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RECURRENCE, TRANSIENCE AND BOUNDED HARMONIC FUNCTIONS FOR DIFFUSIONS IN THE PLANE[1]

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We give conditions for transience and recurrence for certain diffusion processes in the plane which "look" recurrent in certain sectors and transient in others. We also give conditions for the convergence of diffusion paths to paths of the deterministic dynamical system corresponding to the first-order part of L, the generator of the process. This in turn is related to the question of existence of bounded harmonic functions for the operator L. Conditions are given for the existence and nonexistence of bounded harmonic functions for L.

1. Introduction. Let $\omega = x(t)$, $\omega \in (\Omega, \mathscr{F}, \mathscr{F}_t, P_x)$ be a diffusion process in R^2 with generator

$$L = \frac{1}{2}\sum_{i,j=1}^{2} a_{ij}\frac{d^2}{dx_i\,dx_j} + \sum_{i=1}^{2} b_i\frac{d}{dx_i} \equiv \frac{1}{2}L_0 + b\cdot\nabla.$$

The drift, $b\cdot\nabla$, may be written in polar coordinates as

$$c(x)\frac{d}{dr} + d(x)\frac{1}{r}\frac{d}{d\theta},$$

where

$$c(x) = b(x)\left(\frac{x_1}{r}, \frac{x_2}{r}\right) \quad \text{and} \quad d(x) = b(x)\left(\frac{-x_2}{r}, \frac{x_1}{r}\right).$$

From now on, we will write the diffusion in polar coordinates as $(r(t), \theta(t))$. We shall assume:

(i) For sufficiently large r,

$$c(x) = \frac{\gamma_1(\theta)p_1(r)}{r^\delta}$$

and

$$d(x) = \frac{\gamma_2(\theta)p_2(r)}{r^k}, \quad \text{for } \delta, k \in R.$$

(ii) $\gamma_i \in C^1(S^1)$, $i = 1,2$, and $\gamma_2 > 0$.

(iii) $0 < p_i \in C^2(R^+)$, $\liminf_{r\to\infty} r^\varepsilon p_i(r) = \infty$, for all $\varepsilon > 0$ and $\limsup_{r\to\infty} r^\varepsilon p_i(r) = 0$, for all $\varepsilon < 0$, $i = 1,2$.

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(iv) If $k < \delta < 1$, then $p \equiv p_1/p_2$ satisfies $p' = O(r^{-\nu_1})$ and $p'' = O(r^{-\nu_2})$ as $r \to \infty$, where $\nu_1 > \max(1 + k - \delta, (1 + \delta)/2)$ and $\nu_2 > 1 + k$.

(v) If $k = \delta < 1$, then p satisfies $\limsup_{r \to \infty} p(r) < \infty$, $p' = o(r^{-1})$ and $p'' = O(r^{-\nu_2})$ as $r \to \infty$, where $\nu_2 > 1 + k$.

(vi) If $\delta = -1$, then $\limsup_{r \to \infty} p_1(r) < \infty$.

(vii) If $\delta = 1$, then $p_1 = 1$. If $\delta \geq 1$ and $k \leq 1$, then $p_2 = 1$.

(viii) The coefficients $a_{ij}(= a_{ji})$ of the operator L_0 are bounded and Lipschitz on R^2, and the matrix $a(x) = \{a_{ij}(x)\}$ is positive definite for each $x \in R^2$.

(ix) $b(x)$ is measurable and bounded on compacts.

A discussion of these conditions will be given at the end of Section 2. Without loss of generality, we assume that (i) holds for $r \geq 1$.

We wish to investigate the transience and recurrence properties and the invariant σ-field for the above class of diffusion processes. For illustration, consider for a moment the simple cases $\gamma_2 \equiv 0$, $\delta < 1$, $p_1(r) = 1$ and (1) $\gamma_1(x) = \varepsilon > 0$ or (2) $\gamma_1(x) = -\varepsilon < 0$. (Of course, since $\gamma_2 \equiv 0$, these cases do not fall into the above class although we could consider it to be the case $k = \infty$.) It is easy to show that in case (1) the process is transient and in case (2) the process is positive recurrent. In the transient case the process explodes if $\delta < -1$. Furthermore, in the transient case, it is easy to show that $\theta(t)$ converges almost surely as $t \to \infty$ to a nonconstant limiting angle θ. As we will recall below, this implies that nonconstant bounded harmonic functions exist for the operator L. In this simple case, the corresponding deterministic dynamical system obtained by looking only at the first-order terms is $r'(t) = \varepsilon/r^\delta$, $\theta'(t) = 0$, which has

$$\hat{r}(t, r_0, \theta_0) = \left(r_0^{1+\delta} + \varepsilon(\delta + 1)t\right)^{1/(1+\delta)}, \qquad \hat{\theta}(t, r_0, \theta_0) = \theta_0$$

as its solution starting from (r_0, θ_0) in the case $\delta > -1$. For $\delta = -1$ one obtains $\hat{r}(t, r_0, \theta_0) = r_0 e^{\varepsilon t}$, $\hat{\theta}(t, r_0, \theta_0) = \theta_0$, while if $\delta < -1$, one obtains $\hat{r}(t, r_0, \theta_0) = (r_0^{\delta+1} + (\delta + 1)\varepsilon t)^{1/(\delta+1)}$, $\hat{\theta}(t, r_0, \theta_0) = \theta_0$, and the solution runs off to infinity in finite time. Since $\theta(t)$ converges almost surely as $t \to \infty$ to a limiting angle, we see that indeed the paths of the diffusion process converge to paths of the corresponding deterministic dynamical system. Moreover, as we shall show, $r(t)$ almost surely grows on the order $t^{1/(1+\delta)}$ if $\delta > -1$, and exponentially if $\delta = -1$. If $\delta < -1$, the process explodes as we have already remarked. Thus, $r(t)$ almost surely possesses the same order of growth as the paths of the corresponding deterministic system. In this paper, we will investigate these types of properties for diffusions with coefficients as given above. In particular, note that since $\gamma_1(\theta)$ is allowed to vary in sign, it is not obvious whether the process is recurrent or transient. Indeed, most general results on recurrence and transience require that the coefficients satisfy certain properties uniformly in the nonradial variables [1]. This obviously does not hold in our case as the process will "look" recurrent in certain sectors ($\{\theta\colon \gamma_1(\theta) < 0\}$) and transient in others ($\{\theta\colon \gamma_1(\theta) > 0\}$). The one general result we are aware of that does not require this, requires reversibility [6]; our processes are not reversible.

We will prove the following theorems concerning transience and recurrence:

THEOREM 1.1. *Let*

$$L = \frac{1}{2}L_0 + b \cdot \nabla = \frac{1}{2}L_0 + \frac{\gamma_1(\theta)p_1(r)}{r^\delta}\frac{d}{dr} + \frac{\gamma_2(\theta)p_2(r)}{r^k}\frac{1}{r}\frac{d}{d\theta},$$

for $r = |x| \geq 1$, *where* L_0, b, γ_i, p_i, δ *and* k *are as in conditions* (i)–(ix) *above. Assume* $k \leq \delta < 1$. *We have*:

(a) *If* $\gamma \equiv \int_0^{2\pi}(\gamma_1/\gamma_2)(\theta)\,d\theta < 0$, *then the process is positive recurrent.*

(b) *If* $\gamma \equiv \int_0^{2\pi}(\gamma_1/\gamma_2)(\theta)\,d\theta > 0$, *then the process is transient.*

(c) *If* $\gamma > 0$, *then the process explodes if and only if* $\delta < -1$.

(d) *If* $\gamma > 0$ *and* $\delta > -1$, *then for each* $\varepsilon > 0$, *there exists a* $t_\varepsilon(\omega) \geq 0$ *such that* $t^{1/(1+\delta+\varepsilon)} \leq r(t) \leq t^{1/(1+\delta-\varepsilon)}$ *for all* $t \geq t_\varepsilon$, *almost surely* $[P_{r,\theta}]$. *If* $\limsup_{r\to\infty} p_1(r) < \infty$, *then there exist* $\lambda_2(\omega) > 0$ *and* $t_2(\omega) \geq 0$ *such that* $r(t) \leq \lambda_2 t^{1/(1+\delta)}$ *for all* $t \geq t_2$, *almost surely* $[P_{r,\theta}]$. *If* $\liminf_{r\to\infty} p_1(r) > 0$, *then there exist* $\lambda_1(\omega) > 0$ *and* $t_1(\omega) \geq 0$ *such that* $r(t) \geq \lambda_1 t^{1/(1+\delta)}$ *for all* $t \geq t_1$, *almost surely* $[P_{r,\theta}]$. *In particular, if* $0 < \liminf_{r\to\infty} p_1(r) \leq \limsup_{r\to\infty} p_1(r) < \infty$, *then there exist* $\lambda_1(\omega) > 0$, $\lambda_2(\omega) > 0$ *and* $t_0(\omega) \geq 0$ *such that* $\lambda_1 t^{1/(1+\delta)} \leq r(t) \leq \lambda_2 t^{1/(1+\delta)}$ *for all* $t \geq t_0$, *almost surely* $[P_{r,\theta}]$. *If* $\gamma > 0$ *and* $\delta = -1$, *then there exist a* $\lambda_2(\omega) > 0$ *and for each* $N > 0$, *a* $t_N(\omega) \geq 0$ *such that* $t^N \leq r(t) \leq e^{\lambda_2 t}$ *for all* $t \geq t_N$, *almost surely* $[P_{r,\theta}]$. *If* $\liminf_{r\to\infty} p_1(r) > 0$ [*recall that if* $\delta = -1$, *then by assumption,* $\limsup_{r\to\infty} p_1(r) < \infty$], *then there exist* $\lambda_1(\omega) > 0$, $\lambda_2(\omega) > 0$ *and* $t_0(\omega) \geq 0$ *such that* $e^{\lambda_1 t} \leq r(t) \leq e^{\lambda_2 t}$ *for all* $t \geq t_0$, *almost surely* $[P_{r,\theta}]$.

REMARK. An explanation of the transience and recurrence criterion in (a) and (b) can be made similar to the explanation given in Remark 1 following Theorem 1.3.

THEOREM 1.2. *Let* L *be as in Theorem* 1.1. *Assume* $\delta < 1$ *and* $k > \delta$. *Consider* $\theta(t)$ *to be defined on* R *rather than on* S^1. *Relax condition* (ii) *so that* γ_2 *may vary in sign and only assume* $\gamma_i \in C(S^1)$, $i = 1, 2$. *We have*:

(a) *If* $\gamma_1 < 0$, *then the process is positive recurrent.*

(b) *If there exists a* $\theta_0 \in S^1$ *such that* $\gamma_1(\theta_0) > 0$, *then the process is transient and* $\Theta = \lim_{t\to t_\infty}\theta(t)$ *exists and is finite almost surely* $[P_{r,\theta}]$. *Moreover,* $\Theta \bmod 2\pi \in \{\theta \in S^1 : \gamma_1(\theta) \geq 0\}$ *almost surely* $[P_{r,\theta}]$, *and*

$$P_{r,\theta}\big(\Theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon)\big) > 0$$

for all $\varepsilon > 0$ *and* θ_0 *satisfying* $\gamma_1(\theta_0) > 0$.

(c) *If there exists a* $\theta_0 \in S^1$ *such that* $\gamma_1(\theta_0) > 0$, *then if* $\delta \geq -1$ *the process does not explode whereas if* $\delta < -1$ *and* $\gamma_1(\Theta \bmod 2\pi) > 0$, *the process does explode.*

(d) *If there exists a* $\theta_0 \in S^1$ *such that* $\gamma_1(\theta_0) > 0$, *then if* $\gamma_1(\Theta \bmod 2\pi) > 0$, $r(t)$ *satisfies the growth rates as specified in Theorem* 1.1(d).

[t_∞ *in* (b) *is the terminal time as defined at the beginning of Section* 3.]

REMARK. Presumably, $P_{r,\theta}(\gamma_1(\Theta \bmod 2\pi) = 0) = 0$, in which case the extra condition in (c) and (d) above may be dispensed with.

The case $\delta \geq 1$ is more delicate and, unlike the case $\delta < 1$, depends on L_0. Before stating our theorem, we need the following:

CALCULATION. In polar coordinates,

$$L_0 = \sum_{i,j=1}^{2} a_{ij} \frac{d^2}{dx_i\, dx_j}$$

becomes

$$L_0 = e_1 \frac{d^2}{dr^2} + \frac{e_2}{r^2} \frac{d^2}{d\theta^2} + \frac{e_3}{r} \frac{d^2}{d\theta\, dr} + \frac{e_4}{r} \frac{d}{dr} + \frac{e_5}{r^2} \frac{d}{d\theta},$$

where

$$\begin{aligned}
e_1 &= a_{11}\cos^2\theta + a_{22}\sin^2\theta + 2a_{12}\cos\theta\sin\theta,\\
e_2 &= a_{11}\sin^2\theta + a_{22}\cos^2\theta - 2a_{12}\cos\theta\sin\theta,\\
e_3 &= -2a_{11}\cos\theta\sin\theta + 2a_{22}\cos\theta\sin\theta + 2(\cos^2\theta - \sin^2\theta)a_{12},\\
e_4 &= a_{11}\sin^2\theta + a_{22}\cos^2\theta - 2a_{12}\cos\theta\sin\theta,\\
e_5 &= 2a_{11}\cos\theta\sin\theta - 2a_{22}\cos\theta\sin\theta + 2(\sin^2\theta - \cos^2\theta)a_{12}.
\end{aligned}$$

In particular, the e_i are bounded on R^2.

For the case $\delta \geq 1$, we will make the assumption that the a_{ij}, and hence the e_{ij} are functions of θ alone. Also note that condition (vii) requires $p_1 = 1$ if $\delta = 1$ and $p_2 = 1$ if $k \leq 1$.

