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ONE-DIMENSIONAL CIRCUIT-SWITCHED NETWORKS

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This paper is concerned with the stationary distribution of a one-dimensional circuit-switched network. We show that if arrival rates decay geometrically with distance, then under the stationary distribution the number of circuits busy on successive links of the network at a fixed point in time is a Markov chain. When each link of the network has unit capacity we show that translation invariant arrival rates lead to a stationary distribution which can be described in terms of an alternating renewal process.

1. Introduction. We begin by describing a stochastic process which can be envisaged as representing a telephone network or a circuit-switched computer communication network. There are finitely many links, labelled $k = 1, 2, \ldots, K$, and link k comprises C_k circuits. A subset $r \subset \{1, 2, \ldots, K\}$ identifies a route. Calls requesting route r arrive as a Poisson process of rate ν_r, and as r varies it indexes independent Poisson streams. A call requesting route r is blocked and lost if on any link $k \in r$ there are no free circuits. Otherwise the call is connected and simultaneously holds one circuit on each link $k \in r$ for the holding period of the call. The call holding period is independent of earlier arrival times and holding periods; holding periods of calls on route r are identically distributed with unit mean. Let $n_r(t)$ be the number of calls in progress at time t on route r, let R be the set of possible routes, and let $n(t) = (n_r(t),\ r \in R)$. Then the stochastic process $(n(t),\ t \geq 0)$ has a unique stationary distribution and under this distribution $\pi_K(n) = P\{n(t) = n\}$ is given by

$$\pi_K(n) = G^{-1} \prod_r \frac{\nu_r^{n_r}}{n_r!}, \qquad n \in N_K, \tag{1.1}$$

where

$$N_K = \left\{ n \in Z_+^R : \sum_{r:\, k \in r} n_r \leq C_k,\ k = 1, 2, \ldots, K \right\}$$

and G is a normalizing constant (the partition function) chosen so that the distribution (1.1) sums to unity ([1], [2], [4], [6]). Note that π does not depend upon the distributions of call holding periods. If call holding periods are exponentially distributed the stochastic process $(n(t),\ t \geq 0)$ is Markov.

The classical example of the above model is a telephone network, but the recent impetus to its study has been provided by developments in local area networks, multi-processor interconnection architectures, and database structures (see [5], [8], [9], [10]). In computer communication networks, and increasingly in

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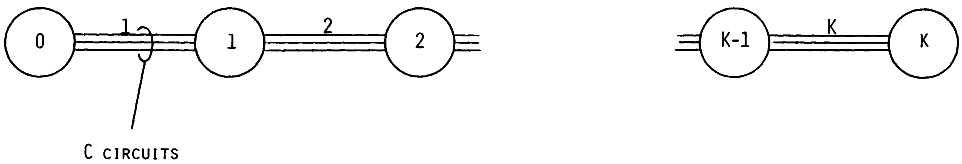

FIG. 1. *A one-dimensional network.*

telephone networks, the circuits are virtual rather than physical: for example a fixed proportion of the transmission capacity of a communication channel. The term "circuit-switched" arises from these application areas, where it is used to describe systems in which before a request (which may be a call, or a task, or a customer) is accepted it is first checked that sufficient resources are available to deal with each stage of the request.

Part of the above model's attraction is that very many generalizations are readily incorporated. For example, if calls requesting route r arrive at rate ν_r/η_r and have holding periods with mean η_r, then the distribution π_K associated with the resulting stochastic process is given by the unaltered expression (1.1). For further generalizations see [4], [6]. The essential features of the model are that a call makes simultaneous use of a number of resources and that blocked calls are lost.

Despite the apparent simplicity of the form (1.1) it is usually difficult to determine from it quantities of interest such as the mean utilization of a link or the loss probability on a route. Our aim in this paper is to explore some of the implications of the form (1.1) when the system represented has an essentially one-dimensional structure. In Section 2 we suppose that each route $r \in R$ is a set of consecutive integers chosen from $\{1, 2, \dots, K\}$ and that $C_k = C$, $k = 1, 2, \dots, K$. One could imagine a cable on which are positioned $K + 1$ stations (Figure 1), and that communication between two stations uses a fraction C^{-1} of the cable's capacity over the section of cable lying between the two stations. Set

$$\nu_r = \lambda \mu^{j-i-1}, \qquad r = \{i, i+1, i+2, \dots, j\}, \tag{1.2}$$

for $\lambda \in (0, \infty)$, $\mu \in (0, 1)$. Thus calls between stations a distance v apart are attempted at rate $\lambda \mu^{v-1}$. Let

$$m_k = \sum_{r:\, k \in r} n_r, \qquad k = 1, 2, \dots, K, \tag{1.3}$$

so that m_k is the number of circuits occupied on link k. Then we show that under the stationary distribution (1.1) the sequence $(m_1, m_2, \dots, m_K)$ has the distribution of an inhomogeneous Markov chain. Further, we exhibit a homogeneous stationary Markov chain $(x_k,\ k \in Z)$ with state space $\{0, 1, \dots, C\}$ and the property that the distribution of $(m_1, m_2, \dots, m_K)$ is the same as the conditional distribution of $(x_1, x_2, \dots, x_K)$ given that $x_0 = x_{K+1} = 0$. The distributions involved provide interesting examples of what have been termed quasi-stationary distributions by Darroch and Seneta [3].

