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DECOMPOSITION OF BINARY RANDOM FIELDS AND ZEROS OF PARTITION FUNCTIONS[1]

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Let $\delta_c(X)$ denote the maximum d in $[0, \frac{1}{2}]$ such that a binary Gibbs random field X can be decomposed as the modulo 2 sum of two independent binary fields, one of which is independent Bernoulli (white binary noise) of weight d. In a recent paper, Hajek and Berger showed, under modest assumptions, that $\delta_c > 0$. We point out here that the decomposition of X is related to the classic statistical mechanics problem of determining zero-free regions of partition functions. A theorem of Ruelle is then applied to obtain improved estimates for δ_c.

1. Decomposition and zeros of partition functions. The primary purpose of this note is to point out that there is a close relation between the decomposition problem for binary random fields studied in a recent paper of Hajek and Berger (1987) and the statistical mechanical problem of determining zero-free regions for partition functions. To demonstrate the usefulness of this relation, we apply to the decomposition problem one theorem from the extensive statistical mechanics literature on zeros of partition functions which has evolved from the pioneering work of Lee and Yang (1952). This theorem, due to Ruelle (1973), leads to (i) an alternate proof of the Hajek–Berger theorem giving sufficient conditions for decomposability, and (ii) improved estimates for the associated "critical distortion" [see Hajek and Berger (1987)].

Let $X = (X_i,\ i \in S)$, Y and U denote 0 or 1 valued random processes with countable index set S. For $D = (D_i,\ i \in S)$ with $0 \le D_i < \frac{1}{2}$, let $U(D)$ denote a process with independent components and with $P(U_i(D) = 1) = D_i$ for each i. The decomposition problem which was analyzed and applied to the calculation of the information theoretic per-site rate-distortion function by Hajek and Berger (1987) may be stated as follows: Given X and D, is there some Y independent of $U(D)$ so that X is equidistributed with $Y \oplus U(D)$? Here, $\oplus$ denotes componentwise modulo 2 addition. When such a decomposition is possible, D is said to be extractable from X.

For the purpose of this paper, the term partition function is merely a synonym for probability generating function. For given X and *finite* $F \subset S$, we define the partition function Z_X^F as the polynomial in the *complex* variables $\mathbf{z} = (z_j,\ j \in F)$,

$$Z_X^F(\mathbf{z}) = E\left(\prod_{j \in F} z_j^{X_j}\right). \tag{1.1}$$

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For $\mathbf{y}$ in $\{0,1\}^F$, we define $X \oplus \mathbf{y}$ as $X \oplus y$, where y in $\{0,1\}^S$ agrees with $\mathbf{y}$ on all components in F and $y_i = 0$ for i not in F.

The following proposition, which is an elementary consequence of Lemma 1 of Hajek and Berger (1987), is the main result of this section.

PROPOSITION 1. *In order that D be extractable from X, it suffices if for every finite $F \subset S$ and every $\mathbf{y}$ in $\{0,1\}^F$,*

$$(1.2) \qquad Z^F_{X\oplus\mathbf{y}}(\mathbf{z}) \neq 0 \quad \textit{when} \quad |z_j| < D_j/(1 - D_j) \quad \textit{for each } j \textit{ in } F.$$

PROOF. Let $\mathbf{X}$ and $\mathbf{D}$ denote the restrictions of X and D to F. If for each finite F, $\mathbf{D}$ is extractable from $\mathbf{X}$, then by standard arguments (involving convergence in distribution of subsequences) D is extractable from X. Lemma 1 of Hajek and Berger (1987) shows that $\mathbf{D}$ is extractable from $\mathbf{X}$ if and only if for all $\mathbf{y} \in \{0,1\}^F$,

$$(1.3) \qquad \sum_{\mathbf{x}\in\{0,1\}^F} \left(\prod_{j\in F} \left[-D_j/(1 - D_j) \right]^{x_j \oplus y_j} \right) P(\mathbf{X} = \mathbf{x}) \geq 0.$$