THEOREM 1.3. *Let L be as in Theorem 1.1. Assume that the a_{ij} are functions of θ alone. Assume $\delta \geq 1$ and let k be arbitrary.*
Define

$$G(\theta) = \begin{cases} \dfrac{1}{\gamma_2(\theta)}, & \text{if } k < 1,\\[2ex] \dfrac{1}{e_2(\theta)} \exp \displaystyle\int_0^\theta \left(\frac{e_5(s) + 2\gamma_2(s)}{e_2(s)} \right) ds, & \text{if } k = 1,\\[2ex] \dfrac{1}{e_2(\theta)} \exp \displaystyle\int_0^\theta \left(\frac{e_5(s)}{e_2(s)} \right) ds, & \text{if } k > 1, \end{cases}$$

and define

$$H(\theta) = \begin{cases} e_4(\theta) - e_1(\theta) + 2\gamma_1(\theta), & \text{if } \delta = 1,\\ e_4(\theta) - e_1(\theta), & \text{if } \delta > 1. \end{cases}$$

Then:

(a) *If* $\int_0^{2\pi} H(\theta)G(\theta)\,d\theta < 0$, *the process is recurrent.*
(b) *If* $\int_0^{2\pi} H(\theta)G(\theta)\,d\theta > 0$, *the process is transient.*

REMARK 1. The conditions in Theorem 1.3 can be explained as follows: Define

$$L_\theta = \begin{cases} \gamma_2(\theta)\dfrac{d}{d\theta}, & \text{if } k < 1, \\ \dfrac{e_2(\theta)}{2}\dfrac{d^2}{d\theta^2} + \left(\dfrac{e_5(\theta)}{2} + \gamma_2(\theta)\right)\dfrac{d}{d\theta}, & \text{if } k = 1, \\ \dfrac{e_2(\theta)}{2}\dfrac{d^2}{d\theta^2} + \dfrac{e_5(\theta)}{2}\dfrac{d}{d\theta}, & \text{if } k > 1. \end{cases}$$

L_θ has been obtained from L as follows: First throw away all terms involving differentiation in r. Then from among the rest of the terms which involve differentiation in θ alone, keep only those for which the power of r appearing in their coefficients is maximal. Then ignore r by setting $r = 1$. Now consider the process generated by L_θ. The density $\phi(\theta)$ of its invariant probability measure satisfies $\tilde{L}_\theta \phi = 0$, where $\tilde{L}_\theta$ is the adjoint of L_θ. One can check that the solution to this (up to a normalization factor) is $\phi(\theta) = G(\theta)$, where $G(\theta)$ is as in Theorem 1.3. Now consider for a moment a diffusion generated by $a(r)(d^2/dr^2) + b(r)(d/dr)$. It is well known that $b(r)/a(r) \le 1/r$ ($b(r)/a(r) \ge (1+\varepsilon)/r$) implies recurrence (transience) where $\varepsilon > 0$ is arbitrary. We can rewrite these conditions as $rb(r) - a(r) \le 0$ and $rb(r) - a(r) \ge \varepsilon a(r)$. In particular, if $a(r)$ does not depend on r, we may rewrite the transience condition as $rb(r) - a > 0$. Now if we ignore differentiation with respect to θ in the operator L, and, among the remaining first-order terms, only consider those for which the power of r appearing in their coefficients is maximal (i.e., those with r^{-1}), we obtain the operator

$$L_r = \begin{cases} \dfrac{e_1(\theta)}{2}\dfrac{d^2}{dr^2} + \left(\dfrac{e_4(\theta)}{2} + \gamma_1(\theta)\right)\dfrac{1}{r}\dfrac{d}{dr}, & \text{if } \delta = 1, \\ \dfrac{e_1(\theta)}{2}\dfrac{d^2}{dr^2} + \dfrac{e_4(\theta)}{2}\dfrac{1}{r}\dfrac{d}{dr}, & \text{if } \delta > 1. \end{cases}$$

Now consider the transience and recurrence conditions with

$$a = \frac{e_1(\theta)}{2} \quad \text{and} \quad b(r) = \begin{cases} \dfrac{1}{r}\left(\dfrac{e_4(\theta)}{2} + \gamma_1(\theta)\right), & \text{if } \delta = 1, \\ \dfrac{e_4(\theta)}{2r}, & \text{if } \delta > 1. \end{cases}$$

Since a does not depend on r, the conditions for transience and recurrence become $H(\theta) > 0$ and $H(\theta) \le 0$, respectively, where $H(\theta)$ is as in Theorem 1.3.

Since H depends on θ, the appropriate condition for transience or recurrence ought to be that $H(\theta)$ integrated against the "invariant" density for $\theta(t)$, that is, against $G(\theta)$, should be positive or nonpositive, respectively. In fact, this is exactly what Theorem 1.3 states, except for the fact that in (a), we have required strict negativity.

REMARK 2. In Theorem 1.1, our method does not cover the case $\gamma = 0$. This is to be expected as can be seen from the following. As we mentioned above, Theorem 1.1 works independently of L_0. One way of obtaining $\gamma = 0$ is by choosing $\gamma_1 \equiv 0$, in which case δ is irrelevant. Hence this case is covered by Theorem 1.3 and, depending on L_0, can be transient or recurrent.

We now focus on the transient cases in Theorems 1.1 and 1.2 (that is, when $\delta < 1$) and consider whether the diffusion trajectories converge to paths of the deterministic dynamical system obtained by looking only at the first-order terms, and whether the invariant σ-field is nontrivial, that is, whether nonconstant bounded harmonic functions exist for the operator L. We cannot expect any of this to occur in the case $\delta \geq 1$. In this case, the drift in the radial direction is on the order $1/r$ and hence is no stronger than the radial drift for d_0-dimensional Brownian motion for some d_0. But Brownian motion in any number of dimensions has a trivial invariant σ-field. We recall the correspondence between nonconstant bounded L-harmonic functions and the invariant σ-field of the process generated by L. The process lives on $\Omega = C([0, \infty), R^2)$, the space of continuous R^2-valued trajectories on $[0, \infty)$. A set $\Lambda \subset \Omega$ is invariant if $\{\omega \in \Lambda\} = \{\theta_t \omega \in \Lambda\}$, where θ_t is the shift operator $(\theta_t \omega(\cdot) = \omega(t + \cdot))$. If $\psi(\omega)$ is a bounded invariant random variable, then $h(x) = E_x \psi(\omega)$ is bounded L-harmonic and, conversely, if $h(x)$ is bounded L-harmonic, then $h(x(t))$ is a bounded martingale which converges almost surely to a limiting random variable $\psi(\omega)$, and in fact $h(x) = E_x \psi(\omega)$. Thus, the invariant σ-field may be identified with the class of bounded harmonic functions and nonconstant bounded L-harmonic functions exist if and only if the invariant σ-field for the process is nontrivial [5]. We need only consider the transient case since the invariant σ-field is always trivial in the recurrent case.

In the next section, we first describe the deterministic dynamical system corresponding to the first-order part of L, and then state our theorems concerning nonconstant bounded harmonic functions and convergence of diffusion paths to deterministic ones.

In Section 3, we collect some lemmas concerning recurrence, transience and explosion for diffusion processes. In order to increase the readability and decrease the length of this paper, we will prove our theorems in the case $p_1 = p_2 = 1$. The proofs of the theorems from Section 1 are given in Section 4 and the proofs of those from Section 2 are given in Section 5. In Section 6, we outline the modifications needed to prove the theorems in the more general context.

It has recently been brought to the author's attention that convergence of diffusion paths to deterministic ones has been studied for another class of multidimensional diffusions by Clark [2].

2. Convergence of diffusion paths to deterministic ones. We consider the deterministic dynamical system

$$r'(t) = \frac{\gamma_1(\theta)p_1(r)}{r^\delta}, \qquad \theta'(t) = \frac{\gamma_2(\theta)p_2(r)}{r^{k+1}},$$

where γ_i and p_i satisfy conditions (ii), (iii), (vi) and the first part of (v). (In what follows, the γ_i are extended periodically from S^1 to R.) We will denote by $(\hat{r}(t, r_0, \theta_0), \hat{\theta}(t, r_0, \theta_0))$ the solution with initial condition (r_0, θ_0) at $t = 0$. Throughout this section we assume that $\delta < 1$, $\gamma = \int_0^{2\pi}(\gamma_1/\gamma_2)(\theta)\,d\theta > 0$ in the case $k \le \delta$ and that there exists a θ_0 with $\gamma_1(\theta_0) > 0$ in the case $k > \delta$. By Theorems 1.1 and 1.2 the diffusion will be transient.

PROPOSITION 2.1. *There exists a $\rho > 0$ such that for $r_0 \ge \rho$ and all $-\infty < \theta_0 < \infty$, the solution $(\hat{r}(t, r_0, \theta_0), \hat{\theta}(t, r_0, \theta_0))$ satisfies*

(1)(a) *$\hat{\theta}(t, r_0, \theta_0)$ is strictly increasing.*

(b) *$\lim_{t\to\infty}\hat{\theta}(t, r_0, \theta_0) = \infty$ if and only if $k \le \delta$.*

(c) *If $k > \delta$, then $\lim_{t\to\infty}\hat{\theta}(t, r_0, \theta_0) \bmod 2\pi \in \{\theta\colon \gamma_1(\theta) \ge 0\}$.*

(2) *$\hat{r}(t, r_0, \theta_0) \ge 1$, for all $t \ge 0$.*

(3) *If $\hat{\theta}(t + q, r_0, \theta_0) = \hat{\theta}(t, r_0, \theta_0) + 2\pi$, for some $q > 0$, then $\hat{r}(t + q, r_0, \theta_0) > \hat{r}(t, r_0, \theta_0)$.*

(4)(a) *If $\delta > -1$, then for each $\varepsilon > 0$, there exists a $t_\varepsilon \ge 0$ depending on r_0 and θ_0 such that*

$$t^{1/(1+\delta+\varepsilon)} \le \hat{r}(t, r_0, \theta_0) \le t^{1/(1+\delta-\varepsilon)}, \quad \textit{for all } t \ge t_\varepsilon.$$

If $\limsup_{r\to\infty}p_1(r) < \infty$, then there exist $\lambda_2 > 0$ and $t_2 \ge 0$ depending on r_0 and θ_0 such that $\hat{r}(t, r_0, \theta_0) \le \lambda_2 t^{1/(1+\delta)}$, for all $t \ge t_2$. If $\liminf_{r\to\infty}p_1(r) > 0$, then there exist $\lambda_1 > 0$ and $t_1 \ge 0$ depending on r_0 and θ_0 such that $\hat{r}(t, r_0, \theta_0) \ge \lambda_1 t^{1/(1+\delta)}$, for all $t \ge t_1$. In particular, if $0 < \liminf_{r\to\infty}p_1(r) \le \limsup_{r\to\infty}p_1(r) < \infty$, then there exist $\lambda_1 > 0$, $\lambda_2 > 0$ and $t_0 \ge 0$ depending on r_0 and θ_0 such that $\lambda_1 t^{1/(1+\delta)} \le \hat{r}(t, r_0, \theta_0) \le \lambda_2 t^{1/(1+\delta)}$, for all $t \ge t_0$.

(b) *If $\delta = -1$, then there exist a $\lambda_2 > 0$ and for each $N > 0$ a $t_N \ge 0$ depending on r_0 and θ_0 such that*

$$t^N \le \hat{r}(t, r_0, \theta_0) \le e^{\lambda_2 t}, \quad \textit{for } t > t_N.$$

If $\liminf_{r\to\infty}p_1(r) > 0$ [recall that $\limsup_{r\to\infty}p_1(r) < \infty$ by assumption], then there exist $\lambda_1 > 0$, $\lambda_2 > 0$ and $t_0 \ge 0$ depending on r_0 and θ_0 such that

$$e^{\lambda_1 t} \le \hat{r}(t, r_0, \theta_0) \le e^{\lambda_2 t}, \quad \textit{for all } t \ge t_0.$$

(c) *If $\delta < -1$, then there exists a $t_\infty < \infty$ depending on r_0 and θ_0 such that $\lim_{t\to t_\infty}\hat{r}(t, r_0, \theta_0) = \infty$.*

PROOF. By solving explicitly, it is easy to check that this is true for the case $\gamma_1 = k_1$, $\gamma_2 = k_2$ and $p_1 = p_2 = 1$. More generally, one can solve explicitly in the case that γ_1, γ_2, p_1 and p_2 are step functions. By approximating the general γ_1, γ_2, p_1 and p_2 by step functions and taking a limit, we obtain a proof for the general case. □

Now consider solutions with initial conditions (ρ, θ) for $-\infty < \theta < \infty$ and ρ as in Proposition 2.1. By the standard uniqueness theorem for ODE's, there exists a unique solution emanating from (ρ, θ) for each $\theta \in (-\infty, \infty)$. Equivalently, integral curves do not cross one another. To each integral curve $(\hat{r}(t, \rho, \theta), \hat{\theta}(t, \rho, \theta))$, let $T(\rho, \theta)$ denote its trajectory, that is, the locus of points $\{(\hat{r}(t, \rho, \theta), \hat{\theta}(t, \rho, \theta)),\ t \geq 0\}$. Note, for example, that by the periodicity of the γ_i, $(r, \theta_1) \in T(\rho, \theta)$ if and only if $(r, \theta_1 + 2n\pi) \in T(\rho, \theta + 2n\pi)$. Also, it is important to observe that because γ_1 may take on negative values, an integral curve $(\hat{r}(t, \rho, \theta), \hat{\theta}(t, \rho, \theta))$ may intersect the circle $r = \rho$ numerous times or even run along a portion of its circumference. If the integral curve $(\hat{r}(t, \rho, \theta), \hat{\theta}(t, \rho, \theta))$ intersects $r = \rho$ at some time t_0 and $\hat{\theta}(t_0, \rho, \theta) = \theta_1$, then in fact $(\hat{r}(t, \rho, \theta_1), \hat{\theta}(t, \rho, \theta_1)) = (\hat{r}(t_0 + t, \rho, \theta), \hat{\theta}(t_0 + t, \rho, \theta))$ for all $t \geq 0$ and hence $T(\rho, \theta_1) \subset T(\rho, \theta)$. Thus, the collection $\{T(\rho, \theta), -\infty < \theta < \infty\}$ of trajectories starting at ρ may contain repetitions in the sense that $T(\rho, \theta_1) \subset T(\rho, \theta)$ for $\theta \neq \theta_1$. Note that, as a consequence of part (3) of Proposition 2.1, this inclusion can never hold if $|\theta - \theta_1| = 2n\pi$ for nonzero integer n.

We note the following fact, which follows from Proposition 2.1 and the above discussions, as

PROPOSITION 2.2. *For each point (r, θ), $r \geq \rho$, $-\infty < \theta < \infty$, there exists a trajectory $T(\rho, \theta_0)$ such that for some $t_0 \geq 0$, $(\hat{r}(t_0, \rho, \theta_0), \hat{\theta}(t_0, \rho, \theta_0)) = (r, \theta)$. This trajectory is unique in the sense that if $T(\rho, \theta_1)$ also satisfies the above statement, then either $T(\rho, \theta_0) \subset T(\rho, \theta_1)$ or $T(\rho, \theta_1) \subset T(\rho, \theta_0)$.*

Now consider the function

$$U(r, \theta) = \int_0^{\theta} \frac{\gamma_1}{\gamma_2}(s)\, ds - \int_1^{r} \frac{p_2}{p_1}(s) s^{\delta - k - 1}\, ds,$$

for $r > 0$ and $-\infty < \theta < \infty$.

PROPOSITION 2.3.

(1) *U is constant along integral curves starting from (ρ, θ) for $-\infty < \theta < \infty$.*
(2) *$U(\rho, \theta_0) = U(\rho, \theta_1)$ if and only if $T(\rho, \theta_0) \subset T(\rho, \theta_1)$ or $T(\rho, \theta_1) \subset T(\rho, \theta_0)$. Thus $U(r, \theta)$ "separates" trajectories.*
(3) *Given any U_0, there exists an integral curve starting at (ρ, θ) for some $\theta \in (-\infty, \infty)$ such that $U = U_0$ along that integral curve.*

PROOF.