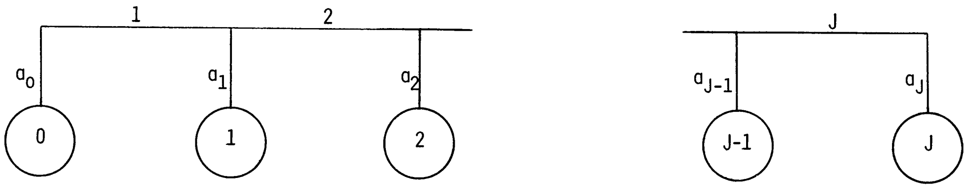

FIG. 2. *A single channel network.*

In Section 3 we suppose that links are labelled $1, 2, \ldots, J$, $a_0, a_1, \ldots, a_J$; that each link comprises a single circuit; and that a route takes the form $r = \{a_i, i+1, i+2, \ldots, j, a_j\}$ for $0 \le i < j \le J$. One could imagine that station j is attached to a single channel cable by an auxiliary link a_j (Figure 2). Set

$$(1.4) \qquad \nu_r = \lambda f(j-i), \quad \text{for } r = \{a_i, i+1, i+2, \ldots, j, a_j\},$$

where $\lambda \in (0, \infty)$ and f is a bounded nonnegative function, not identically zero. Thus we have relaxed the assumption that call arrival rates decay geometrically with distance covered, but have ensured that no station or link is involved in more than one call. Let $m_j \in \{0, 1\}$ be the number of circuits occupied on link j, $j = 1, 2, \ldots, J$. Then we show that the distribution of the sequence $(m_1, m_2, \ldots, m_J)$ is the same as the conditional distribution of $(x_1, x_2, \ldots, x_J)$ given that $x_0 = x_{J+1} = 0$, where $(x_k, k \in Z)$ is a stationary alternating renewal process. The process $(x_k, k \in Z)$ consists of blocks of consecutive ones alternating with blocks of consecutive zeros: the lengths of successive blocks of ones have distribution

$$(1.5) \qquad g_1(v) = \lambda(1-\rho)^{-1}\rho^{v+1}f(v), \qquad v = 1, 2, \ldots,$$

and the lengths of the intervening blocks of zeros have the geometric distribution

$$(1.6) \qquad g_0(u) = (1-\rho)\rho^{u-1}, \qquad u = 1, 2, \ldots;$$

here ρ is the unique solution in the interval $(0, 1)$ of the equation

$$(1.7) \qquad (1-\rho) = \lambda \sum_{v=1}^{\infty} \rho^{v+1} f(v).$$

The alternating renewal process defined by (1.5), (1.6), and (1.7) has many interesting features, some of which are briefly indicated in Section 4. For example, if $f(v) = 1$, $v = 1, 2, \ldots$, then the distributions (1.5) and (1.6) are both geometric with the same parameter ρ: hence the mean length of a block of ones is the same as the mean length of a block of zeros whatever the value of the arrival rate parameter λ.

This paper is primarily concerned with systems comprising a finite number of links. Some of the questions which arise in infinite link systems are described in [7].

2. The Markov distribution. Write Z for the one-dimensional integer lattice. For $i < j$ let

$$[i; j] = \{k \in Z: i < k \leq j\},$$

and let

$$R = \{[i; j]: i < j, i, j \in Z\}.$$

Write Z_+ for the nonnegative integers. Let $Y = Z_+^R$ with the product topology and with measurable structure given by the σ-algebra of Borel sets. Write $y = (y_r, r \in R)$ for a typical element of Y. Let π be the probability measure on Y defined by

$$\pi(y: y_r = n_r, r \in R') = \prod_{r \in R'} e^{-\nu_r} \frac{\nu_r^{n_r}}{n_r!}, \tag{2.1}$$

for any finite subset $R' \subset R$. Assume that

$$\sum_{r: k \in r} \nu_r < \infty, \qquad k \in Z. \tag{2.2}$$

If we regard y_r as the number of calls on route r then under π the number of calls on distinct routes are independent Poisson random variables, and condition (2.2) ensures that the number of calls on each link is finite with probability one.