But the l.h.s. of (1.3) equals $Z^F_{X+\mathbf{y}}(\mathbf{z})$ with $z_j = -D_j/(1 - D_j)$. Let $Z(t)$ denote the same partition function but with $z_j = -tD_j/(1 - D_j)$, so that the l.h.s. of (1.3) equals $Z(1)$. $Z(t)$ is real for real t, $Z(0) \geq 0$ and by (1.2), $Z(t) \neq 0$ for $0 \leq t < 1$; hence $Z(1) \geq 0$, which is just (1.3). □

2. Ruelle's theorem. There is a substantial body of work, in the rigorous statistical mechanics literature, on the problem of determining zero-free regions for partition functions. This work originated with the classic Lee–Yang circle theorem [Lee and Yang (1952)]. Among its generalizations [e.g., Newman (1974) and Lieb and Sokal (1981)] is a theorem of Ruelle (1973) which is particularly useful in the present context.

For simplicity, we will restrict attention to Gibbs distributions with only single site and pair potentials. Ruelle's theorem can be applied to more general distributions, as in the work of Monroe (1983). Since Gibbs distributions with S infinite can be obtained as a limit of finite S distributions, we will further restrict ourselves to finite S (our notation below implicitly sets $F = S$). The distributions we consider then have the general form,

$$(2.1) \qquad P(\mathbf{X} = \mathbf{x}) = \text{const.}\exp\left(2\sum_j \tilde{H}(j)x_j + 2\sum_{\{i,j\}} \tilde{V}(i,j)(x_i \oplus x_j) \right),$$

where the $\tilde{H}(j)$'s and $\tilde{V}(i,j)$'s are real. This leads to

$$(2.2) \qquad Z_{\mathbf{X}+\mathbf{y}}(\mathbf{z}) = \text{const.}\sum_{\mathbf{w}} \exp\left(2\sum_{\{i,j\}} V(i,j)(w_i \oplus w_j) \right) \prod_j \left(e^{2H(j)} z_j \right)^{w_j},$$

where

$$(2.3) \qquad H(j) = (1 - 2y_j)\tilde{H}(j) = \pm\tilde{H}(j),$$

$$(2.4) \qquad V(i,j) = (1 - 2y_i)(1 - 2y_j)\tilde{V}(i,j) = \pm\tilde{V}(i,j).$$

Ruelle's analysis allows one to determine zero-free regions for multivariate partition functions such as (2.2) in terms of zero-free regions for bivariate partition functions,

$$(2.5)\qquad K_2(z_1, z_2) = 1 + e^{2V}\left(e^{2H}z_1 + e^{2H}z_2\right) + e^{4H}z_1 z_2$$

with $z_1 = z_2$. Elementary calculations show that $K_2(z, z)$ does not vanish in the complex disk

$$(2.6)\quad |z| < e^{-2|H|}\left(e^{2|V|} - \left[e^{4|V|} - 1\right]^{1/2}\right) = \exp\left[-2|H| - \cosh^{-1}\left(e^{2|V|}\right)\right].$$

THEOREM 2 [Ruelle (1973)]. *The partition function* (2.2) *does not vanish in the complex polydisk*

$$(2.7)\qquad \left\{\mathbf{z}\colon |z_j| < \exp\left[-2H^* - (M-1)\cosh^{-1}\left(e^{2V^*}\right)\right] \text{ for all } j\right\},$$

where

$$(2.8)\qquad H^* = \max_j \{|H(j)|\}, \qquad V^* = \max_{\{i, j\}} \{|V(i, j)|\}$$

and

$$(2.9)\qquad M - 1 = \max_j \{\text{number of } i\text{'s with } \tilde{V}(i, j) = \tilde{V}(j, i) \text{ nonzero}\}.$$

PROOF. This is an immediate consequence of Theorems 1.1 and 1.3 of Ruelle (1973) and (2.6). □

3. Extraction from Gibbs distributions. As an immediate corollary of Proposition 1 and Theorem 2, we obtain a new estimate for

$$(3.1)\qquad \begin{aligned}\delta_c &= \delta_c(X)\\ &= \sup\left\{d \in \left[0, \tfrac{1}{2}\right]\colon D \text{ with each } D_i = d \text{ is extractable from } X\right\}.\end{aligned}$$

THEOREM 3. *For a Gibbs distribution as in* (2.1),

$$(3.2)\qquad \delta_c(X) \geq \frac{1 - \tanh\left[H^* + (1/2)(M-1)\cosh^{-1}\left(e^{2V^*}\right)\right]}{2},$$

where H^*, V^* and M are defined in (2.8)–(2.9).