(1) Obvious by differentiation.
(2) As in the proof of Proposition 2.1, one can show this explicitly in the case that γ_1, γ_2, p_1 and p_2 are step functions. This is enough.

(3) Since $\int_0^{2\pi}(\gamma_1/\gamma_2)(s)\,ds = \gamma > 0$, there exists a unique θ such that $U(\rho, \theta) = U_0$. Then by part (1), $U = U_0$ along the integral curve starting at ρ and this θ. □

We now return to our diffusion process and consider $\theta(t)$ to be defined on R rather than on S^1. We will prove the following theorems:

THEOREM 2.1. *Let L be as in Theorem 1.1. If $k \le \delta$, assume $\gamma > 0$. If $k > \delta$, assume there exists a θ_0 with $\gamma_1(\theta_0) > 0$. For all $r \ge 0$ and $-\infty < \theta < \infty$, we have*:

(a) *If $-1 \le \delta < 1$ and $\delta - \frac{1}{2}(1-\delta) < k \le \delta$, then almost every diffusion path $(r(t), \theta(t))$ converges as $t \to \infty$ to a trajectory of the corresponding deterministic dynamical system in the sense that $\mathscr{U}(\omega) = \lim_{t\to\infty} U(r(t), \theta(t))$ exists and is finite almost surely $[P_{r,\theta}]$. Furthermore, the distribution of $\mathscr{U}(\omega)$ under $P_{r,\theta}$ converges to the atom at U_0 as r and θ go to infinity in such a way that $U(r,\theta) \to U_0$. That is $P_{r,\theta}(\mathscr{U}(\omega) \in dx) \Rightarrow \delta_{U_0}(dx)$ as r and θ go to infinity with $U(r,\theta) \to U_0$.*

(b) *If $-1 < \delta < 1$ and $k < \delta - \frac{1}{2}(1-\delta)$ or $k = \delta - \frac{1}{2}(1-\delta)$ and $0 < \liminf_{r\to\infty} p_1(r)$, then almost every diffusion path crosses every deterministic trajectory infinitely often. That is, $\limsup_{t\to\infty} U(r(t), \theta(t)) = \infty$ almost surely $[P_{r,\theta}]$ and $\liminf_{t\to\infty} U(r(t), \theta(t)) = -\infty$ almost surely $[P_{r,\theta}]$. If $\delta = -1$, the same result holds with the additional requirement $k > \delta - (1-\delta) = -3$.*

(c) *If $\delta < 1$ and $k > \delta$, then almost every diffusion path $(r(t), \theta(t))$ converges as $t \to t_\infty$ to a trajectory of the corresponding deterministic dynamical system in the sense that $\mathscr{U}(\omega) = \lim_{t\to t_\infty} U(r(t), \theta(t))$ exists and is finite almost surely $[P_{r,\theta}]$. In fact,*

$$\mathscr{U}(\omega) = \int_0^{\Theta(\omega)} \frac{\gamma_1}{\gamma_2}(s)\,ds - \int_1^\infty \frac{p_2}{p_1}(s) s^{\delta-k-1}\,ds,$$

where $\Theta(\omega) = \lim_{t\to t_\infty} \theta(t)$. Furthermore, the distribution of $\Theta(\omega)$ under $P_{r,\theta}$ converges to the atom at θ_0 as $r \to \infty$ and $\theta \to \theta_0$ with $\gamma_1(\theta_0) > 0$. That is, $P_{r,\theta}(\Theta(\omega) \in dx) \Rightarrow \delta_{\theta_0}(dx)$ as $r \to \infty$ and $\theta \to \theta_0$ with $\gamma_1(\theta_0) > 0$.

REMARK. The above theorem may be interpreted as follows. If $k \le \delta - \frac{1}{2}(1-\delta)$, then the vector field in the θ-direction is large compared to the vector field in the r-direction and, consequently, the spirals of the deterministic trajectories wrap around tightly. If $k > \delta - \frac{1}{2}(1-\delta)$, then the spirals of the deterministic trajectories wrap around more loosely and there is more "space" between the individual deterministic trajectories. In the former case, the noise of the diffusion process prevents individual diffusion paths from settling down and following particular deterministic trajectories. In the latter case, the noise can operate in the "space" between individual trajectories and thus does not prevent the individual diffusion paths from converging to deterministic ones.

Theorem 2.2. *Let L be as in Theorem 1.1. Assume $-1 \le \delta < 1$, $k \le \delta$ and $\gamma > 0$. If $k \le \delta - \frac{1}{2}(1-\delta)$, also assume $\delta \ne -1$. For all $r > 0$ and $-\infty < \theta < \infty$, we have*

$$\lim_{t\to\infty} \frac{1}{\theta(t)} \int_1^{r(t)} \frac{p_2}{p_1}(s) s^{\delta-k-1}\, ds = \frac{\gamma}{2\pi} \quad \text{almost surely } [P_{r,\theta}].$$

In particular, if $p = 1$, then we have

$$\lim_{t\to\infty} \frac{r^{\delta-k}}{\theta(t)} = \frac{\gamma(\delta-k)}{2\pi} \quad \text{almost surely } [P_{r,\theta}], \text{ if } k \ne \delta,$$

$$\lim_{t\to\infty} \frac{\log r(t)}{\theta(t)} = \frac{\gamma}{2\pi} \quad \text{almost surely } [P_{r,\theta}], \text{ if } k = \delta.$$

Concerning nonconstant bounded harmonic functions, we have:

Theorem 2.3. *Let L be as in Theorem 1.1. If $k \le \delta$, assume $\gamma > 0$. If $k > \delta$, assume there exists a θ_0 with $\gamma_1(\theta_0) > 0$. We have:*

(a) *If $\delta < 1$ and $k > \delta$ or if $-1 \le \delta < 1$ and $k > \delta - \frac{1}{2}(1-\delta)$, then there exist nonconstant bounded harmonic functions for L.*

(b) *If $-1 < \delta < 1$, $k \le \delta - (1-\delta) = 2\delta - 1$, and $0 < \liminf_{r\to\infty} p_1(r) \le \limsup_{r\to\infty} p_1(r) < \infty$, then there are no nonconstant bounded harmonic functions for L.*

Remark 1. We believe that Theorems 2.1–2.3 also hold in the case $\delta < -1$, the case of explosion. These theorems depend heavily on the growth rates in Theorem 1.1(d). We believe, but have not been able to prove, that in the case of explosion, analogous to Theorem 1.1(d), $r(t)$ exceeds $(t_\infty - t)^{1/(1+\delta+\varepsilon)}$ and is dominated by $(t_\infty - t)^{1/(1+\delta-\varepsilon)}$, for any $\varepsilon > 0$ as $t \to t_\infty$, the time of explosion. Using this, one can show that Theorems 2.1–2.3 hold in the case $\delta < -1$. The reason that $\delta \ne -1$ has been excluded for certain values of k in each theorem, is that the exponential growth rates given in Theorem 1.1(d) are not sufficiently precise to allow the analysis to work.

We make the following remark under the assumption $0 < \liminf_{r\to\infty} p_1(r) \le \limsup_{r\to\infty} p_1(r) < \infty$.

Remark 2. Theorem 2.3 leaves open the question of nonconstant bounded harmonic functions in the gap $\delta - (1-\delta) < k \le \delta - \frac{1}{2}(1-\delta)$, if $-1 < \delta < 1$ and in the half line $k \le \delta - \frac{1}{2}(1-\delta) = -2$ in the case $\delta = -1$. We believe that there is none in these cases. Let $\tilde{\mathscr{U}}(\omega) = \mathscr{U}(\omega) \bmod \gamma$ in the case that $\mathscr{U}(\omega)$ exists and $\tilde{\mathscr{U}}(\omega) \equiv 0$, otherwise. Corresponding to $\tilde{\mathscr{U}}$ are the equivalence classes $\tilde{T}(\rho,\theta)$, $\theta \in S^1$, of trajectories given by $\tilde{T}(\rho,\theta) = \{T(\rho,\hat{\theta}),\ -\infty < \hat{\theta} < \infty,\ \hat{\theta} \bmod 2\pi = \theta\}$. We conjecture that $\tilde{\mathscr{U}}$ in fact generates the entire invariant σ-field. Note that by Theorem 2.1, if $-1 < \delta < 1$, then $\tilde{\mathscr{U}}$ is not almost surely constant if and only if $k > \delta - \frac{1}{2}(1-\delta)$. If $\delta = -1$, then $\tilde{\mathscr{U}}$ is not almost surely constant if $k > \delta - \frac{1}{2}(1-\delta) = -2$ and is almost surely constant if

$-3 = \delta - (1-\delta) < k \le \delta - \frac{1}{2}(1-\delta) = -2$. The theorem does not cover the case $\delta = -1$ and $k \le \delta - (1-\delta) = -3$. Thus, excluding the case $\delta = -1$ and $k \le -3$, our conjecture implies that there are no nonconstant bounded harmonic functions if $-1 \le \delta < 1$ and $k \le \delta - \frac{1}{2}(1-\delta)$. Our conjecture also implies that convergence of diffusion paths to deterministic trajectories $\tilde{T}$ gives the minimal Martin boundary for the process. To this end, one should consult the interesting paper by Cranston [4] which was the impetus for the investigations of Section 2 and in which the entire invariant σ-field and the Martin boundary are given for certain two-dimensional diffusions.

REMARK 3. Theorem 2.3 in the case $k > \delta$ follows immediately from Theorem 1.2 and its proof. In Theorem 1.2 we allowed γ_2 to vary sign, thus Theorem 2.3 in the case $k > \delta$ also holds with no restriction on the sign of γ_2.

We now make some comments on conditions (i)–(ix) given in Section 1. Cases where p_1 and p_2 are linear combinations of powers of logarithms and iterated logarithms are covered by conditions (iii) and (iv). The most conspicuous exception to condition (iv) is the case of an oscillating p. For example, if $p_1 = 2 + \sin r$ and $p_2 = 1$, then condition (iv) fails. Yet it seems intuitive that Theorem 1.1 and, consequently, the theorems of Section 2 which depend on Theorem 1.1 should hold in this case. We believe that the theorems hold without condition (iv). In terms of the stochastic differential equations we employ, this would mean that certain integrals whose integrands change sign satisfy certain bounds due to mass cancellation of positive and negative parts. Of course, this is very difficult to prove directly. More generally, we believe that all the theorems should hold without such rigid growth and decay rates as dictated by conditions (i)–(iii). The case $\delta \ge 1$ (Theorem 1.3) is much more delicate. Condition (vii) allows the proof given in the special case to work in the general case. The first part of condition (v) guarantees that the order of magnitude of the drift in the d/dr direction does not exceed that of the drift in the $(1/r)d/d\theta$ direction. In light of the dramatically different behavior in the case $k > \delta$ as compared to $k \le \delta$, this condition is natural. The requirement $p' = o(r^{-1})$ in condition (v) unfortunately excludes certain combinations of logarithms. As in condition (iv), the decay rates of the derivatives in condition (v) are probably not necessary. Condition (vi) is given to preclude worrying about a possible borderline case of explosion. In condition (ii), $\gamma_2 > 0$ is important in the case of $k \le \delta$ as can be seen from the fact that transience or recurrence depends on the sign of $\gamma = \int_0^{2\pi} (\gamma_1/\gamma_2)(\theta)\, d\theta$. If γ_2 is allowed to change sign and $k \le \delta$, the behavior changes dramatically. To get an idea of what would occur, see [4]. Similar behavior would occur in our context. If $k > \delta$, then as already noted, Theorems 1.2 and 2.3 go through even if γ_2 does change sign.

3. Transience and recurrence. In this section only, let

$$L = \sum_{i,j=1}^{d} a_{ij} \frac{d^2}{dx_i\, dx_j} + \sum_{i=1}^{d} b_i \frac{d}{dx_i}, \quad \text{with } a_{ij} (= a_{ji}),$$

and b_i continuous on R^d. Also assume $a(x) = \{a_{ij}(x)\}$ is positive definite for each $x \in R^d$. We present several propositions concerning transience, recurrence and explosion of diffusion processes in R^d generated by an operator L as above which we will utilize in the next section. Let $\tau_r = \inf\{t \geq 0: |x(t)| = r\}$. Let τ_∞ be the terminal time for the process. That is, τ_∞ is the time of explosion if the process explodes and $\tau_\infty = \infty$ otherwise. For our diffusions, which are strictly elliptic with bounded coefficients on compacts, transience (recurrence) is equivalent to the condition $P_x(\tau_{r_0} < \infty) < 1$ $[P_x(\tau_{r_0} < \infty) = 1]$ for some $r_0 > 0$ and some $x \in R^d$ with $|x| > r_0$. Positive recurrence is equivalent to the condition $E_x\tau_{r_0} < \infty$ for some $r_0 > 0$ and some $x \in R^d$ with $|x| > r_0$ [1]. We will use the above equivalences to verify the four following propositions.