Let

$$[i; k+] = \{[i; j]: j \geq k\}$$

and let

$$R(k) = \{[i; k+]: i < k\}.$$

For $y \in Y$ define

$$y_{[i; k+]} = \sum_{r \in [i; k+]} y_r$$

and

$$y(k) = (y_{r^+}, r^+ \in R(k)).$$

We can regard $y_{[i, k+]}$ as the number of calls with left end i and right end $\geq k$, and the collection $y(k)$ as giving the left ends of all calls using link k. Let

$$Y(k) = \left\{ y(k): y \in Y, \sum_{r^+ \in R(k)} y_{r^+} < \infty \right\}.$$

Note that $Y(k)$ is countable, and that by condition (2.2)

$$\pi(y: y(k) \in Y(k)) = 1.$$

Further, under the probability measure π, the sequence $(y(k), k \in Z)$ is a Markov chain. Its transition matrix is determined, in generating function form, by the identity

$$E\left(s_{k+1}^{y[k;\,k+1+]}\prod_{i=-\infty}^{k-1}s_i^{y[i;\,k+1+]}\middle|\,y(k)\right)$$

$$(2.3)\qquad = \exp\left\{-(1-s_{k+1})\sum_{j=k+1}^{\infty}\nu_{[k;\,j]}\right\}$$

$$\times\prod_{i=-\infty}^{k-1}\left\{1-(1-s_i)\frac{\sum_{j=k+1}^{\infty}\nu_{[i;\,j]}}{\sum_{j=k}^{\infty}\nu_{[i;\,j]}}\right\}^{y[i;\,k+]}.$$

Next let

$$(2.4)\qquad x_k(y)=\sum_{r:\,k\in r}y_r,\qquad k\in Z,$$

and write $x(y)=(x_k(y),\ k\in Z)$. Let $X=Z_+^Z$ with the product topology and with measurable structure given by the σ-algebra of Borel sets. Then the construction (2.4) induces a probability measure σ over X given by the natural relation

$$\sigma(A)=\pi(y\colon x(y)\in A).$$

Observe that condition (2.2) ensures $x_k<\infty$ a.s. Under the probability measure σ the sequence $(x_k,\ k\in Z)$ will not, in general, be a Markov chain. The next result concerns a special case when it is.

LEMMA 2.1. *If*

$$(2.5)\qquad \nu_{[i;\,j]}=\lambda\mu^{j-i-1},\qquad i<j,$$

where $\lambda\in(0,\infty)$, $\mu\in(0,1)$, *then*

$$\sigma(x\colon x_k=m_k,\ i<k\le j)=\tilde{P}(m_i)\prod_{k=i}^{j-1}\tilde{p}(m_k,m_{k+1}),\qquad i<j,$$

where $\tilde{p}(\cdot,\cdot)$ *is a transition matrix on* $Z_+\times Z_+$ *with unique invariant probability distribution* $\tilde{P}(\cdot)$.

PROOF. From (2.5)

$$\sum_{r:\,k\in r}\nu_r=\lambda/(1-\mu)^2,$$

and thus x_k has the Poisson distribution

$$\sigma(x\colon x_k=m)=\exp\{-\lambda/(1-\mu)^2\}\left[\lambda/(1-\mu)^2\right]^m/m!$$
$$\triangleq\tilde{P}(m).$$

Further

$$x_k=\sum_{r:\,k\in r}y_r=\sum_{i=-\infty}^{k-1}\sum_{j=k}^{\infty}y_{[i;\,j]}$$
$$=\sum_{i=-\infty}^{k-1}y_{[i;\,k+]}=\sum_{r^+\in R(k)}y_{r^+}.$$

This makes explicit the representation of x_k as a function of $y(k)$. Note that, by (2.5),

$$\sum_{j=k+1}^{\infty} \nu_{[k;\,j]} = \lambda/(1-\mu)$$

and

$$\frac{\sum_{j=k+1}^{\infty} \nu_{[i;\,j]}}{\sum_{j=k}^{\infty} \nu_{[i;\,j]}} = \mu, \qquad i<k.$$

Hence, from (2.3),

$$\begin{aligned} E(s^{x_{k+1}}|y(k)) &= E\left(s^{y_{[k;\,k+1+]}} \prod_{i=-\infty}^{k-1} s^{y_{[i;\,k+1+]}} \middle| y(k) \right) \\ &= \exp\{-(1-s)\lambda/(1-\mu)\} \prod_{i=-\infty}^{k-1} \{1-(1-s)\mu\}^{y_{[i;\,k+]}} \qquad (2.6) \\ &= \exp\{-(1-s)\lambda/(1-\mu)\}\{1-(1-s)\mu\}^{x_k}. \end{aligned}$$

