A. (1975). A note on the lack of memory property of the exponential distribution. *Ann. Probab.* **3** 352–354.

RAO, C. R. and RUBIN, H. (1964). On a characterization of the Poisson distribution. *Sankhyā, Ser. A* **26** 295–298.

RAO, C. R., SRIVASTAVA, R. C., TALWALKER, S. and EDGAR, G. A. (1980). Characterization of probability distributions based on generalized Rao–Rubin condition. *Sankhyā, Ser. A* **42** 161–169.

SENETA, E. (1973). *Non-negative Matrices.* George Allen and Unwin, London.

SHANBHAG, D. N. (1974). An elementary proof for the Rao–Rubin characterization of the Poisson distribution. *J. Appl. Probab.* **11** 211–215.

SPITZER, F. (1967). Two explicit Martin boundary constructions. *Symposium on Probability Methods in Analysis. Lecture Notes in Math.* **31** 296–298. Springer, Berlin.

TALWALKER, S. (1980). A note on the generalized Rao–Rubin condition and characterization of certain discrete distributions. *J. Appl. Probab.* **17** 563–569.

WIDDER, D. V. (1946). *The Laplace Transform.* Princeton Univ. Press.

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