PROPOSITION 3.1. *Let $u \in C^2(R^d)$ be bounded and satisfy $Lu(x) \leq 0$ for all x with $|x| \geq r_0 > 0$. Also assume there exists an x_0 with $|x_0| > r_0$ such that $u(x_0) < \inf_{|x|=r_0} u(x)$. Then the diffusion generated by L is transient.*

PROOF. Let x_0 be as in the statement of the theorem. Now $u(x(t \wedge \tau_\infty)) - \int_0^{t \wedge \tau_\infty} Lu(x(s))\,ds$ is a local P_{x_0}-martingale. Since $Lu \leq 0$ for $|x| \geq r_0$, and u is bounded, $u(x(t \wedge \tau_\infty \wedge \tau_{r_0}))$ is a P_{x_0}-supermartingale. Hence, $E_{x_0}u(x(t \wedge \tau_\infty \wedge \tau_{r_0})) \leq u(x_0)$. Now assume $x(t)$ is recurrent, i.e., $P_{x_0}(\tau_{r_0} < \infty) = 1$. Then $t \wedge \tau_\infty \wedge \tau_{r_0} = t \wedge \tau_{r_0} \to \tau_{r_0}$ almost surely $[P_{x_0}]$ as $t \to \infty$, and by the boundedness of u, we obtain $E_{x_0}u(x(\tau_{r_0})) \leq u(x_0)$. This is a contradiction since $|x(\tau_{r_0})| = r_0$ and $\inf_{|x|=r_0} u(x) > u(x_0)$. □

Similarly, we can show

PROPOSITION 3.2. *Let $u \in C^2(R^d)$ satisfy $u(x) \to \infty$ as $|x| \to \infty$ and $Lu(x) \leq 0$ for x satisfying $|x| \geq r_0 > 0$. Then the process is recurrent.*

PROPOSITION 3.3. *Let $u \in C^2(R^d)$ satisfy $u(x) \to \infty$ as $|x| \to \infty$ and $Lu(x) \leq -\varepsilon < 0$ for x satisfying $|x| \geq r_0 > 0$. Then the process is positive recurrent.*

PROOF. From Proposition 3.2, the process is recurrent, so $\tau_\infty = \infty$. Pick x_0 with $|x_0| > r_0$. We have for $N > |x_0|$,

$$E_{x_0}u\big(x(t \wedge \tau_{r_0} \wedge \tau_N)\big) = u(x_0) + E_{x_0}\int_0^{t \wedge \tau_{r_0} \wedge \tau_N} Lu(x(s))\,ds$$
$$\leq u(x_0) - \varepsilon E_{x_0}\big(t \wedge \tau_{r_0} \wedge \tau_N\big)$$

or

$$E_{x_0}\big(t \wedge \tau_{r_0} \wedge \tau_N\big) \leq \varepsilon^{-1}\Big(u(x_0) - E_{x_0}u\big(x(t \wedge \tau_{r_0} \wedge \tau_N)\big)\Big).$$

Letting $N \to \infty$ and using the monotone convergence theorem and the fact that u is bounded from below and then letting $t \to \infty$ and again using the monotone convergence theorem and the fact that u is bounded from below, we see that $E_{x_0}\tau_{r_0} < \infty$. □

Proposition 3.4. (a) *Let* $u \in C^2(R^d)$ *satisfy* $Lu + \lambda u \le 0$ *for* x *satisfying* $|x| \ge r_0 > 0$ *and some* $\lambda > 0$. *Also assume* $\inf_x u(x) > 0$. *Then if the process is transient, it explodes.*

(b) *Let* $u \in C^2(R^d)$ *satisfy* $Lu - \lambda u \le 0$ *for* x *satisfying* $|x| \ge r_0 > 0$ *and some* $\lambda > 0$. *Also assume* $u(x) \to \infty$ *as* $|x| \to \infty$. *Then the process does not explode.*

Proof. Pick x_0 with $|x_0| > r_0$ and let u be as in (a). Then

$$e^{\lambda(t \wedge \tau_\infty \wedge \tau_{r_0})} u\big(x(t \wedge \tau_\infty \wedge \tau_{r_0})\big) - \int_0^{t \wedge \tau_\infty \wedge \tau_{r_0}} e^{\lambda s}(Lu + \lambda u)(x(s))\, ds$$

is a local P_{x_0}-martingale, and it follows that

$$E_{x_0} e^{\lambda(t \wedge \tau_\infty \wedge \tau_{r_0})} u\big(x(t \wedge \tau_\infty \wedge \tau_{r_0})\big) \le u(x_0). \tag{3.1}$$

Since the process is transient,

$$P_{x_0}\Big(\lim_{t \to \infty} t \wedge \tau_\infty \wedge \tau_{r_0} = \tau_\infty\Big) > 0. \tag{3.2}$$

Now let $t \to \infty$ in (3.1). Using (3.2) and the fact that $\inf_x u(x) > 0$, we come to a contradiction unless $\tau_\infty < \infty$. Part (b) is proved similarly. □

4. Proof of the theorems from Section 1. The diffusion matrix in polar coordinates is given by

$$e \equiv \begin{pmatrix} e_1 & \dfrac{e_3}{2r} \\ \dfrac{e_3}{2r} & \dfrac{e_2}{r^2} \end{pmatrix}.$$

In this section and in the next one, we will be employing stochastic differential equations. Hence, we need the unique positive square root σ of e, which will be as smooth as e is since e is positive definite at each point. It is easy to see that σ_{11}, $r\sigma_{12}$ and $r\sigma_{22}$ are bounded as $r \to \infty$. We will use this fact below. Recall that we are giving our proofs under the assumption $p_1 = p_2 = 1$. Itô's formula in polar coordinates allows us to write the following stochastic differential equation for $f(r(t), \theta(t))$, where $f \in C^2(R^+ \times S^1)$:

$$\begin{aligned} f(r(t), \theta(t)) = f(r(0), \theta(0)) &+ \int_0^t f_r(r(s), \theta(s))\big(\sigma_{11}(r(s), \theta(s))\, dB_1(s) \\ &\qquad + \sigma_{12}(r(s), \theta(s))\, dB_2(s)\big) \\ &+ \int_0^t f_\theta(r(s), \theta(s))\big(\sigma_{12}(r(s), \theta(s))\, dB_1(s) \\ &\qquad + \sigma_{22}(r(s), \theta(s))\, dB_2(s)\big) \\ &+ \int_0^t Lf(r(s), \theta(s))\, ds, \end{aligned}$$

where $B(t) = (B_1(t), B_2(t))$ is a two-dimensional Brownian motion.

PROOF OF THEOREM 1.1. We have to break the proof into the two cases $k = \delta$ and $k < \delta$.

(a) and (b) First assume $k = \delta$ and $\gamma = \int_0^{2\pi} \gamma_1(s)/\gamma_2(s)\, ds > 0$. We consider a function $u_1(r, \theta)$ of the form $u_1 = r^{-1} f_1(\theta)$ with $f_1(\theta) > 0$. If we show that $Lu_1 \le 0$ for all sufficiently large r, we will have proved transience by Proposition 3.1. We have

$$Lu_1 = O(r^{-3}) + r^{-2-\delta}(-\gamma_1(\theta) f_1(\theta) + \gamma_2(\theta) f_1'(\theta)).$$

Since $\delta < 1$, we only need to pick $f_1 \in C^2(S^1)$ satisfying $-\gamma_1(\theta) f_1(\theta) + \gamma_2(\theta) f_1'(\theta) < 0$. Solving $-\gamma_1(\theta)\tilde{f}(\theta) + \gamma_2(\theta)\tilde{f}'(\theta) = 0$ gives $\tilde{f}(\theta) = \exp(\int_0^\theta (\gamma_1/\gamma_2)(s)\, ds)$. Now $\tilde{f}(\theta)$ is not continuous on S^1, but $f_1(\theta) = \exp(\int_0^\theta (\gamma_1/\gamma_2)(s)\, ds - (\gamma/2\pi)\theta)$ is, and in fact, $f_1 \in C^2(S^1)$ and satisfies $-\gamma_1(\sigma) f_1(\theta) + \gamma_2(\theta) f_1'(\theta) = -(\gamma/2\pi)\gamma_2(\theta) f_1(\theta) < 0$. For the positive recurrent case, let $u_2 = (r f_2(\theta))^m$, where

$$f_2(\theta) = \exp\left(\frac{\gamma}{2\pi}\theta - \int_0^\theta \frac{\gamma_1}{\gamma_2}(s)\, ds\right) = (f_1(\theta))^{-1}$$

and m is a positive number satisfying $m \ge 1 + \delta$. Then

$$Lu_2 = O(r^{m-2}) + m\frac{\gamma}{2\pi}\gamma_2(\theta) f_2^m(\theta) r^{m-\delta-1}$$

and we have $Lu_2 \le -\varepsilon$ for large r and some $\varepsilon > 0$. By Proposition 3.3, the process is positive recurrent. This proves (a) and (b) in the case $\delta = k$. Now assume $k < \delta$. According to whether the transient or the recurrent case is being considered, after trial and error we were led to the functions $u_1 = r^{-1}(f_1(\theta))^{r^q}$ and $u_2 = r^m (f_2(\theta))^{mr^q}$ with $q = k - \delta < 0$, m a positive number satisfying $m \ge 1 + \delta$, and f_1 and f_2 as above. We have

$$Lu_1 = -\frac{\gamma}{2\pi}\gamma_2(\theta)(f_1(\theta))^{r^q} r^{-2-\delta} + \text{lower-order terms}.$$

This proves (b) by Proposition 3.1. For u_2, we have

$$Lu_2 = m\frac{\gamma}{2\pi}\gamma_2(\theta)(f_2(\theta))^{mr^q} r^{m-1-\delta} + \text{lower-order terms}.$$

This proves (a) by Proposition 3.3.

(c) If $\delta \ge -1$, the process does not explode. It is well known that explosion can only occur if the drift is larger than linear. Now we show that if $\delta < -1$ and $\gamma > 0$, the process will explode. First take the case $\delta = k$. Let $u = c + (r^{-1} f_1(\theta))^\nu$ with $c > 0$ and $0 < \nu < -\delta - 1$. Then for $\lambda > 0$, we have

$$Lu + \lambda u = -\nu\frac{\gamma}{2\pi}\gamma_2(\theta) f_1^\nu(\theta) r^{-\nu-\delta-1} + c\lambda + \lambda r^{-\nu} f_1^\nu(\theta) + O(r^{-\nu-2}).$$

Since $-\nu - \delta - 1 > 0$, we see that $Lu + \lambda u < 0$ for all large r. By Proposition 3.4, the process explodes. If $k < \delta$, one proceeds as above, using the function $u = c + r^{-\nu}(f_1(\theta))^{\nu r^q}$ with ν as above and again, $q = k - \delta$. This proves part (c).

(d) To prove this part, we must treat the cases $k = \delta$ or $k < \delta$ and $\delta \le 0$ or $\delta > 0$ separately. We will first prove completely the case $k = \delta$ and $\delta > 0$. Then we will comment on the other cases as we deem appropriate. Let $u = rf_2(\theta)$. We have

$$Lu = \frac{\gamma}{2\pi} r^{-\delta} f_2(\theta)\gamma_2(\theta) + O(r^{-1}). \tag{4.1}$$

Thus, we can find a $c \geq 1$ and positive numbers ε and N such that

$$\varepsilon r^{-\delta} \leq Lu \leq N r^{-\delta}, \quad \text{for } r \geq c. \tag{4.2}$$

(We pick $c \geq 1$ because L only has the prescribed form for $r \geq 1$.) Let $T = \sup\{t \geq 0\colon r(t) \leq c\}$ be the last exit time from the region $r \leq c$. Define $T = 0$ if $r(t) > c$ for all $t \geq 0$. Since the process is transient, $T < \infty$ almost surely $[P_{r,\theta}]$. Itô's formula gives

$$\begin{aligned} u(r(t),\theta(t)) &= u(r_0,\theta_0) + \int_0^t f_2(\theta(s))(\sigma_{11}(r(s),\theta(s))\,dB_1(s) \\ &\qquad\qquad + \sigma_{12}(r(s),\theta(s))\,dB_2(s)) \\ &\quad + \int_0^t r(s) f_2'(\theta(s))(\sigma_{12}(r(s),\theta(s))\,dB_1(s) \\ &\qquad\qquad + \sigma_{22}(r(s),\theta(s))\,dB_2(s)) + \int_0^t Lu(r(s),\theta(s))\,ds \\ &= u(r(T),\theta(T)) + m_1(t) - m_1(T) + m_2(t) \\ &\quad - m_2(T) + \int_T^t Lu(r(s),\theta(s))\,ds, \end{aligned} \tag{4.3}$$

for $t \geq T$, where $B(t) = (B_1(t), B_2(t))$ is a two-dimensional Brownian motion on a probability space $(\Omega, \mathscr{F}, P)$,

$$m_1(t) = \int_0^t f_2(\theta(s))(\sigma_{11}(r(s),\theta(s))\,dB_1(s) + \sigma_{12}(r(s),\theta(s))\,dB_2(s))$$

and

$$m_2(t) = \int_0^t r(s) f_2'(\theta(s))(\sigma_{12}(r(s),\theta(s))\,dB_1(s) + \sigma_{22}(r(s),\theta(s))\,dB_2(s)).$$

We will use ω to denote a point in Ω. Now

$$(r(t),\theta(t)) = (r(t,r_0,\theta_0,\omega),\theta(t,r_0,\theta_0,\omega))$$

is the solution to a stochastic differential equation. In the sequel, when we refer to the measure P_{r_0,θ_0} on $(r(t),\theta(t)) \in C([0,\infty), R^2)$, we will mean the measure on $(r(t,r_0,\theta_0,\omega),\theta(t,r_0,\theta_0,\omega))$ induced by P. Pick $\nu > 0$ such that $(1+\nu)/2 < 1/(1+\delta)$. Since $m_1(t)$ and $m_2(t)$ are time changes of Brownian motion, and since all the coefficients in the stochastic integrals are bounded on R^2, there exist positive constants $C_1(\omega)$ and $C_2(\omega)$ such that

$$|m_1(t) - m_1(T)| \leq C_1(\omega) t^{(1+\nu)/2} + C_1(\omega), \quad \text{for all } t \geq T(\omega)$$

and

$$|m_2(t) - m_2(T)| \leq C_2(\omega) t^{(1+\nu)/2} + C_2(\omega), \quad \text{for all } t \geq T(\omega).$$

Using this along with (4.2) and (4.3), we obtain for $t > T$ the inequality

$$\begin{aligned}
&r(T)f_2(\theta(T)) - C(\omega)t^{(1+\nu)/2} - C(\omega) + \varepsilon\int_T^t r^{-\delta}(s)\,ds \\
(4.4)\qquad &\le r(t)f_2(\theta(t)) \\
&\le r(T)f_2(\theta(T)) + C(\omega)t^{(1+\nu)/2} + C(\omega) + N\int_T^t r^{-\delta}(s)\,ds,
\end{aligned}$$

where $C = C_1 + C_2$. Let $R(t) = \int_T^t r^{-\delta}(s)\,ds$. Then $r(t) = (R'(t))^{-1/\delta}$ and the right-hand inequality in (4.4) may be written as

$$f_2^\delta(\theta(t)) \le \big(r(T)f_2(\theta(T)) + C(\omega)t^{(1+\nu)/2} + C(\omega) + NR(t)\big)^\delta R'(t).$$

In this inequality, replace the factor $R'(t)$ by $(R'(t) + ((1+\nu)/2N)C(\omega)t^{(\nu-1)/2})$ which only makes the right-hand side larger. Integrating this new inequality gives

$$\begin{aligned}
&\int_T^t f_2^\delta(\theta(s))\,ds \\
&\quad\le N^{-1}(1+\delta)^{-1}\big(r(T)f_2(\theta(T)) + C(\omega)t^{(1+\nu)/2} + C(\omega) + NR(t)\big)^{1+\delta} \\
&\quad\quad - N^{-1}(1+\delta)^{-1}\big(r(T)f_2(\theta(T)) + C(\omega)T^{(1+\nu)/2} + C(\omega)\big)^{1+\delta}.
\end{aligned}$$

Let $f_0 = \min_{\theta\in S^1} f_2(\theta)$, replace the left-hand side above by $f_0^\delta(t - T)$, and do some algebra to obtain

$$\begin{aligned}
&R(t) \ge N^{-1}\Big(\big(r(T)f_2(\theta(T)) + C(\omega)T^{(1+\nu)/2} \\
(4.5)\qquad &\qquad + C(\omega)\big)^{1+\delta} + N(1+\delta)f_0^\delta(t-T)\Big)^{1/(1+\delta)} \\
&\quad - N^{-1}\big(r(T)f_2(\theta(T)) + C(\omega)t^{(1+\nu)/2} + C(\omega)\big).
\end{aligned}$$

