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ON A PROBLEM OF KAHANE ABOUT THE IMAGE OF GAUSSIAN TAYLOR SERIES

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We prove that a Gaussian Taylor series takes almost surely every complex value with at most one exception. This presents an answer to a problem of Kahane. We also present an example to show some differences between Gaussian and classical Taylor series.

1. Introduction. Let $\{Z_n\}$ be a sequence of complex random variables which are independently and normally distributed in standard form (see Kahane [6], page 118). As introduced by Kahane ([6], page 125), a power series is called a Gaussian Taylor series if it can be written as

$$F(z) = \sum_{n=0}^{\infty} a_n Z_n z^n, \tag{1}$$

where $a_n > 0$, $\limsup_{n\to\infty} a_n^{1/n} = 1$, and z is a complex variable. Since $Z_n = O(\log n)^{1/2}$ a.s. (almost surely) ([6], page 121, Proposition 3), it follows that a.s. (1) admits the unit circle $C = \{z\colon |z| = 1\}$ as a natural boundary ([6], page 32, Theorem 1).

Let D be the unit disk, $C = \partial D$, and E a subset of D. We say that $F(z)$ has the recurrence property on E if for each complex number w, we have a.s. $\liminf|F(z) - w| = 0$, as $|z| \to 1$, $z \in E$, and the transience property if we have a.s. $\lim|F(z)| = \infty$, as $|z| \to 1$, $z \in E$.

With the above two definitions, Kahane has proved the following two theorems ([6], pages 132–137).

THEOREM K1. *If the sequence a_n is monotonic and satisfies*

$$\sum_{n=1}^{\infty} a_n^2 = \infty \quad \textit{and} \quad a_n = o\left(\frac{n}{\log n}\right)^{1/2},$$

then $F(z)$ has the recurrence property on any circular set (i.e., a union of circles $|z| = r$ with $r \to 1$).

THEOREM K2. *If the coefficients a_n satisfy*

$$\lim_{n\to\infty} a_n (n \log n)^{-1/2} = \infty, \tag{2}$$

then $F(z)$ has the transience property on some circular sets.

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By applying the Rouché theorem, Kahane has obtained the following:

THEOREM K3. *If* (2) *holds, then* $F(z)$ *takes a.s. every complex value.*

Clearly, if $a_n = n^\alpha$, $\alpha > \frac{1}{2}$, then the above theorem says that $F(z)$ takes a.s. every complex value. From this, Kahane ([6], page 137) asked whether the image of $F(z)$ fills the plane a.s., when $a_n = n^\alpha$, $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$. In this paper, we shall present an answer in the following affirmative sense. For simplicity, we say that $F(z)$ takes a.s. every complex value with at most one exception if the probability of $F(z)$ omitting any two complex values is zero.

THEOREM 1. *Let* $\{a_n\}$ *be a real sequence in the* $\Sigma a_n^2 = \infty$. *Then* $F(z)$ *takes a.s. every complex value with at most one exception.*

2. Preliminary lemmas. Before proving Theorem 1, we shall state some necessary results from probability and function theory. We first need the following assertions of Kahane ([6], page 126, Theorems d′ and e′), which are derived from the Paley–Zygmund inequalities ([6], page 24) and the Poisson method of summation ([6], page 26).

LEMMA 1. *If* $\Sigma a_n^2 = \infty$, *where* a_n *are real, then we have:*

(i) $F(re^{it})$ *diverges a.s. as* $r \to 1$, *for almost every* t.
(ii) *Given any real* t, $F(re^{it})$ *diverges a.s. as* $r \to 1$.

Note that the assertion (i) is an immediate consequence of (ii), but for our convenience, we list both of them here. Also notice that if the Gaussian Taylor series $F(z)$ is replaced by a standard Taylor series, then there is a series

$$f(z) = \sum a_n z^n, \quad \text{where} \quad \sum a_n^2 = \infty,$$

such that the limit $f(re^{it})$ diverges as $r \to 1$, for almost every t, but it is not necessary that for any t, $f(re^{it})$ diverges as $r \to 1$. In other words, the assertion (i) is true, but not (ii) for such a series. An example will be given at the end of this paper.