Thus the distribution of x_{k+1} conditional on $y(k)$ depends on $y(k)$ only through x_k. It follows (see, for example, [11], Chapter IIId) that the sequence $(x_k,\ k \in Z)$ is a Markov chain. From (2.6) we can derive an explicit expression for its transition matrix,

$$(2.7)\quad \tilde{p}(u,v) = \exp\left(\frac{-\lambda}{1-\mu}\right) \sum_{w=\max\{0,\,u-v\}}^{u} \binom{u}{w} \mu^{u-w}(1-\mu)^{u-v} \frac{\lambda^{v+w-u}}{(v+w-u)!}.$$

It follows, and can be checked by an explicit calculation, that $\tilde{P}(\cdot)$ is the unique invariant probability distribution for the transition matrix $\tilde{p}(\cdot,\cdot)$. □

We now turn to the finite network described in the Introduction and illustrated in Figure 1. Let

$$N_K = \left\{ n = \left(n_{[i;\,j]},\ 0 \le i \le j \le K\right) : n_{[i;\,j]} \in Z_+,\ 0 \le i < j \le K, \right.$$
$$\left. \sum_{i=0}^{k-1} \sum_{j=k}^{K} n_{[i;\,j]} \le C,\ k = 1,2,\dots,K \right\},$$

and let

$$M_K = \{m = (m_k,\ k = 0,1,\dots,K+1) : m_0 = m_{K+1} = 0;$$
$$m_k \in \{0,1,\dots,C\},\ k = 1,2,\dots,K\}.$$

Thus N_K is the set of possible configurations for the network illustrated in Figure 1. For $n \in N_K$ let

$$(2.8)\qquad m_k(n) = \sum_{r:\,k \in r} n_r, \qquad k = 0,1,\dots,K+1,$$

and let $m(n) = (m_k(n),\ k = 0, 1, \dots, K+1)$. Thus $m_k(n)$ is the number of circuits in use on link k under configuration n. Note that M_K is the image of N_K under the mapping $n \to m$ defined by (2.8).

Let Φ be the projection mapping on X which sends x to $(x_k,\ k = 0, 1, \dots, K+1)$ and Ψ the projection mapping on Y which sends y to $(y_r,\ r\colon r \cap \{0, 1, \dots, K\} \neq \phi)$. Define a probability distribution over N_K by

$$\pi_K(n) = \frac{\pi(y\colon \Psi(y) = n)}{\pi(y\colon \Psi(y) \in N_K)}, \qquad n \in N_K$$

$$= G^{-1} \prod_{0 \le i < j \le K} \frac{\nu_{[i;\,j]}{}^{n_{[i;\,j]}}}{n_{[i;\,j]}!}, \qquad n \in N_K,$$

where G is a normalizing constant chosen so that the distribution sums to unity. Thus π_K is exactly the distribution (1.1) for the network illustrated in Figure 1. The distribution π_K over N_K induces a distribution σ_K over M_K by the natural relation

$$\sigma_K(m) = \sum_{n \in N_K\colon m(n) = m} \pi_K(n), \qquad m \in M_K.$$

LEMMA 2.2.

$$\sigma_K(m) = \frac{\sigma(x\colon \Phi(x) = m)}{\sigma(x\colon \Phi(x) \in M_K)}, \qquad m \in M_K.$$

PROOF. For $m \in M_K$

$$\begin{aligned}\sigma(x\colon \Phi(x) = m) &= \pi(y\colon \Phi(x(y)) = m)\\ &= \pi(y\colon m(\Psi(y)) = m)\\ &= \sum_{n \in N_K\colon m(n) = m} \pi(y\colon \Psi(y) = n).\end{aligned}$$

Further

$$\sigma(x\colon \Phi(x) \in M_K) = \pi(y\colon \Psi(y) \in N_K)$$

and so the desired result follows from the definition of σ_K and π_K. □

THEOREM 2.3. *If*

$$\nu_{[i;\,j]} = \lambda \mu^{j-i-1}, \qquad 0 \le i < j \le K,$$

then

(i) *there exist transition matrices* $p_{K-k}(\cdot, \cdot)$, $k = 0, 1, \dots, K$, *over* $\{0, 1, \dots, C\}^2$ *such that*

$$\sigma_K(m) = \prod_{k=0}^{K} p_{K-k}(m_k, m_{k+1}), \qquad m \in M_K;$$

(ii) *there exists a transition matrix* $p(\cdot,\cdot)$ *over* $\{0,1,\dots,C\}^2$ *such that for* $k = 1,2,\dots$

$$\lim_{K\to\infty} p_{K-k}(u,v) = p(u,v);$$

(iii)
$$\sigma_K(m) = \frac{\prod_{k=0}^{K} p(m_k, m_{k+1})}{p^{K+1}(0,0)}, \qquad m \in M_K.$$