Since $(1+\nu)/2 < 1/(1+\delta)$, there exists a constant $k(\omega) > 0$ and a $t_1(\omega) > T$ such that

$$\int_T^t r^{-\delta}(s)\,ds = R(t) \ge kt^{1/(1+\delta)}, \quad \text{for all } t > t_1.$$

The left-hand inequality in (4.4) may now be written as

$$r(t)f_2(\theta(t)) \ge r(T)f_2(\theta(T)) - C(\omega)t^{(1+\nu)/2} - C(\omega) + \varepsilon kt^{1/(1+\delta)}, \quad \text{for } t > t_1.$$

From this, it is clear that there exists a $\lambda_1(\omega) > 0$ and a $t_2(\omega) \ge t_1(\omega)$ such that $r(t) \ge \lambda_1 t^{1/(1+\delta)}$ for all $t > t_2$. This proves the lower bound. To get the upper inequality, use the upper bound in (4.4) again. Write

$$\int_T^t r^{-\delta}(s)\,ds = \int_T^{t_2} r^{-\delta}(s)\,ds + \int_{t_2}^t r^{-\delta}(s)\,ds$$

and replace $r(s)$ in the second term by $\lambda_1 s^{1/(1+\delta)}$. Since $\delta > 0$, this just makes the right-hand side of (4.4) even larger. Upon integrating this term, one sees that there exists a $\lambda_2(\omega)$ and a $t_0(\omega) \ge t_2(\omega)$ such that $r(t) \le \lambda_2 t^{1/(1+\delta)}$ for $t > t_0$. Thus, $\lambda_1 t^{1/(1+\delta)} \le r(t) \le \lambda_2 t^{1/(1+\delta)}$ for $t > t_0$. This completes the proof of (d) in the case $\delta = k > 0$. Now consider $-1 \le \delta = k \le 0$. (4.4) still holds. We run into a problem if we try to proceed exactly as we did in the above case. For this time, at the point we would like to replace the factor $R'(t)$ by $(R'(t) + ((1+\nu)/2N)C(\omega)t^{(\nu-1)/2})$, the inequality goes the wrong way. We proceed as follows. Consider (4.4) with ν picked so that $(1+\nu)/2 < 1$. Since $\delta \le 0$, $r^{-\delta}(s) \ge 1$ for $s \ge T$ and thus $\int_T^t r^{-\delta}(s)\,ds \ge (t-T)$. Thus, the term $t^{(1+\nu)/2}$ is a priori dominated by $\int_T^t r^{-\delta}(s)\,ds$ and we can rewrite (4.4) in the form

$$\begin{aligned} r(T)f_2(\theta(T)) - D(\omega) + \frac{\varepsilon}{2}\int_T^t r^{-\delta}(s)\,ds \\ \le r(t)f_2(\theta(t)) \\ \le r(T)f_2(\theta(T)) + D(\omega) + (N+1)\int_T^t r^{-\delta}(s)\,ds, \end{aligned}$$

for some $D(\omega) > 0$. The proof now proceeds more simply than in the previous case since $\delta \le 0$. For the case $k < \delta$, we again consider separately the two possibilities $\delta > 0$ or $-1 \le \delta \le 0$. We handle this situation similarly to the way we treated the previous one. This time we apply Itô's formula to $u = r(f_2(\theta))^{r^q}$, where, as before, $q = k - \delta < 0$. The two martingales that come up this time are

$$\begin{aligned} m_1(t) = \int_0^t (f_2(\theta(s)))^{r^q(s)}(1 + qr^q(s)\log f_2(\theta(s))) \\ \times(\sigma_{11}(r(s),\theta(s))\,dB_1(s) + \sigma_{12}(r(s),\theta(s))\,dB_2(s)) \end{aligned}$$

and

$$\begin{aligned} m_2(t) = \int_0^t r^{q+1}(s)(f_2(\theta(s)))^{r^q(s)-1} f_2'(\theta(s))(\sigma_{12}(r(s),\theta(s))\,dB_1(s) \\ +\sigma_{22}(r(s),\theta(s))\,dB_2(s)). \end{aligned}$$

Since $q < 0$, these martingales may be handled like those in the previous case. In place of (4.1), we will have

$$Lu = \frac{\gamma}{2\pi}r^{-\delta}(f_2(\theta))^{r^q}\gamma_2(\theta) + \text{lower-order terms.}$$

Everything else goes through exactly as in the previous case. This completes the proof of Theorem 1.1. □

PROOF OF THEOREM 1.2. (a) Use $u(r,\theta) = r^m$ for a positive number $m \ge 1 + \delta$ and apply Proposition 3.3.

(b) Let θ_0, η and ε be such that $\gamma_1(\theta_0) > 0$ and $\inf_{|\theta-\theta_0|\le\eta}\gamma_1(\theta) \ge \varepsilon$. Let

$$W_{\theta_0,\eta} = \{(r,\theta): r > 1,\ \theta_0 - \eta < \theta < \theta_0 + \eta\}$$

and

$$\tau_{\theta_0,\eta} = \inf\{t \ge 0: (r(t),\theta(t)) \notin W_{\theta_0,\eta}\}.$$

We now show that there exists an $r_1 = r_1(\theta_0,\eta,q)$ such that for $r \ge r_1$,

$$P_{r,\theta_0}(\tau_{\theta_0,\eta} = \infty) \ge \frac{r_1^{-q} - r^{-q}}{r_1^{-q}}, \tag{4.6}$$

where q is any number satisfying $0 < q < \min(1-\delta, k-\delta)$.

Let $u(r,\theta) = (1 - r^{-q})f(\theta)$, where $0 < q < \min(1-\delta, k-\delta)$ and $f(\theta) = 2\eta^2 - (\theta - \theta_0)^2$. Then

$$Lu = qr^{-q-1-\delta}\gamma_1(\theta)f(\theta) + \text{lower-order terms},$$

and thus $Lu(r,\theta) \geq 0$ for $\theta \in [\theta_0 - \eta, \theta_0 + \eta]$ and $r \geq r_1(\theta_0, \eta, q)$, for some $r_1(\theta_0, \eta, q)$. Let $\tau_{r_1} = \inf\{t \geq 0\colon\ r(t) = r_1\}$ and let $\tau = \tau_{r_1} \wedge \tau_{\theta_0,\eta}$. Then $u(r(t\wedge\tau), \theta(t\wedge\tau))$ is a P_{r,θ_0}-submartingale for $r > r_1$, giving us

$$E_{r,\theta_0}u(r(t\wedge\tau), \theta(t\wedge\tau)) \geq u(r,\theta_0) = (1 - r^{-q})2\eta^2.$$

Without loss of generality, assume $1 - r_1^{-q} > \frac{1}{2}$. Then the above inequality gives us

$$(1 - r_1^{-q})2\eta^2 P_{r,\theta_0}(\tau \leq t) + 2\eta^2 P_{r,\theta_0}(\tau > t) \geq (1 - r^{-q})2\eta^2.$$

Letting $t \to \infty$ gives

$$P_{r,\theta_0}(\tau = \infty) \geq \frac{r_1^{-q} - r^{-q}}{r_1^{-q}}.$$

Since $\tau \leq \tau_{\theta_0,\eta}$, this proves (4.6). This shows that the process is transient. We now show that if the process starts in $W_{\theta_0,\eta}$ and $\tau_{\theta,\eta} = \infty$, then in fact $\theta(t)$ converges to a limiting angle. For if $\theta(t)$ does not converge to a limiting angle, then we can find a θ_1 and θ_2 with $\theta_2 - \theta_1 = \eta_1 > 0$ and $\{\theta\colon |\theta - \theta_2| \leq \eta_1\} \subset (\theta_0 - \eta, \theta_0 + \eta)$ such that

$$\theta(t) = \theta_i, \qquad i = 1, 2, \tag{4.7}$$

for arbitrarily large values of t and, consequently, for arbitrarily large values of r. Yet by (4.6),

$$P_{r,\theta_2}(\tau_{\theta_2,\eta_1} = \infty) \geq \frac{r_1^{-q}(\theta_2,\eta_1,q) - r^{-q}}{r_1^{-q}(\theta_2,\eta_1,q)}, \quad \text{for } r > r_1(\theta_2,\eta_1,q).$$

This, coupled with the strong Markov property, shows that if $\theta(t) = \theta_2$ for arbitrarily large values of r, then with probability 1, $\theta(t)$ is eventually larger than θ_1, contradicting (4.7). This argument, coupled with (4.6) shows that $P_{r,\theta}(\Theta \bmod 2\pi \in (\theta_1 - \varepsilon, \theta_1 + \varepsilon)) > 0$ for all $r \geq 0$, $-\infty < \theta < \infty$, $\varepsilon > 0$, and θ_1 satisfying $\gamma_1(\theta_1) > 0$. We are left with showing that Θ exists with probability one and that almost surely $\Theta \bmod 2\pi \in \{\theta \in S^1\colon \gamma_1(\theta) \geq 0\}$. We argue as follows.

Define

$$A_\varepsilon = \{\theta \in S^1\colon \gamma_1(\theta) > \varepsilon\} \quad \text{and} \quad B_\varepsilon = \{\theta \in S^1\colon \gamma_1(\theta) < -\varepsilon\}.$$

Since the process is transient, it cannot eventually remain in B_ε for any $\varepsilon > 0$, where it "looks" like a recurrent process. In fact, the process cannot remain in A_0^c. For in this region, the drift in the r direction is

$$\frac{e_4}{2r} + \frac{\gamma_1}{r^\delta} \leq \frac{e_4}{2r}.$$

The magnitude of the diffusion in the r direction is $e_1/2$. In general, if one has a diffusion $a(d^2/dr^2) + b(d/dr)$, for $b > 0$, then the "magnitude" of transience is given by b/a. In particular, for d-dimensional Brownian motion, this ratio is $(d-1)/r$. Let

$$d_0 = \left[\sup_{r,\theta} \frac{e_4}{e_1}(r,\theta)\right] + 2,$$

where the brackets represent the greatest integer function. Then our ratio is no more than $(d_0 - 1)/r$. Since d-dimensional Brownian motion for any d is recurrent on S^{d-1}, our process must leave the region $\{\theta: \gamma_1(\theta) \le 0\}$. In fact, then, the transience of the process, (4.6), and the argument following it can be used to show that $\theta(t)$ converges to a limiting angle in $\{\theta: \gamma_1(\theta) \ge 0\}$.

(c) As mentioned in the proof of Theorem 1.1(c), there is a chance for explosion only in the case $\delta < -1$. We must show that if $\delta < -1$, $\lim_{t\to t_\infty}\theta(t) = \theta_0$ and $\gamma_1(\theta_0) > 0$, then the process explodes. Say $\gamma_1(\theta_0) = 2\varepsilon$ and pick $\eta > 0$ such that $|\theta - \theta_0| \le \eta$ implies $\gamma_1(\theta) > \varepsilon$. Now compare this process with the process generated by $L_0 + \varepsilon/r^\delta(d/dr)$. This process can be shown to explode by Proposition 3.4 using $u = c + r^{-\nu}$, for $0 < \nu < -\delta - 1$. Since our process eventually remains in a region where $\gamma_1(\theta) \ge \varepsilon$, it also explodes. Alternatively, one could show explosion by the method we employ in (d) to find the growth rates.

(d) Say $\Theta(\omega) \bmod 2\pi = \theta_0$. Let ε and η correspond to θ_0 as in (c). Let $T_1 = \sup\{t \ge 0: |\theta(t) - \theta_0| \ge \delta\}$, $T_2 = \sup\{t: r(t) \le 1\}$ and set $T = T_1 \vee T_2$. Now proceed as in the proof of Theorem 1.1(d), using the above T instead of the T used there and using the function $u = r$ rather than $u = rf_2(\theta)$. □

PROOF OF THEOREM 1.3. We first handle the case $k < 1$, which is simpler. Let $u(r,\theta) = r^m \exp(r^q \rho(\theta))$ for $q = k - 1 < 0$, and $\rho(\theta)$ and m as yet unspecified. We have

$$Lu = \left[r^{m-2}\left(\frac{m(m-1)}{2}e_1(\theta) + \frac{m}{2}e_4(\theta) + \gamma_2(\theta)\rho'(\theta)\right)\right.$$
$$\left. + m\gamma_1(\theta)r^{m-1-\delta} + O(r^{m-2+q})\right]\exp(r^q\rho(\theta)).$$

In order that Lu be nonpositive for all large r, we require

$$\frac{m(m-1)}{2}e_1(\theta) + \frac{m}{2}e_4(\theta) + \gamma_2(\theta)\rho'(\theta) < 0, \quad \text{if } \delta > 1$$

and

$$\frac{m(m-1)}{2}e_1(\theta) + \frac{m}{2}e_4(\theta) + \gamma_2(\theta)\rho'(\theta) + m\gamma_1(\theta) < 0, \quad \text{if } \delta = 1.$$

We continue now under the assumption $\delta = 1$, the case $\delta > 1$ being handled identically. We want

$$(4.8) \qquad \rho'(\theta) < \frac{m}{2}\frac{(e_1(\theta) - e_4(\theta) - 2\gamma_1(\theta))}{\gamma_2(\theta)} - \frac{m^2}{2}\frac{e_1(\theta)}{\gamma_2(\theta)}.$$

Now in order to invoke Proposition 3.1 for transience or Proposition 3.2 for recurrence, we need $m < 0$ or $m > 0$, respectively. First consider $m > 0$, and assume the assumption of (a), that

$$\nu \equiv \int_0^{2\pi} H(\theta)G(\theta)\, d\theta = \int_0^{2\pi} \frac{e_4(\theta) - e_1(\theta) + 2\gamma_1(\theta)}{\gamma_2(\theta)}\, d\theta < 0.$$

Then integrating the right-hand side of (4.8), we obtain

$$\eta(m) = \frac{-m}{2}\nu - \frac{m^2}{2}\int_0^{2\pi} \frac{e_1(\theta)}{\gamma_2(\theta)}\, d\theta$$

and thus $\eta(m) > 0$ for sufficiently small m. For such an m, let

$$(4.9)\quad \rho(\theta) = \frac{m}{2}\int_0^{\theta} \frac{e_1(\theta) - e_4(\theta) - 2\gamma_1(\theta)}{\gamma_2(\theta)}\, d\theta - \frac{m^2}{2}\int_0^{\theta} \frac{e_1(\theta)}{\gamma_2(\theta)}\, d\theta - \eta(m)\frac{\theta}{2\pi}.$$