Next, we shall need a theorem of Collingwood and Cartwright (see [2], Theorem 4.2). For this, we say that a function $f(z)$ has the angular limit v at a point e^{it} on C if $f(z)$ tends to v as $z \to e^{it}$ inside any angular domain lying in D and having e^{it} as vertex. A special case is that of a radial limit; that $\lim_{r\to 1} f(re^{it}) = f(e^{it})$ exists.

LEMMA 2. *If* $f(z)$ *is analytic and omits two values in* D, *and if* $f(z)$ *tends to a limit* v *along a path* $\Gamma \subset D$, *then this path* Γ *must terminate at a point* e^{it} *on* C *and* $f(z)$ *has the angular limit* v *at* e^{it}.

Note that the notions of radial limits and angular limits are the same for functions which are analytic and bounded in D, due to Lindelöf's theorem ([2], Theorem 2.3). This together with Fatou's theorem ([2], Theorem 2.1) yields that if $f(z)$ is analytic and bounded in D then the angular limits exist almost everywhere on C.

With the above definition and remark, we can now state the following extension of Löwner's theorem, due to Ohtsuka and Tsuji (see [9], Theorem VIII.30).

LEMMA 3. *If $f(z)$ is analytic and bounded by one in D, and if $E = \{e^{it}: |f(e^{it})| = 1\}$ and $E^* = \{f(e^{it}): e^{it} \in E\}$, then both E and E^* are measurable and their measures satisfy $mE \leq mE^*$.*

Finally, we shall need the following strong form of the maximum principle (see Collingwood and Lohwater [2], Theorem 5.3).

LEMMA 4. *If $f(z)$ is analytic and bounded by M in D and if the radial limits $|f(e^{it})| \leq m < M$ almost everywhere on C, then $|f(z)| < m$ everwhere in D, unless $f(z)$ is a constant of modulus m.*

To close this section, we remark that if $f(z)$ is analytic and omits two complex values in D, then $f(z)$ is a normal function in the Montel sense, due to Lehto and Virtanen ([7], page 53). From this, we can see that the technique of Bagemihl and Seidel [1] is applicable in proving our Theorem 1, cf. Hwang [4], [5].

3. Proof of Theorem 1. Suppose on the contrary that the assertion is false; then with a positive probability the series $F(z)$ omits two complex values, say a_1 and a_2. In view of the remark made in (1), we have a.s. $F(z) \neq \infty$ in D. It follows that with a positive probability, $F(z)$ omits three values a_1, a_2, and ∞.

We now consider the function

$$(*) \qquad G(z) = 1/(F(z) - a_1), \qquad z \in D.$$

Then with a positive probability, $G(z)$ is analytic in D. We shall construct a path γ such that with a positive probability $G(z)$ tends to ∞ along γ. This together with Lemma 2 will contradict Lemma 1.

For more precise demonstration, we denote by $(\Omega, \mathscr{A}, \mathbf{P})$ the probability space and we write $F(z, Z(\omega))$ in place of $F(z)$, where $\omega \in \Omega$. Let A be the event in $\mathscr{A}$ defined by

$$A = \{\omega\colon F(z, Z(\omega)) \neq a_i(\omega), \text{for all } z \in D\},$$

where $a_i(\omega) = a_i$, a.s., $a_1 \neq a_2$ are constant, and $a_3 = \infty$. Then the probability $\mathbf{P}(A) > 0$ and the function $G(z, Z(\omega))$, where G is as in $(*)$, is analytic in D for any $\omega \in A$. To construct the desired path, we fix an $\omega \in A$ and write $G^{\omega}(z) = G(z, Z(\omega))$. We then define the region

$$R_n^{\omega} = \{z\colon z \in D \text{ and } |G^{\omega}(z)| > n\}, \qquad n = 1, 2, \dots.$$

Let z_n be a point in R_n^{ω} and let S_n^{ω} be the component of R_n^{ω} containing the point

z_n. Denote by $\bar{S}_n^\omega$ the closure of S_n^ω. Then by the maximum principle, we find that the intersection $\bar{S}_n^\omega \cap C$ is not empty.