PROOF. Define a matrix $q(\cdot,\cdot)$ on $\{0,1,\dots,C\}^2$ by

$$q(u,v) = \tilde{p}(u,v), \qquad u,v \in \{0,1,\dots,C\}, \tag{2.9}$$

where $\tilde{p}(\cdot,\cdot)$ is the transition matrix (2.7) identified in Lemma 2.1. Define $q^k(\cdot,\cdot)$ by matrix multiplication, with $q^0(\cdot,\cdot)$ the identity matrix. Define the matrix $p_{K-k}(\cdot,\cdot)$ on $\{0,1,\dots,C\}^2$ by

$$p_{K-k}(u,v) = q(u,v)\frac{q^{K-k}(v,0)}{q^{K-k+1}(u,0)} \tag{2.10}$$

and observe that $p_{K-k}(\cdot,\cdot)$ is a transition matrix. Now by Lemma 2.1

$$\sigma(x\colon \Phi(x) = m) = \tilde{P}(m_0)\prod_{k=0}^{K}\tilde{p}(m_k, m_{k+1}).$$

Hence for $m = (m_0 = 0, m_1, m_2, \dots, m_K, m_{K+1} = 0) \in M_K$,

$$\begin{aligned}\sigma(x\colon \Phi(x) = m) &= \tilde{P}(0)\prod_{k=0}^{K} q(m_k, m_{k+1})\\ &= \tilde{P}(0)q^{K+1}(0,0)\prod_{k=0}^{K} p_{K-k}(m_k, m_{k+1}),\end{aligned}$$

and

$$\sigma(x\colon \Phi(x) \in M_K) = \tilde{P}(0)q^{K+1}(0,0).$$

Part (i) now follows from Lemma 2.2.

Since $q(\cdot,\cdot)$ is a primitive nonnegative matrix

$$q^j(u,v) \sim e^j c(u)d(v), \quad \text{as } j \to \infty, \tag{2.11}$$

where e is the largest eigenvalue of $q(\cdot,\cdot)$ and $c(\cdot)$ and $d(\cdot)$ are its associated right and left eigenvectors, normalized so that $\Sigma_u c(u)d(u) = 1$ [12]. Let

$$p(u,v) = \frac{q(u,v)c(v)}{ec(u)} \tag{2.12}$$

and observe that this defines a transition matrix on $\{0,1,\dots,C\}^2$. Part (ii) now follows from (2.10) and (2.11).

Finally, for $m = (m_0 = 0, m_1, m_2, \dots, m_K, m_{K+1} = 0) \in M_K$

$$\begin{aligned}\sigma_K(m) &= \frac{\prod_{k=0}^{K} q(m_k, m_{k+1})}{q^{K+1}(0,0)}\\ &= \frac{\prod_{k=0}^{K} p(m_k, m_{k+1})}{p^{K+1}(0,0)}\end{aligned}$$

by (2.12). □

Part (i) of Theorem 2.3 shows that under σ_K the vector $m = (0, m_1, m_2, \ldots, m_K, 0)$ has the distribution of an inhomogeneous Markov chain. Let $(x_k,\ k \in Z)$ be a homogeneous stationary Markov chain with transition matrix $p(\cdot,\cdot)$. Then Part (iii) of Theorem 2.3 shows that the conditional distribution of $(x_1, x_2, \ldots, x_K)$ given that $x_0 = x_{K+1} = 0$ is σ_K.

Note that relations (2.7), (2.9), and (2.12) provide a construction of the matrix $p(\cdot,\cdot)$. The unique invariant probability distribution for $p(\cdot,\cdot)$ is $(c(u)d(u),\ u = 0, 1, \ldots, C)$. Darroch and Seneta [3] call this a quasi-stationary distribution: see [3] for its interpretation in terms of a Markov chain with transition matrix $\tilde{p}(\cdot,\cdot)$ conditioned to avoid the set of states $\{C+1, C+2, \ldots\}$.