Now, if $e_1, e_4 \in C^1(S^1)$, then $\rho(\theta) \in C^2(S^1)$ and satisfies (4.8). This shows recurrence. In the general case, a standard mollification procedure applied to ρ works, since e_1 and e_4 are assumed to be Lipschitz. Similarly, if the assumption of (b) holds, that is, $\nu > 0$, then $\eta(m) > 0$ for $m < 0$ and $|m|$ sufficiently small. Define $\rho(\theta)$ by (4.9) for such an m. Again, if $e_1, e_4 \in C^1(S^1)$, then $\rho(\theta) \in C^2(S^1)$ and satisfies (4.8). This shows transience. The general case is proved by mollification. Now we turn to the case $k \geq 1$. Let $u = r^m\rho(\theta)$ with m and $\rho(\theta) > 0$ as yet unspecified. We have

$$Lu = \tfrac{1}{2}e_2(\theta)r^{m-2}(\rho'' + g(\theta, m)\rho' + h(\theta, m)\rho) + \text{lower-order terms},$$

where

$$g(\theta, m) = \begin{cases} \dfrac{me_3(\theta) + e_5(\theta) + 2\gamma_2(\theta)}{e_2(\theta)}, & \text{if } k = 1, \\[2ex] \dfrac{me_3(\theta) + e_5(\theta)}{e_2(\theta)}, & \text{if } k > 1, \end{cases}$$

and

$$h(\theta, m) = \begin{cases} \dfrac{m(m-1)e_1(\theta) + me_4(\theta) + 2m\gamma_1(\theta)}{e_2(\theta)}, & \text{if } \delta = 1, \\[2ex] \dfrac{m(m-1)e_1(\theta) + me_4(\theta)}{e_2(\theta)}, & \text{if } \delta > 1. \end{cases}$$

To prove the theorem, it suffices by Propositions 3.1 and 3.2 to show that under the assumption of (a), we can find an $m > 0$ and a $\rho > 0$ for which $\rho'' + g\rho' + h\rho < 0$, and, under the assumption of (b), we can find an $m < 0$ and a $\rho > 0$ for which $\rho'' + g\rho' + h\rho < 0$. We consider the eigenvalue problem

$$(4.10)\quad -(\rho'' + g\rho' + h\rho) = \lambda\rho, \quad \text{with } \rho(0) = \rho(2\pi) \text{ and } \rho'(0) = \rho'(2\pi).$$

By the Sturm–Liouville theory, the eigenfunction ρ_0 (normalized so that

$\int_0^{2\pi}\rho_0^2\,d\theta = 1$) corresponding to the smallest eigenvalue λ_0 is strictly positive (see [3], Chapter 8, Section 3). Thus, we can complete the proof by showing that under the assumption of (a), $\lambda_0(m) > 0$ for sufficiently small $m > 0$, and under the assumption of (b), $\lambda_0(m) > 0$ for $m < 0$ with $|m|$ sufficiently small. Now by the Rayleigh–Ritz variational formula, we have

$$(4.11)\quad \lambda_0(m) = \inf_{\rho\in\Lambda}\int_0^{2\pi} e^{V(\theta,m)}(\rho'(\theta))^2\,d\theta - \int_0^{2\pi} e^{V(\theta,m)}h(\theta,m)\rho^2(\theta)\,d\theta,$$

with $\Lambda = \{\rho \in C^1(S^1)\colon \int_0^{2\pi}e^{V(\theta,m)}\rho^2(\theta)\,d\theta = 1\}$ and $V(\theta,m) = \int_0^\theta g(s,m)\,ds$. It is clear from (4.11), or by inspection from (4.10) that $\lambda_0(0) = 0$. The corresponding eigenfunction is $\rho_0(\theta,0) \equiv 1/\sqrt{2\pi}$. Also, since g and h are analytic in m, so are $\lambda_0(m)$ and $\rho_0(\theta,m)$. We will write $\lambda_0(m) = \sum_{j=1}^\infty \lambda_j m^j$ and

$$\rho_0(\theta,m) = \frac{1}{\sqrt{2\pi}} + \sum_{j=1}^\infty \rho_j(\theta)m^j.$$

To complete the proof, we need to show that $\lambda'_0(0) = \lambda_1 > 0$ under the assumption in (a), and $\lambda'_0(0) = \lambda_1 < 0$ under the assumption in (b). Plugging the power series for $\rho_0(\theta,m)$ and for $\lambda_0(m)$ into (4.11) and collecting terms shows that

$$\lambda_1 = -\frac{1}{2\pi}\int_0^{2\pi}H(\theta)G(\theta)\,d\theta.$$

This completes the proof. □

We remark that in the case $\int_0^{2\pi}H(\theta)G(\theta)\,d\theta = 0$, it is natural to consider $\lambda''_0(0) = \lambda_2$. However, after a somewhat involved calculation, one finds that in this case $\lambda_2 < 0$. Thus our method cannot handle the borderline case.

5. Proofs of the theorems from Section 2. Recall that we are assuming $p_1 = p_2 = 1$. We first give the

PROOF OF THEOREM 2.1. We have

$$U(r,\theta) = \int_0^\theta \frac{\gamma_1}{\gamma_2}(s)\,ds - \int_1^r s^{\delta-k-1}\,ds.$$

[Throughout this section, we refrain from integrating $\int_1^r s^{\delta-k-1}\,ds$ since in the general case this term will be $\int_1^r (p_2/p_1)(s)s^{\delta-k-1}\,ds$ and cannot be integrated.] We will treat parts (a) and (b) of the theorem first; hence in what follows, it is assumed that $k \le \delta$. Applying Itô's formula yields

$$\begin{aligned}
&U(r(t),\theta(t))\\
&\quad = U(r_0,\theta_0) + \int_0^t O(r^{\delta-k-2}(s))\,ds\\
(5.1)\qquad &\qquad + \int_0^t \frac{\gamma_1}{\gamma_2}(\theta(s))(\sigma_{12}(r(s),\theta(s))\,dB_1(s) + \sigma_{22}(r(s),\theta(s))\,dB_2(s))\\
&\qquad - \int_0^t r^{\delta-k-1}(s)(\sigma_{11}(r(s),\theta(s))\,dB_1(s) + \sigma_{12}(r(s),\theta(s))\,dB_2(s)).
\end{aligned}$$

(The σ_{ij} and the $B_i(t)$ are as in Section 4.) From Theorem 1.1(d), we have

$r(t) \le \lambda_2(\omega)t^{1/(1+\delta)}$ for large t if $\delta > -1$ and $r(t) \le e^{\lambda_2(\omega)t}$ for large t if $\delta = -1$. From this, one sees that the nonstochastic integral in (5.1) converges if $2\delta - k - 1 < 0$. Let

$$m_1(t) = \int_0^t r^{\delta-k-1}(s)\big(\sigma_{11}(r(s),\theta(s))\,dB_1(s) + \sigma_{12}(r(s),\theta(s))\,dB_2(s)\big)$$

and

$$m_2(t) = \int_0^t \frac{\gamma_1}{\gamma_2}(\theta(s))\big(\sigma_{12}(r(s),\theta(s))\,dB_1(s) + \sigma_{22}(r(s),\theta(s))\,dB_2(s)\big).$$

The variance processes of these martingales are

$$V_1(t) = \int_0^t r^{2\delta-2k-2}(s)\big(\sigma_{11}^2(r(s),\theta(s)) + \sigma_{12}^2(r(s),\theta(s))\big)\,ds$$

and

$$V_2(t) = \int_0^t \frac{\gamma_1^2}{\gamma_2^2}(\theta(s))\big(\sigma_{12}^2(r(s),\theta(s)) + \sigma_{22}^2(r(s),\theta(s))\big)\,ds.$$

Define $\tau_i(t)$, $i = 1,2$, by $V_i(\tau_i(t)) = t$. Then, as is well known, $z_i(t) \equiv m_i(\tau_i(t))$ is a Brownian motion up to time $\tau_i^{-1}(\infty) = V_i(\infty)$. Since $m_i(t) = z_i(\tau_i^{-1}(t))$, to show that $m_i(t)$ converges as $t \to \infty$, it suffices to show that $\tau_i^{-1}(\infty) < \infty$ almost surely. Recall that σ_{11} is bounded and that σ_{12} and σ_{22} are $O(r^{-1})$ as $r \to \infty$. Since $\delta \ge k$, we need only show that

$$\int_0^\infty r^{2\delta-2k-2}(s)\,ds < \infty \quad \text{almost surely.} \tag{5.2}$$

Using the asymptotic rates from Theorem 1.1(d) again, one finds that (5.2) holds if $3\delta - 2k - 1 < 0$. Since $\delta \ge k$, $3\delta - 2k - 1 < 0$ implies that $2\delta - k - 1 < 0$. Thus, if $3\delta - 2k - 1 < 0$, or equivalently, $k > \delta - \frac{1}{2}(1 - \delta)$, then the right-hand side of (5.1) converges. Thus,

$$\mathscr{U}(\omega) = \lim_{t\to\infty} U(r(t),\theta(t))$$

exists and is finite almost surely $[P_{r,\theta}]$. This proves the first contention in (a). For the second claim in (a), we utilize the analysis in Theorem 1.1(d). We only worked that out in detail for the case $\delta > 0$ and $k = \delta$. Hence, we shall prove the second claim in (a) for this case. The other cases are proven similarly, using the corresponding analysis in Theorem 1.1(d). One should convince oneself by reviewing the just completed proof, that it is sufficient to show that

$$\limsup_{r_0\to\infty} \sup_{\theta\in S^1} P_{r_0,\theta}\left(\int_0^\infty r^{2\delta-2k-2}(s)\,ds > \varepsilon\right) = 0, \quad \text{for all } \varepsilon > 0.$$

Now, $\lim_{t\to\infty}\theta(t) = \infty$ almost surely, by the first part of (a) and the fact that this is true for the deterministic trajectories. From this and from the transience of the process, we can conclude that

$$\lim_{r_0\to\infty} \sup_{\theta\in S^1} P_{r_0,\theta}\big(r(t) \ge c, \text{for all } 0 \le t < \infty\big) = 1,$$

where c is as in (4.2). Thus it suffices to show that

$$\limsup_{r_0\to\infty}\sup_{\theta\in S^1} P_{r_0,\theta}\left(\int_0^\infty r^{2\delta-2k-2}(s)\,ds > \varepsilon \,|\, r(t)\ge c, \text{ for all } 0\le t<\infty\right) = 0.$$

Now consider the analysis from (4.3) to (4.5) under the condition $r(t)\ge c$ for all $0\le t<\infty$. Then $\sigma=0$ and $r(\sigma)=r_0$. (4.5) becomes

$$R(t)\ge N^{-1}\Big(\big(r_0 f_2(\theta_0)+C(\omega)\big)^{1+\delta}+N(1+\delta)f_0^{\delta}t\Big)^{1/(1+\delta)}$$
$$-N^{-1}\big(r_0 f_2(\theta_0)+C(\omega)t^{(1+\nu)/2}+C(\omega)\big).$$

Plugging this back into (4.4) and letting $f_m=\sup_{\theta\in S^1} f_2(\theta)$, we have

$$\begin{aligned} r(t)\ge{}& f_m^{-1}N^{-1}\varepsilon\Big(\big(r_0 f_2(\theta_0)+C(\omega)\big)^{1+\delta}+N(1+\delta)f_0^{\delta}t\Big)^{1/(1+\delta)}\\ &+f_m^{-1}\Big(1-\frac{\varepsilon}{N}\Big)r_0 f_2(\theta_0)-f_m^{-1}\Big(1+\frac{\varepsilon}{N}\Big)C(\omega)t^{(1+\nu)/2}-C(\omega)f_m^{-1}\Big(1+\frac{\varepsilon}{N}\Big). \end{aligned} \tag{5.3}$$

Replace $f_2(\theta_0)$ by f_0 $(=\inf_{\theta\in S^1} f_2(\theta))$ on the right-hand side of (5.3) and call the resulting expression $\tilde r(t, r_0, \omega)$. What we have now is that for every $\theta_0\in S^1$, $r(t)=r(t,r_0,\theta_0,\omega)\ge\tilde r(t,r_0,\omega)$, if $r(t)=r(t,r_0,\theta_0,\omega)\ge c$, for all $0\le t<\infty$. Recalling the remark about P and P_{r_0,θ_0} which can be found between (4.3) and (4.4), and noting that $2\delta-2k-2<0$, we have,

$$P_{r_0,\theta_0}\left(\int_0^\infty r^{2\delta-2k-2}(s)\,ds>\varepsilon\,|\,r(t)\ge c, 0\le t<\infty\right)$$
$$\le P\left(\int_0^\infty \tilde r^{2\delta-2k-2}(s,r_0,\omega)\,ds>\varepsilon\right)$$

and

$$\begin{aligned} &\limsup_{r_0\to\infty}\sup_{\theta_0\in S^1} P_{r_0,\theta_0}\left(\int_0^\infty r^{2\delta-2k-2}(s)\,ds>\varepsilon\,|\,r(t)\ge c, 0\le t<\infty\right)\\ &\quad\le \limsup_{r_0\to\infty} P\left(\int_0^\infty \tilde r^{2\delta-2k-2}(s,r_0,\omega)\,ds>\varepsilon\right). \end{aligned} \tag{5.4}$$

But as $\tilde r(s, r_0, \omega)$ is given by the right-hand side of (5.3) with $f_2(\theta_0)$ changed to f_0, it is easy to see that the right-hand side of (5.4) is zero. This completes the proof of (a). For (b), consider (5.1). First assume $\delta\ne-1$. Since $r(t)$ grows on the order $t^{1/(1+\delta)}$ the nonstochastic integral in (5.1) grows on no larger an order than $t^{(2\delta-k-1)/(1+\delta)}$ if $2\delta-k-1>0$, and $\log t$, if $2\delta-k-1=0$. If $2\delta-k-1<0$, then this integral is bounded on $(0,\infty)$. The integrand in the variance process $V_2(t)$ of the martingale $m_2(t)$ is on the order $r^{-2}(t)$. In terms of t, this integral is on the order $t^{(\delta-1)/(\delta+1)}$. Hence $V_2(\infty)<\infty$ almost surely and $m_2(t)$ converges almost surely to a finite limit. The integrand of the variance process $V_1(t)$ of $m_1(t)$ is on the order $r^{2\delta-2k-2}(t)$ or $t^{(2\delta-2k-2)/(1+\delta)}$. Hence, $V_1(t)$ grows on the order of $t^{(3\delta-2k-1)/(1+\delta)}$ if $3\delta-2k-1>0$ and on the order of $\log t$ if $3\delta-2k-1=0$ (by assumption $3\delta-2k-1\ge0$). Recalling from (a) the relationship between $z_1(t)$, $m_1(t)$, $V_1(t)$ and $\tau_1(t)$, we see that from the law of the iterated logarithm, $m_1(t)$ will exceed $t^{(3\delta-2k-1)/(2(1+\delta))}$ [or $(\log t)^{1/2}$ if $3\delta - 2k - 1 = 0$] and dip below $-t^{(3\delta-2k-1)/(2(1+\delta))}$ [or $-(\log t)^{1/2}$ if $3\delta - 2k - 1 = 0$] for arbitrarily large values of t. Now

$$\frac{3\delta - 2k - 1}{2(1+\delta)} > \frac{2\delta - k - 1}{1+\delta},$$

since a little algebra shows this to be equivalent to $\delta < 1$. (b) now follows immediately. For later use in the proof of Theorem 2.2, note that by the law of the iterated logarithm, for $3\delta - 2k - 1 \geq 0$, $|m_1(t)|$ will grow more slowly than $t^{(3\delta-2k-1)/(2(1+\delta))+\eta}$, for any $\eta > 0$. In fact then, the right-hand side of 5.1 will grow more slowly than this. Now consider the case $\delta = -1$. By assumption, $k > \delta - (1-\delta) = -3$. Thus $\delta - k - 2 = -3 - k < 0$. Since $r(t)$ grows exponentially, the nonstochastic integral in (5.1) converges. The integrand of the variance process $V_2(t)$ of the martingale $m_2(t)$ is on the order $r^{-2}(t)$. Again, since $r(t)$ grows exponentially, $V_2(\infty) < \infty$ almost surely and $m_2(t)$ converges. The integrand of the variance process $V_1(t)$ of the martingale $m_1(t)$ is on the order $r^{2\delta-2k-2}(t)$. By assumption, $2\delta - 2k - 2 = 3\delta - 2k - 1 \geq 0$. Thus, $V_1(t)$ grows at least linearly and $m_1(t)$ will infinitely often exceed $t^{1/2}$ and dip below $-t^{1/2}$. (b) for the case $\delta = -1$ now follows. For (c), there is not much to prove. The existence of $\Theta(\omega)$ was proven in Theorem 1.2. This coupled with the fact that $k > \delta$ shows that

$$\mathscr{U}(\omega) \text{ exists and } \mathscr{U}(\omega) = \int_0^{\Theta(\omega)} \frac{\gamma_1}{\gamma_2}(s)\,ds - \int_1^{\infty} s^{\delta-k-1}\,ds.$$