We shall prove that the function $G^\omega(z)$ is unbounded in S_n^ω. For this, we let $S_n^\omega{}^*$ be the smallest simply connected region containing S_n^ω. Then by the Riemann mapping theorem, there is a function $z = \phi^\omega(w)$ that maps D_w ($|w| < 1$) conformally onto $S_n^\omega{}^*$. It is easy to see that $\partial S_n^\omega \cap C = \partial S_n^\omega{}^* \cap C$. Clearly, the function $\phi^\omega(w)$ is analytic and bounded by one in D_w. Let C_w be the unit circle in w plane. Then by Fatou's theorem ([2], Theorem 2.1), $\phi^\omega(w)$ has a radial limit at almost all points of C_w. As before, we let

$$E = \left\{e^{it}: |\phi^\omega(e^{it})| = 1\right\} \quad \text{and} \quad E^* = \left\{\phi^\omega(e^{it}): e^{it} \in E\right\}.$$

Then by Lemma 3, we know that $mE \leq mE^*$.

We now have two cases to be considered: either $mE > 0$ or $mE = 0$. In either case, we are going to prove that the function

$$H^\omega(w) = G^\omega(\phi^\omega(w))$$

is unbounded in D_w. Clearly, from the definition of R_n^ω, we have

$$|H^\omega(w)| > n \quad \text{for } w \in D_w, \text{ where } \phi^\omega(w) \in R_n^\omega.$$

Suppose on the contrary that $H^\omega(w)$ is bounded in D_w and assume the first case that $mE > 0$. Let E_0 be a Borel subset of positive measure of E at each point of which $H^\omega(w)$ possesses a radial limit, and let E_0^* be the image of E_0 under the mapping $z = \phi^\omega(w)$. It follows from Lemma 3 again that $0 < mE_0 \leq mE_0^*$.

Let e^{it} be an arbitrary point on E_0^* and let e^{is} be the preimage of e^{it} under the mapping $z = \phi^\omega(w)$. Denote by r_s the radius in D_w ending at e^{is} and $r_s^* = \phi^\omega(r_s)$. Since $H^\omega(w)$ has the radial limit, say v at e^{is} it follows that the function $G^\omega(z)$ tends to v along the path r_s^* and hence the function $F^\omega(z)$ tends to the value $v_1 = a_1 + 1/v$ along the path r_s^* due to (*). By Lemma 2, the path r_s^* ends at e^{it} and the function $F^\omega(z)$ has the angular limit v_1 at e^{it}. Since $mE_0^* > 0$, $\omega \in A$ is arbitrary, and $\mathbf{P}(A) > 0$, this certainly contradicts Lemma 1(i).

Turning to the second case $mE = 0$, we shall prove that the function $H^\omega(w)$ has radial limits of modulus n almost everywhere on C_w. For this, we let $e^{is} \notin E$, then the modulus of the image $|\phi^\omega(e^{is})| < 1$, so that the point $z = \phi^\omega(e^{is})$ lies on the boundary of S_n^ω. This yields that $|G^\omega(z)| = n$ or $|H^\omega(e^{is})| = n$. Since e^{is} is an arbitrary point not in E, we thus conclude that the function $H^\omega(w)$ has radial limits of modulus n almost everywhere on C_w. It follows from Lemma 4 that if the function $H^\omega(w)$ is bounded in D_w, then we have $|H^\omega(w)| < n$ for all $w \in D_w$, a contradiction. This concludes that the function $H^\omega(w)$ is unbounded in D_w; so is the function $G^\omega(z)$ in S_n^ω.

Applying the method in [4], a path γ ending at e^{it} can be constructed such that the function $G^\omega(z) \to \infty$, as $z \to e^{it}$, $z \in \gamma$, so that $F^\omega(z) \to a(\omega)$. It then follows from Lemma 2 that the radial limit

$$F^\omega(re^{it}) \to a(\omega), \quad \text{as } r \to 1, \text{ for each } \omega \in A.$$

Since the probability $\mathbf{P}(A) > 0$, the above limit contradicts Lemma 1(ii). This concludes that the function $F(z)$ takes a.s. every complex value with at most one exception and the proof is complete. □

4. Extension. As an immediate consequence of Theorem 1, we obtain the following theorem of Kahane and Zygmund ([6], page 127, Theorem 1).

THEOREM 2. *If $\Sigma a_n^2 = \infty$, then $F(z)$ has the recurrence property on D.*

Instead of the whole disk D, recently [3], we have extended Theorem 2 to the following rectifiable subregion of D.