3. The renewal distribution. Let a_i, $i \in Z$, be a set of labels, distinct from each other and from the set Z. We can regard a_i as labelling the auxiliary link from station i (Figure 2). Let

$$[i; j] = \{a_i, i+1, i+2, \ldots, j, a_j\}$$

and let

$$R = \{[i; j]: i < j,\ i, j \in Z\}.$$

Set

$$\nu_{[i;\,j]} = \lambda f(j - i), \qquad i < j, \tag{3.1}$$

where $\lambda \in (0, \infty)$ and f is a function $\{1, 2, \ldots\} \to [0, \infty)$ not identically zero. We shall assume that

$$1 - \rho = \lambda \sum_{v=1}^{\infty} \rho^{v+1} f(v) \tag{3.2}$$

has a solution $\rho \in (0, 1)$ and that

$$\sum_{v=1}^{\infty} v \rho^{v+1} f(v) < \infty. \tag{3.3}$$

Note that (3.2) can have at most one solution $\rho \in (0, 1)$, and that both assumptions are certainly satisfied when f is bounded above.

Define the geometric distribution

$$g_0(u) = (1 - \rho)\rho^{u-1}, \qquad u = 1, 2, \ldots. \tag{3.4}$$

Let

$$g_1(v) = \lambda(1 - \rho)^{-1} \rho^{v+1} f(v), \qquad v = 1, 2, \ldots. \tag{3.5}$$

This also defines a distribution, by (3.2). Define p by

$$(1 - p)/p = \sum_{u=1}^{\infty} u g_0(u) \Big/ \sum_{v=1}^{\infty} v g_1(v). \tag{3.6}$$

Then

$$1 - p = \left[1 + \lambda \sum_{v=1}^{\infty} v \rho^{v+1} f(v)\right]^{-1} > 0,$$

by (3.4), (3.5), and the assumption (3.3).

Let $X = \{0,1\}^Z$ with the product topology and with measurable structure given by the σ-algebra of Borel sets. Construct a probability measure σ on X as follows. Let

$$A(\tau; u_1, u_2, \dots, u_{n+1}; v_1, v_2, \dots, v_n)$$
$$= \Big\{x: x_k = 0,\ \tau + \sum_{t=1}^{s} (u_t + v_t) \le k < \tau + \sum_{t=1}^{s} (u_t + v_t) + u_{s+1},$$
$$s = 0,1,\dots,n;$$
$$x_k = 1,\ \tau + \sum_{t=1}^{s} (u_t + v_t) + u_{s+1} \le k < \tau + \sum_{t=1}^{s+1} (u_t + v_t),$$
$$s = 0,1,\dots,n-1\Big\}.$$

For $\tau \in Z$, $n > 0$, $u_1, u_2, \dots, u_n, u_{n+1}, v_1, v_2, \dots, v_n > 0$, set

$$\sigma(A(\tau; u_1, u_2, \dots, u_{n+1}; v_1, v_2, \dots, v_n))$$
$$= (1-p)\left(\prod_{s=1}^{n} g_0(u_s)\right)\left(\prod_{s=1}^{n} g_1(v_s)\right)\rho^{u_{n+1}-1}. \tag{3.7}$$

This defines a measure σ corresponding to a stationary alternating renewal process: the lengths of successive blocks of ones have distribution (3.5), and the lengths of the intervening blocks of zeros have the geometric distribution (3.4). Note that p, given by (3.6), is the stationary probability of a one for this process:

$$p = \sigma(x: x_k = 1), \qquad k \in Z. \tag{3.8}$$

We now turn to the finite network described in the Introduction and illustrated in Figure 2. Let

$$N_{(J)} = \Big\{n = (n_{[i;j]}, 0 \le i < j \le J): n_{[i;j]} = 0 \text{ or } 1, 0 \le i < j \le J;$$
$$\sum_{i=0}^{j-1} n_{[i;j]} + \sum_{k=j+1}^{J} n_{[j;k]} \le 1,\ j = 0,1,\dots,J;$$
$$\sum_{i=0}^{k-1} \sum_{j=k}^{J} n_{[i;j]} \le 1,\ k = 1,2,\dots,J\Big\},$$

and let

$$M_{(J)} = \{m = (m_k, k = 0,1,\dots,J+1): m_0 = m_{J+1} = 0;$$
$$m_k \in \{0,1\}, k = 1,2,\dots,J\}.$$

Then $N_{(J)}$ is the set of possible configurations for the network illustrated in Figure 2. For $n \in N_{(J)}$ let

$$m_k(n) = \sum_{r:\, k \in r} n_r, \qquad k = 0,1,\dots,J+1, \tag{3.9}$$

and let $m(n) = (m_k(n),\ k = 0, 1, \dots, J+1)$. Thus $m_k(n)$ is the number of circuits in use on link k under configuration n. Note that (3.9) defines a one-to-one correspondence between $N_{(J)}$ and $M_{(J)}$. Define a probability distribution $\pi_{(J)}$ over $N_{(J)}$ by