That $P_{r,\theta}(\Theta(\omega) \in dx) \Rightarrow \delta_{\theta_0}(dx)$ as $r \to \infty$ and $\theta \to \theta_0$ with $\gamma_1(\theta_0) > 0$, comes directly from (4.6). □

Proof of Theorem 2.2. Divide both sides of (5.1) by $\int_1^{r(t)} s^{\delta-k-1}\,ds$ and call the resulting equation (5.1′). Using the fact that $\int_0^{2\pi}(\gamma_1/\gamma_2)(s)\,ds = \gamma$, the theorem will follow if we show that the right-hand side of (5.1′) goes to zero almost surely as $t \to \infty$. We know from Theorem 2.1(a) that if $k > \delta - \frac{1}{2}(1-\delta)$, then the right-hand side of (5.1) is bounded, for a.e. ω, and hence the right-hand side of (5.1′) goes to zero a.s. as $t \to \infty$. For $k \leq \delta - \frac{1}{2}(1-\delta)$, we noted at the end of the proof of Theorem 2.1(b) that the right-hand side of (5.1) grows more slowly than $t^{(3\delta-2k-1)/(2(1+\delta))+\eta}$, for any $\eta > 0$. Now $\int_1^{r(t)} s^{\delta-k-1}\,ds$ grows on the order $t^{(\delta-k)/(1+\delta)}$ and

$$\frac{\delta - k}{1+\delta} > \frac{3\delta - 2k - 1}{2(1+\delta)}$$

since this is equivalent to $\delta < 1$. Thus again, the right-hand side of (5.1′) goes to zero a.s. as $t \to \infty$. □

For (b) of Theorem 2.3, we will need a couple of lemmas. If a path has the property that there exist times $t_1 < t_2$ such that $\theta(t_2) = \theta(t_1) + 2\pi$ and $r(t_1) = r(t_2)$, we will say that the path makes a loop. Define the radius of such a loop by $\inf_{t_1 \leq t \leq t_2} r(t)$.

LEMMA 5.1. *Let* $-1 < \delta < 1$ *and assume* $\gamma > 0$. *If* $k \leq \delta - (1-\delta) = 2\delta - 1$, *and* $0 < \liminf_{r\to\infty} p_1(r) \leq \limsup_{r\to\infty} p_1(r) < \infty$, *then almost every* $[P_{r,\theta}]$ *path makes loops with arbitrarily large radii.*

PROOF. Define $\tau(t)$ by

$$\int_0^{\tau(t)} \frac{\gamma_2(\theta(s))}{r^{k+1}(s)}\, ds = t.$$

Since $\gamma_2 > 0$, $\tau(t)$ is continuous in t. Since $r(t)$ grows on the order $t^{1/(1+\delta)}$, $\tau(t)$ grows on the order $t^{(1+\delta)/(\delta-k)}$. Consider the time changed process $(r'(t), \theta'(t)) = (r(\tau(t)), \theta(\tau(t)))$. The generator for this process is

$$\tilde{L} = \frac{r^{k+1}}{\gamma_2} L = \frac{r^{k+1}}{\gamma_2} L_0 + r^{k+1-\delta}\frac{\gamma_1}{\gamma_2}\frac{d}{dr} + \frac{d}{d\theta}.$$

The new σ-matrix is

$$\tilde{\sigma} = \frac{r^{(k+1)/2}}{\gamma_2^{1/2}}\sigma.$$

It suffices to prove the lemma for $(r'(t), \theta'(t))$. Let $(\tilde{r}(t), \tilde{\theta}(t))$ be the solution to the stochastic differential equation

$$\tilde{r}(t) = r_0 + \int_0^t \tilde{r}^{k+1-\delta}(s)\frac{\gamma_1}{\gamma_2}(\tilde{\theta}(s))\, ds + \int_0^t \left(\tilde{\sigma}_{11}(\tilde{r}(s), \tilde{\theta}(s))\, dB_1(s) + \tilde{\sigma}_{12}(\tilde{r}(s), \tilde{\theta}(s))\, dB_2(s)\right),$$

$$\tilde{\theta}(t) = \theta_0 + t + \int_0^t \left(\tilde{\sigma}_{12}(\tilde{r}(s), \tilde{\theta}(s))\, dB_1(s) + \tilde{\sigma}_{22}(\tilde{r}(s), \tilde{\theta}(s))\, dB_2(s)\right).$$

Since $(\tilde{r}(\cdot), \tilde{\theta}(\cdot))$ has the same distribution as $(r'(\cdot), \theta'(\cdot))$, it suffices to prove the lemma for $(\tilde{r}(t), \tilde{\theta}(t))$. Call the probability measure associated with this process $\tilde{P}_{r_0, \theta_0}$. Let

$$M(t) = \int_0^t \left(\tilde{\sigma}_{11}(\tilde{r}(s), \tilde{\theta}(s))\, dB_1(s) + \tilde{\sigma}_{12}(\tilde{r}(s), \tilde{\theta}(s))\, dB_2(s)\right)$$

and

$$N(t) = \int_0^t \left(\tilde{\sigma}_{12}(\tilde{r}(s), \tilde{\theta}(s))\, dB_1(s) + \sigma_{22}(\tilde{r}(s), \tilde{\theta}(s))\, dB_2(s)\right).$$

Note that there exist constants $\lambda_1(\omega) > 0$, $\lambda_2(\omega) > 0$ and $t_0(\omega) \geq 0$ such that

$$\lambda_1 t^{1/(\delta-k)} \leq \tilde{r}(t) \leq \lambda_2 t^{1/(\delta-k)}, \quad \text{for all } t \geq t_0. \tag{5.5}$$

This comes from Theorem 1.1(d), the growth rate on $\tau(t)$ and the fact that $(r'(\cdot), \theta'(\cdot))$ has the same distribution as $(\tilde{r}(\cdot), \tilde{\theta}(\cdot))$.

We now show that

$$\limsup_{s\to\infty} \sup_{t_1, t_2 \geq s} \left|\tilde{\theta}(t_2) - \tilde{\theta}(t_1) - (t_2 - t_1)\right| = 0 \quad \text{almost surely } \left[\tilde{P}_{r_0, \theta_0}\right]. \tag{5.6}$$

The variance process for $N(t)$ is

$$V_N(t) = \int_0^t \left(\tilde{\sigma}_{12}^2(\tilde{r}(s), \tilde{\theta}(s)) + \tilde{\sigma}_{22}^2(\tilde{r}(s), \tilde{\theta}(s))\right) ds,$$

and $\tilde{\sigma}_{12}(\tilde{r}, \tilde{\theta})$ and $\tilde{\sigma}_{22}(\tilde{r}, \tilde{\theta})$ grow on the order $\tilde{r}^{(k-1)/2}$. This combined with (5.5) and the fact that $\delta < 1$ shows that $V_N(\infty) < \infty$ almost surely $[\tilde{P}_{r_0, \theta_0}]$ and hence that $N(t)$ is a convergent martingale. This is enough to give (5.6). Now define the stopping time $\rho(t) = \inf\{s \geq 0: \tilde{\theta}(s) = \tilde{\theta}(0) + t\}$. From (5.6) we have

$$\limsup_{s \to \infty} \sup_{t_1, t_2 > s} |\rho(t_2) - \rho(t_1) - (t_2 - t_1)| = 0. \tag{5.7}$$

Define

$$H_n = \begin{cases} 1, & \text{if } \tilde{r}(\rho(2n\pi + 2\pi)) > \tilde{r}(\rho(2n\pi)), \\ 0, & \text{if } \tilde{r}(\rho(2n\pi + 2\pi)) = \tilde{r}(\rho(2n\pi)), \\ -1, & \text{if } \tilde{r}(\rho(2n\pi + 2\pi)) < \tilde{r}(\rho(2n\pi)). \end{cases}$$

Of course, $\tilde{P}_{r_0, \theta_0}(H_n = 0, \text{ for some } n) = 0$ so we can ignore the possibility $H_n = 0$. Let $A = \{H_n = 1, \text{ i.o.}\}$ and let $B = \{H_n = -1, \text{ i.o.}\}$. Since the process is transient, $\tilde{P}_{r_0, \theta_0}(A) = 1$. We can prove the lemma by showing that $\tilde{P}_{r_0, \theta_0}(B) = 1$. For if both A and B occur infinitely often, then for infinitely many n, we will have $\tilde{r}(\rho(2n\pi) + 2\pi)) > \tilde{r}(\rho(2n\pi))$ and $\tilde{r}(\rho(2n\pi + 4\pi)) < \tilde{r}(\rho(2n\pi + 2\pi))$.

Yet, for each n that this occurs, the only way the path can avoid crossing itself (since it eventually must run off to infinity) is by "unwinding", that is, by having its θ-component decrease by at least 2π. By (5.6) such an unwinding can occur only finitely often. This then shows that the path makes an infinite number of loops and by transience, it must make loops of arbitrarily large radii. We are left with showing $\tilde{P}_{r_0, \theta_0}(B) = 1$. By the strong Markov property, it suffices to show that

$$\tilde{P}_{r_0, \theta_0}(H_n = -1, \text{for some } n) = 1, \quad \text{for all } (r_0, \theta_0).$$

Let

$$S(t) = r_0 + \int_0^t \tilde{r}^{k+1-\delta}(s) \frac{\gamma_1}{\gamma_2}(\tilde{\theta}(s))\, ds,$$

let

$$S_n = S(\rho(2n\pi + 2\pi)) - S(\rho(2n\pi)), \qquad n = 0, 1, 2, \dots,$$

and let

$$M_n = M(\rho(2n\pi + 2\pi)) - M(\rho(2n\pi)), \qquad n = 0, 1, 2, \dots.$$

Then we can write

$$r(\rho(2n\pi + 2\pi)) - r(\rho(2n\pi)) = S_n + M_n.$$

We need to show that $P_{r_0, \theta_0}(S_n + M_n < 0, \text{ for some } n) = 1$.

From (5.5) and (5.7), there exists a $c(\omega) > 0$ such that

$$|S_n| \leq c n^{(k+1-\delta)/(\delta-k)}, \qquad n = 1, 2, \dots.$$

Let

$$\tilde{M}_n = \frac{M_n}{n^{(k+1)/(2(\delta-k))}} \quad \text{and} \quad \tilde{S}_n = \frac{S_n}{n^{(k+1)/(2(\delta-k))}}.$$

Then for $n \geq 1$, we have $|\tilde{S}_n| \leq c(\omega) n^{(k+1-2\delta)/(2(\delta-k))} \leq c(\omega)$ since by assumption, $k + 1 - 2\delta \leq 0$. Thus,

$$\begin{aligned}\tilde{P}_{r_0,\theta_0}(S_n + M_n < 0, \text{for some } n) &\geq \tilde{P}_{r_0,\theta_0}\big(\tilde{M}_n < -c(\omega), \text{for some } n \geq 1\big)\\ &\geq \tilde{P}_{r_0,\theta_0}\Big(\inf_{n\geq 1} \tilde{M}_n = -\infty\Big).\end{aligned}$$

To complete the proof, we will show that

$$\tilde{P}_{r_0,\theta_0}\Big(\inf_{n\geq 1} \tilde{M}_n = -\infty\Big) = 1.$$

Define $\zeta(t)$ by

$$\int_0^{\zeta(t)} \big(\tilde{\sigma}_{11}^2(\tilde{r}(s),\tilde{\theta}(s)) + \tilde{\sigma}_{12}^2(\tilde{r}(s),\tilde{\theta}(s))\big)\, ds = t.$$

Then $B(t) \equiv M(\zeta(t))$ is a Brownian motion and $M(t) = B(\zeta^{-1}(t))$. Note that for each $s \geq 0$, $\zeta^{-1}(s)$ is a stopping time relative to the filtration $\{\mathscr{F}_{\zeta(t)}, t \geq 0\}$ to which the Brownian motion $B(t)$ is adapted. We have

$$\tilde{M}_n = \frac{B\big(\zeta^{-1}(\rho(2n\pi + 2\pi))\big) - B\big(\zeta^{-1}(\rho(2n\pi))\big)}{n^{(k+1)/(2(\delta-k))}}. \tag{5.8}$$

From (5.5), (5.7), the fact that $\tilde{\sigma}_{11}^2(\tilde{r},\tilde{\theta}) + \tilde{\sigma}_{12}^2(\tilde{r},\tilde{\theta})$ grows on the order $\tilde{r}^{k+1}$ and the fact that

$$\begin{aligned}&\zeta^{-1}(\rho(2n\pi + 2\pi)) - \zeta^{-1}(\rho(2n\pi))\\ &\quad = \int_{\rho(2n\pi)}^{\rho(2n\pi+2\pi)} \big(\tilde{\sigma}_{11}^2(\tilde{r}(s),\tilde{\theta}(s)) + \tilde{\sigma}_{12}^2(\tilde{r}(s),\tilde{\theta}(s))\big)\, ds,\end{aligned}$$

there exist $c_0(\omega)$ and $c_1(\omega)$ such that

$$\begin{aligned}c_0(\omega) n^{(k+1)/(\delta-k)} &\leq \zeta^{-1}(\rho(2n\pi + 2\pi)) - \zeta^{-1}(\rho(2n\pi))\\ &\leq c_1(\omega) n^{(k+1)/(\delta-k)}.\end{aligned} \tag{5.9}$$