THEOREM 3. *If $\Sigma a_n^2 = \infty$, then $F(z)$ has the recurrence property on any subregion R of D, bounded by a rectifiable Jordan curve J such that the measure $mJ \cap C > 0$.*

In contrast to Theorem 3, we shall now extend Theorem 1. Here for the definition of accessibility we refer to [2], page 168.

THEOREM 4. *If $\Sigma a_n^2 = \infty$ and if R is a simply connected subregion of D whose boundary contains an arc Γ of C such that every point of Γ is accessible, then $F(z)$ takes a.s. every complex value with at most one exception inside R.*

PROOF. The method here is the same as in Theorem 1 and we sketch the details. We first observe from the hypothesis that every point of Γ is accessible, so that the endpoints of Γ can be joined by an arc Γ' contained in R except for the endpoints. Let S be the region bounded by Γ and Γ'. Then $S \subset R$, since the subregion R is simply connected. Instead of A as defined in Theorem 1, we now define

$$A_S = \{\omega\colon F(z, Z(\omega)) \neq a_i(\omega), \text{for } z \in S\}.$$

Let e^{it} be an arbitrary interior point of Γ and $D(e^{it})$ be an arbitrary disk with center at e^{it}. Then the relative neighborhood $R^* = D(e^{it}) \cap S$ contains a Jordan region R_1 whose boundary J is rectifiable and consists of an arc of C. It follows from Theorem 3 that $F(z)$ has the recurrence property on R_1 and therefore on R^*. Thus, given a complex number a, we can choose a sequence $z_n \in R^*$ for which $F(z_n)$ tends to a a.s. as $n \to \infty$. Let $G(z) = 1/(F(z) - a)$; then the sequence $G(z_n)$ tends to ∞ a.s. as $n \to \infty$. Therefore by the same argument as in Theorem 1, there can be constructed a path γ terminating either at e^{it} or at a point $\zeta \in J$ as close to e^{it} as we please such that on the event A_S, the function $G(z)$ tends to ∞ along γ or equivalently the function $F(z)$ tends to a along γ. Since e^{it} is an arbitrary interior point of Γ, the point ζ can be required to lie on Γ, so that $\zeta \in C$. Let $\zeta = e^{i\theta}$; then by the same argument as in Theorem 1, we conclude that

$$\lim_{r \to 1} F(re^{i\theta}) = a \quad \text{on the event } A_S.$$

Since the probability $\mathbf{P}(A_S) > 0$, the above limit contradicts Lemma 1(ii). This completes the proof. □

5. Remark. We state two conjectures which would extend Theorems 1 and 4.

CONJECTURE 1. *If* $\Sigma a_n^2 = \infty$, *then* $F(z)$ *takes a.s. every complex value.*

CONJECTURE 2. *If* $\Sigma a_n^2 = \infty$ *and if* R *is a subregion of* D, *bounded by a rectifiable Jordan curve* J *such that the measure* $mJ \cap C > 0$, *then* $F(z)$ *should take a.s. every complex value with at most one exception inside* R.

Note that the same argument as in Theorem 1 can only give that if $F(z)$ omits two values a and b, then with a positive probability $F(z)$ tends to a along a path terminating at a point $\zeta \in J$. The second conjecture would be true if one could prove that such a ζ point can always be chosen on $J \cap C$.

Finally, we observe that Gaussian Taylor series behave differently than do ordinary Taylor series. Thus both conjectures fail if we replace $F(z)$ by the Taylor series

$$f(z) = \sum a_n z^n, \quad \text{where } \sum a_n^2 = \infty.$$

For instance, if $\mu(z)$ is a Schwarzian triangle function (see Montel [8], Chapter II) for which $\mu(z)$ omits these three values 0, 1, and ∞, and $\mu(x)$ can be required to be real whenever x is real, then the coefficients a_n of the Taylor series of $\mu(z)$ are real. This function $\mu(z)$ is analytic in D and omits two finite values 0 and 1. Furthermore, $\mu(z)$ can have only radial limits 0, 1, and ∞ at a countable subset of C, so that the series $\Sigma a_n^2 = \infty$. Thus each conjecture is false for Taylor series. Meanwhile, this function $\mu(z)$ shows that both Lemma 1(ii) and Theorem 4 are no longer true if the Gaussian Taylor series is replaced by Taylor series. Finally, the radial limits of $\mu(z)$ diverge almost everywhere, but converge at a countable dense subset of C. This property explains our remark which followed the statement of Lemma 1.

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