$$(3.10)\qquad \pi_{(J)}(n) = G^{-1} \prod_{0 \le i < j \le J} \nu_{[i;\, j]}{}^{n_{[i;\, j]}}, \qquad n \in N_J,$$

where G is a normalizing constant, chosen so that the distribution sums to unity. Thus $\pi_{(J)}$ is the distribution (1.1) for the network of Figure 2 (a network with a total of $K = 2J + 1$ links). Let $\sigma_{(J)}$ be the distribution over $M_{(J)}$ induced by $\pi_{(J)}$ and the correspondence (3.9). Note that since this correspondence is one-to-one between $N_{(J)}$ and $M_{(J)}$ the distribution $\pi_{(J)}$ is determined by $\sigma_{(J)}$. Let Φ be the projection mapping on X which sends x to $(x_k,\ k = 0, 1, \dots, J+1)$.

THEOREM 3.1.

$$\sigma_{(J)}(m) = \frac{\sigma(x\colon \Phi(x) = m)}{\sigma(x\colon \Phi(x) \in M_{(J)})}, \qquad m \in M_{(J)}.$$

PROOF. For an element $n \in N_{(J)}$ with coordinate $n_{[k;\, l]} = 1$ write $n - [k; l]$ for the element $n' \in N_{(J)}$ with

$$\begin{aligned} n'_{[i;\, j]} &= n_{[i;\, j]}, \qquad [i; j] \ne [k; l], \\ &= 0, \qquad [i; j] = [k; l]. \end{aligned}$$

Then the form (3.10) is determined up to a multiplicative constant by the requirement that it satisfy the detailed balance condition

$$(3.11)\qquad \pi_{(J)}(n) = \pi_{(J)}(n - [k; l])\nu_{[k;\, l]},$$

for every $n \in N_{(J)}$ and every $[k; l]$ such that $n_{[k;\, l]} = 1$. It remains, therefore, to show that the proposed form for $\sigma_{(J)}$, and hence $\pi_{(J)}$, is consistent with (3.11). Let $m = m(n)$ and write $m - [k; l]$ for $m(n - [k; l])$. From (3.1) and (3.11) it is thus sufficient to check that

$$\sigma_{(J)}(m) = \sigma_J(m - [k; l])\lambda f(k - l).$$

But this follows from (3.4), (3.5) and (3.7) since

$$\begin{aligned} \frac{\sigma_{(J)}(m)}{\sigma_{(J)}(m - [k; l])} &= \frac{\sigma(x\colon \Phi(x) = m)}{\sigma(x\colon \Phi(x) = m - [k; l])} \\ &= \frac{(1 - \rho) g_1(k - l)}{\rho^{k - l + 1}} \\ &= \lambda f(k - l). \end{aligned}$$

□

Let $(x_k,\ k \in Z)$ be the stationary alternating renewal process with probability measure σ. Then Theorem 3.1 shows that the conditional distribution of $(x_1, x_2, \dots, x_J)$ given that $x_0 = x_{J+1} = 0$ is $\sigma_{(J)}$. Note that for any fixed value of

J the distribution $\sigma_{(J)}$ is unaffected by the values of $f(v)$ for $v > J$; in particular $f(v)$, $v > J$, could be chosen to ensure that (3.2) has a solution $\rho \in (0,1)$ that satisfies (3.3). The probability measure σ does depend on $f(v)$ for all values of $v \geq 1$. Our development has assumed a function f defined on the entire set $\{1,2,\dots\}$ since this has enabled us to relate the entire family $(\sigma_{(J)}, J = 1,2,\dots)$ to a common probability measure, σ.

4. Concluding remarks. In Section 3 we assumed that a station could be involved in at most one call (Figure 2), while in Section 2 we allowed a station to be the end point of up to $2C$ calls (Figure 1). These differing assumptions were made for ease of exposition. If in Section 3 we had allowed a station to be involved in up to two calls (by increasing the capacity of the auxiliary links in Figure 2, or omitting them altogether), then very similar results could have been obtained but with the complication that the associated alternating renewal process would allow contiguous blocks of ones [the revised geometric distribution corresponding to (3.4) would have support $\{0,1,\dots\}$]. Similarly, if in Section 2 we had insisted that a station be the end point of at most one call then a somewhat lengthier development could have established a version of Theorem 2.3 with the matrices $p_{K-k}(\cdot,\cdot)$, $p(\cdot,\cdot)$ constructed from (2.10), (2.12), and the revised matrix

$$
\begin{aligned}
q(u,v) &= u(1-\mu)\mu^{u-1}, && u = 1,2,\dots,C;\ v = u-1,\\
&= \mu^u, && u = 0,1,\dots,C;\ v = u,\\
&= \lambda\mu^u(1-\mu)^{-1}, && u = 1,2,\dots,C-1;\ v = u+1,\\
&= 0, && \text{otherwise.}
\end{aligned}
$$

The work of Lagarias, Odlyzko and Zagier [5] can be viewed as an investigation of this system under the limiting regime $\mu \to 1$. [Note that while $q(\cdot,\cdot)$ degenerates under this limit, $p(\cdot,\cdot)$ does not. Note also that while the assumption of [5] that all realizable configurations are equally likely to occur corresponds to $\lambda = 1$, $\mu \to 1$, their methods extend to the case $\lambda \neq 1$.]