Although this theorem is stated for the case of finite S, it extends to infinite S by standard arguments (involving convergence in distribution of subsequences) provided the infinite S distribution is the limit of finite Gibbs distributions for which (3.2) is valid. As an example, consider the n-dimensional Ising model with zero external magnetic field. Here, $S = \mathbb{Z}^n$, $\tilde{H}(j) = 0$ and $\tilde{V}(i, j) = -\gamma < 0$ (for a ferromagnet) or $+\gamma > 0$ (for an antiferromagnet) when the Euclidean distance between i and j is exactly 1 (otherwise $\tilde{V} = 0$). In this case (3.2) becomes

$$(3.3)\qquad \delta_c(\text{Ising}_n) \geq \frac{1 - \tanh\left[n \cosh^{-1}\left(e^{2\gamma}\right)\right]}{2}.$$

For the one-dimensional case ($n = 1$), which corresponds to a binary symmetric Markov chain, the r.h.s. of (3.3) can be simplified to

$$\tfrac{1}{2}\left(1 - \left(1 - e^{-4\gamma}\right)^{1/2}\right), \tag{3.4}$$

which coincides with the exact result obtained by Gray (1970).

How do our estimates for δ_c compare with those of Hajek and Berger? We will discuss this issue only briefly, since the main purpose of this paper is to present a new approach to the decomposition problem rather than to obtain specific estimates. For simplicity, we restrict attention to the case where all $\tilde{H}(j)$'s are zero and all $\tilde{V}(i, j)$'s have the same modulus (as in Ising_n). Then the estimates of Theorem 1 and Section 4 of Hajek and Berger (1987) (other than those mentioned at the end of this section) may be expressed as

$$\delta_c \geq \frac{1 - \left[\tanh((M-1)V^*)\right]^{1/\bar{K}}}{2}, \tag{3.5}$$

where $\bar{K}$ is in general M^{M-1}, but may be smaller in special cases such as Ising_2 where $\bar{K} = 27$ (rather than 5^4). It is always the case that $\bar{K} \geq 2^{M-1}$ and usually the inequality is strict.

Both estimates (3.5) and (3.2) (with $H^* = 0$) tend to $\frac{1}{2}$ as $V^* \to 0$ and to 0 as $V^* \to \infty$. However the rates of approach are different. As $V^* \to 0$, our estimate is asymptotic to $(\frac{1}{2}) - \text{const.}(V^*)^{1/2}$, while the other estimate approaches $\frac{1}{2}$ much more slowly (for $\bar{K} > 2$), i.e., like $(\frac{1}{2}) - \text{const.}(V^*)^{1/\bar{K}}$. (The two constants differ.) As $V^* \to \infty$ both estimates are asymptotically proportional to $\exp(-2(M-1)V^*)$, but the proportionality constant is smaller in (3.5) when $\bar{K} > 2^{M-1}$.

To compare the two estimates over the entire range of possible values of V^*, we may define

$$K_M = \sup_{0<v<\infty} \frac{\left|\ln\left[\tanh((M-1)v)\right]\right|}{\left|\ln\left[\tanh((1/2)(M-1)\cosh^{-1}(e^{2v}))\right]\right|}. \tag{3.6}$$

Then our estimate is an improvement for all V^* if $K_M \leq \bar{K}$. It is easy to see (by letting $v \to \infty$ in the definition of K_M) that $K_M \geq 2^{M-1}$. We conjecture that $K_M = 2^{M-1}$. This is supported by numerical computations for $M - 1 = 4$ which give $K_5 = 16$ for Ising_2 compared to $\bar{K} = 27$. It can also be shown that the conjecture is asymptotically valid for large M in the sense that $K_M/2^{M-1} \to 1$ as $M \to \infty$.

We conclude by noting that in Section 4 of Hajek and Berger (1987) there are certain estimates better than (3.5) which lead, for example, to the exact Ising_1 value of δ_c. We have not investigated whether their corresponding implicit Ising_2 estimate would improve (3.3).

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