Now leave the above problem for a moment and consider the situation where a sequence $\eta_1 < \eta_2 < \cdots$ of stopping times satisfying

$$c_0 n^{(k+1)/(\delta-k)} \leq \eta_{n+1} - \eta_n \leq c_1 n^{(k+1)/(\delta-k)},$$

for constants c_0 and c_1, is adapted to a Brownian motion $\mathscr{B}(t)$ living on a probability space $(\hat{\Omega}, \hat{P})$. Then for any $K > 0$,

$$\hat{P}\left(\inf_{n>0} \frac{\mathscr{B}(\eta_{n+1}) - \mathscr{B}(\eta_n)}{n^{(k+1)/(2(\delta-k))}} < -K\right) = 1. \tag{5.10}$$

This follows directly from the fact that

$$\hat{P}\left(\frac{\mathscr{B}(\eta_{n+1}) - \mathscr{B}(\eta_n)}{n^{(k+1)/(2(\delta-k))}} < -K \right) > \varepsilon(K) > 0,$$

where $\varepsilon(K)$ is independent of n. To prove this inequality, we write

$$\begin{aligned}\hat{P}\left(\frac{\mathscr{B}(\eta_{n+1}) - B(\eta_n)}{n^{(k+1)/(2(\delta-k))}} < -K \right) &\geq \hat{P}\left(\frac{B\left(c_0 n^{(k+1)/(\delta-k)}\right)}{n^{(k+1)/(2(\delta-k))}} < -2K \right) \\ &\quad \times \hat{P}\left\{ \sup_{0 \leq s \leq (c_1 - c_0) n^{(k+1)/(2(\delta-k))}} \frac{B(s)}{n^{(k+1)/(2(\delta-k))}} < K \right\} \\ &> \varepsilon(K) > 0,\end{aligned}$$

where the two inequalities are a consequence of Brownian scaling. Returning to our problem, let

$$\begin{aligned}\mathscr{E} = \{\omega\colon \exists \text{ constants } c_0(\omega) \text{ and } c_1(\omega) \text{ such that} \\ c_0(\omega) n^{(k+1)/(\delta-k)} \leq \zeta^{-1}(\rho(2n\pi + 2\pi)) - \zeta^{-1}(\rho(2n\pi)) \\ \leq c_1(\omega) n^{(k+1)/(\delta-k)},\ n \geq 1\}.\end{aligned}$$

From (5.9), $\tilde{P}_{r_0, \theta_0}(\mathscr{E}) = 1$. Hence

$$\tilde{P}_{r_0, \theta_0}\left(\inf_{n \geq 1} \tilde{M}_n = -\infty \right) = \tilde{P}_{r_0, \theta_0}\left(\inf_{n \geq 1} \tilde{M}_n = -\infty | \mathscr{E} \right). \tag{5.11}$$

Now (5.8), (5.10) and (5.11) show that

$$\tilde{P}_{r_0, \theta_0}\left(\inf_{n \geq 1} \tilde{M}_n = -\infty \right) = 1.$$

This completes the proof of the lemma. □

Let $\Sigma = C([0, \infty), R^2)$, define $Q_{r,\theta} = P_{r,\theta} \times P_{r,\theta}$ on $\Sigma \times \Sigma$ and denote points in $\Sigma \times \Sigma$ by (ω_1, ω_2). Say that two paths ω_1 and ω_2 intersect at arbitrarily large times if there exist sequences $\{s_n\}$ and $\{t_n\}$ with $s_n \to \infty$ and $t_n \to \infty$ as $n \to \infty$ such that $\omega_1(s_n) = \omega_2(t_n)$, $n = 1, 2, \dots$. Let $\mathscr{A} \subset \Sigma \times \Sigma$ be defined by

$$\mathscr{A} = \{(\omega_1, \omega_2) \in \Sigma \times \Sigma\colon \omega_1 \text{ and } \omega_2 \text{ intersect at arbitrarily large times}\}.$$

LEMMA 5.2. *Under the conditions of Lemma* 5.1, $Q_{r,\theta}(\mathscr{A}) = 1$.

PROOF. Let $C_1 = \{\omega \in \Sigma\colon \omega \text{ makes loops of arbitrarily large radii}\}$ and let $C_2 = \{\omega \in \Sigma\colon |\omega(t)| \to \infty \text{ as } t \to \infty\}$. Then by Lemma 5.1, $P_{r,\theta}(C_1) = 1$ and by transience, $P_{r,\theta}(C_2) = 1$. Thus $Q_{r,\theta}(C_1 \times C_2) = 1$. But $C_1 \times C_2 \subset \mathscr{A}$. □

We can now give the

PROOF OF THEOREM 2.3. (a) First consider the case $k > \delta$. Then $\Theta(\omega)$ exists by Theorem 1.2(b). Let $\tilde{\Theta}(\omega) = \Theta(\omega) \bmod 2\pi$ and let $g(\theta)$ be any bounded measurable function on S^1 which is not almost surely constant (with respect to

Lebesgue measure on S^1) on the set $\{\theta: \gamma_1(\theta) > 1\}$. Let $H(r, \theta) = E_{r,\theta} g(\tilde{\Theta}(\omega))$. Then $H(r, \theta)$ is bounded and periodic of period 2π in θ and is harmonic since $\tilde{\Theta}(\omega)$ is an invariant random variable. That H is nonconstant comes from the last statement in Theorem 2.1(c). In the case $-1 \le \delta < 1$ and $\delta - \frac{1}{2}(1-\delta) < k \le \delta$, $\mathscr{U}(\omega)$ exists by Theorem 2.1(a). Let $\tilde{\mathscr{U}}(\omega) = \mathscr{U}(\omega) \bmod \gamma$ and let $g(\theta)$ be a bounded measurable function on S^1 which is not almost surely constant (with respect to Lebesgue measure on S^1). Let $H(r, \theta) = E_{r,\theta} g((2\pi/\gamma)\tilde{\mathscr{U}}(\omega))$. Then $H(r, \theta)$ is bounded and periodic of period 2π in θ and is harmonic since $\mathscr{U}(\omega)$ is an invariant random variable. H is nonconstant by the last statement in Theorem 2.1(a).

(b) Say that $h(r, \theta)$ is a nonconstant bounded harmonic function. Then $H(\omega) \equiv \lim_{t \to \infty} h(r(t), \theta(t))$ exists and is nonconstant almost surely $[P_{r,\theta}]$. [That $H(\omega)$ is nonconstant almost surely $[P_{r,\theta}]$ can be proved using the strong Markov property and the fact that $h(r, \theta) = E_{r,\theta} H$.] Thus, there exists a number b such that $0 < P_{r,\theta}(H(\omega) > b) < 1$ and $0 < P_{r,\theta}(H(\omega) < b) < 1$. Let $D_1 = \{\omega: H(\omega) > b\}$ and let $D_2 = \{\omega: H(\omega) < b\}$. Then $Q_{r,\theta}(D_1 \times D_2) > 0$, where $Q_{r,\theta}$ is as in Lemma 5.2. Now if $\omega_1 \in D_1$ and $\omega_2 \in D_2$, then ω_1 and ω_2 cannot intersect each other at arbitrarily large times. Thus $(D_1 \times D_2) \cap \mathscr{A} = \varnothing$. This coupled with $Q_{r,\theta}(D_1 \times D_2) > 0$ contradicts Lemma 5.2. Thus, there can be no nonconstant bounded harmonic functions. □

6. Modifications in the general ($p \not\equiv 1$) case. In this section we sketch the modifications needed to prove the general case. For Theorem 1.1, first take $\delta = k$. In (a) and (b), we show transience and recurrence by replacing the test functions $u_1 = r^{-1} f_1(\theta)$ and $u_2 = r^m f_2^m(\theta)$ by $u_1 = r^{-1} f_1^{p(r)}(\theta)$ and $u_2 = r^m(f_2(\theta))^{mp(r)}$. Conditions (iii) and the second and third parts of condition (v) guarantee that $Lu_1 < 0$ and $Lu_2 < 0$ for large r. The first part of condition (v) guarantees that u_1 is bounded as is required. For (c), to show explosion, we replace the test function $u = c + (r^{-1} f_1(\theta))^\nu$ by $u = c + r^{-\nu}(f_1(\theta))^{\nu p(r)}$. The above comments on the conditions again apply; now the operator is $L + \lambda$. To show the absence of explosion, we use condition (vi) and the original argument. For (d), replace the test function $u = r f_2(\theta)$ at the beginning of the proof by $u = r(f_2(\theta))^{p(r)}$. The rest of the proof follows similarly except for one problem in the case $\delta > 0$. Recall from the proof, that the basic idea is to get an inequality which, without the additional frills, looks like

$$c + \varepsilon \int_0^t r^{-\delta}(s)\, ds \le r(t) \le c + N \int_0^t r^{-\delta}(s)\, ds. \tag{6.1}$$

Now, if $\delta > 0$, then as we saw, a Gronwall inequality type of analysis on the right-hand inequality in (6.1) gives a lower bound on $\int_0^t r^{-\delta}(s)\, ds$ and such an analysis on the left-hand inequality gives an upper bound on $\int_0^t r^{-\delta}(s)\, ds$. Since these inequalities go the wrong way, neither the right-hand nor the left-hand inequality alone provides either an upper bound or a lower bound for $r(t)$. However, the two inequalities in tandem, provide both the upper and the lower bound. Now in the case $\limsup_{r \to \infty} p_1(r) = \infty$, the right-hand side of (6.1) must be replaced by $\int_0^t r^{-\delta + \varepsilon}(s)\, ds$ for arbitrary $\varepsilon > 0$ and in the case

$\liminf_{r\to\infty} p_1(r) = 0$, the left-hand side of (6.1) must be replaced by $\int_0^t r^{-\delta-\varepsilon}(s)\,ds$ for arbitrary $\varepsilon > 0$. Thus, consider

$$c + \int_0^t r^{-\delta-\varepsilon}(s)\,ds \le r(t) \le c + \int_0^t r^{-\delta+\varepsilon}(s)\,ds. \tag{6.2}$$

The right-hand inequality provides the bound

$$\int_0^t r^{-\delta+\varepsilon}(s)\,ds \ge c_1 t^{1/(1+\delta-\varepsilon)}, \quad \text{for some } c_1 > 0, \tag{6.3}$$

while the left-hand inequality provides the bound

$$\int_0^t r^{-\delta-\varepsilon}(s)\,ds \le c_2 t^{1/(1+\delta+\varepsilon)}, \quad \text{for some } c_2 > 0. \tag{6.4}$$

To translate this into bounds on $r(t)$, we need inequalities like (6.3) and (6.4) going the other way. Using Hölder's inequality, we have

$$\int_0^t r^{-\delta+\varepsilon}(s)\,ds \le t^{2\varepsilon/(\delta+\varepsilon)}\left(\int_0^t r^{-\delta-\varepsilon}(s)\,ds\right)^{(\delta-\varepsilon)/(\delta+\varepsilon)}.$$

Combining this with (6.3) gives

$$c_1 t^{1/(1+\delta-\varepsilon)} \le t^{2\varepsilon/(\delta+\varepsilon)}\left(\int_0^t r^{-\delta-\varepsilon}(s)\,ds\right)^{(\delta-\varepsilon)/(\delta+\varepsilon)},$$

and then using (6.2) gives

$$r(t) \ge c + \int_0^t r^{-\delta-\varepsilon}(s)\,ds \ge c_1^{(\delta+\varepsilon)/(\delta-\varepsilon)} t^{(\delta-\varepsilon-2\varepsilon\delta+2\varepsilon^2)/((\delta-\varepsilon)(1+\delta-\varepsilon))} + c.$$

As $\varepsilon \to 0$, the exponent on the right-hand side above approaches $1/(1+\delta)$. This is exactly the inequality we want. Similar use of Hölder's inequality gives one the corresponding upper bound.

For the case $k < \delta$, replace $u = r^{-1}(f_1(\theta))^{r^q}$ and $u_2 = r^m(f_2(\theta))^{mr^q}$ by $u_1 = r^{-1}(f_1(\theta))^{r^q p(r)}$ and $u_2 = r^m(f_2(\theta))^{mr^q p(r)}$ and use conditions (iii) and (iv) to get (a) and (b). For (c), to show explosion, replace $u = c + r^{-\nu}(f_1(\theta))^{\nu r^q}$ by $u = c + r^{-\nu}(f_1(\theta))^{\nu r^q p(r)}$ and use conditions (iii) and (iv). For this theorem, from condition (iv) we need $\nu_1 > \max(1 + k - \delta, (1+k)/2)$ and $\nu_2 > 1 + k$. To show the absence of explosion, use condition (vi) and the original argument. For (d), replace $u = r(f_2(\theta))^{r^q}$ by $u = r(f_2(\theta))^{r^q p(r)}$. The remarks above regarding (d) apply here too.

Now consider Theorem 1.2. Part (a) is trivial and no adaptation nor condition is necessary. For (b), the key estimate (4.6) goes through as before with no changes; we rely on condition (iii). The proof of (c) goes through using conditions (iii) and (vi). (d) goes through with the same test function $u = r$. The remarks above concerning the proof of (d) also apply here.

Theorem 1.3 goes through as before without any changes, using condition (vii). In fact, if the drift $c(x)$ in the d/dr direction [or the drift $d(x)$ in the $(1/r)d/d\theta$ direction] is $o(1/|x|)$ as $|x| \to \infty$, then its specific form is immaterial and conditions (i)–(vii) with respect to c (or with respect to d) may be dispensed with.

The theorems of Section 2 in the general case follow from the general case version of Theorem 1.1(d) and conditions (iii), (iv) and the first part of condition (v). In conditions (iv) and (v), the condition on p'' is not used. For Theorem 2.1(a) we need $\nu_1 > 1 + k - \delta$ and for (b) we need $\nu_1 > (1 + \delta)/2$. In Theorem 2.2, we need $\nu_1 > \delta$. Theorem 2.3(a) follows from Theorem 2.1(a). Condition (iv) is not used in Theorem 2.3(b). It should be noted that although, given the validity of Theorem 1.1(d), specific parts of certain conditions are not needed in the proofs of the theorems of Section 2, they were used, at least implicitly, in the proof of Theorem 1.1(d). Thus, they are required implicitly in the proofs of the theorems of Section 2. For example, the stipulations on p'' in conditions (iv) and (v) are only required explicitly in the proof of transience—Theorem 1.1(b). Given transience, (d) follows without any reference to p'', and given (d), the theorems of Section 2 also follow without reference to p''. Without the assumption $0 < \liminf_{r \to \infty} p_1(r) \le \limsup_{r \to \infty} p_1(r) < \infty$, the general version of Theorem 1.1(d) is weaker than the version in the special case $p_1 = p_2 = 1$ and causes a problem in certain cases. This is why an extra condition concerning p_1 was included in (b) of Theorem 2.3 and in the border line case of (b) of Theorem 2.1.

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