Some simplifications arise if in the model of Section 2 we take the limit $K \to \infty$, $\mu \to 1$, $\lambda \to 0$, with $K(1-\mu)$ and $K^2\lambda$ held fixed. This corresponds to a finite length of cable with a call able to connect any two points on the cable; the distinction between whether or not a station can be involved in more than one call disappears. Ziedins [13] has considered this limiting case, investigating the form of the quasi-stationary distribution and the way in which the probability a call is accepted depends upon the distance covered by the call.

We shall conclude by discussing briefly some of the interesting features of the alternating renewal measure σ defined in Section 3. In the model of Section 3 the probability that a call attempting to cover a distance v is accepted is approximately $(1-p)\rho^{v-1}$: more precisely, if j, $J - j \to \infty$, then

$$
(4.1) \qquad \sum_{m \in M_{(J)}} I[m_i = 0,\ i = j, j+1,\dots, j+v-1]\sigma_{(J)}(m) \to (1-p)\rho^{v-1}.
$$

Since the offered traffic may to some extent attempt to compensate for this exponential decay in acceptance probability, it is of interest to consider a wide variety of choices for the function f, including examples where f is unbounded. Recall that p, given by (3.6) and (3.8), is the probability of a one under the stationary measure σ.

EXAMPLE 1. $f(v) = v^{-1}$, $v = 1, 2, \ldots$. Then as the arrival rate parameter λ decreases to zero the solution ρ to Equation (3.2) increases to one, and p decreases to zero. Limiting acceptance probabilities, given by (4.1), all increase to one, as we might expect.

EXAMPLE 2. $f(v) = 1$, $v = 1, 2, \ldots$. The solution to (3.2) is $\rho = (1 + \lambda^{1/2})^{-1}$, and the distributions (3.4) and (3.5) are both geometric with parameter ρ. Every choice $\lambda \in (0, \infty)$ gives the same value $p = 0.5$. An increase in λ decreases the limiting acceptance probabilities (4.1) for $v \geq 2$, and especially for large values of v. That p is unaffected by λ is perhaps surprising. For the end effects introduced in obtaining $\sigma_{(J)}$ from σ to be limited it is necessary that J be large in comparison with $\lambda^{-1/2}$: provided this is true changes in λ will have very little influence on $E_{(J)}(\sum_{j=1}^{J} m_j)$, the expected utilization of links $1, 2, \ldots, J$ under the distribution $\sigma_{(J)}$. An algebraically tractable generalization of this example is

$$f(v) = (v-1)(v-2)\cdots(v-l), \qquad v = l+1, l+2, \ldots,$$

which leads to the solution $p = (l+1)/(l+2)$ whatever the value of $\lambda \in (0, \infty)$.

EXAMPLE 3. $f(v) = \alpha^{v-1}$, $v = 1, 2, \ldots,$ where $\alpha \in (1, \infty)$. Then as λ decreases to zero the solution ρ increases to α^{-1}; but p increases to one, and hence $(1-p)\rho^{v-1}$ eventually decreases to zero for any fixed value of v. It is perhaps surprising that decreasing the arrival rate parameter λ should cause the probability p to increase. Suppose J is large in comparison with $(1 - \alpha\rho)^{-2}$: then under the distribution $\sigma_{(J)}$ the effect of decreasing the arrival rate parameter λ is to let in calls covering great distances and, counter-intuitively, to *increase* the expected utilization of links $1, 2, \ldots, J$.

A variant of the above example shows that it is not always possible to satisfy (3.2) and (3.3). If

$$f(v) = v^{-\alpha}\alpha^{v-1}, \qquad v = 1, 2, \ldots,$$

where $\alpha \in (1, \infty)$ then as λ decreases to the critical value

$$\lambda_c = \alpha(\alpha - 1)\left(\sum_{v=1}^{\infty} v^{-2}\right)^{-1}$$

the probability p increases to one. For $\lambda \in (0, \lambda_c)$ Equation (3.2) has no solution $\rho \in (0,1)$. For $\lambda = \lambda_c$ Equation 3.2 has the solution $\rho = \alpha^{-1}$, but condition (3.3) is violated.

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