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A LAW OF THE ITERATED LOGARITHM FOR SUMS OF EXTREME VALUES FROM A DISTRIBUTION WITH A REGULARLY VARYING UPPER TAIL

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Let $X_1, X_2, \ldots$ be independent observations from a distribution with a regularly varying upper tail with index a greater than 2. For each $n \geq 1$, let $X_{1,n} \leq \cdots \leq X_{n,n}$ denote the order statistics based on $X_1, \ldots, X_n$. Choose any sequence of integers $(k_n)_{n \geq 1}$ such that $1 \leq k_n \leq n$, $k_n \to \infty$, and $k_n/n \to 0$. It has been recently shown by S. Csörgő and Mason (1986) that the sum of the extreme values $X_{n,n} + \cdots + X_{n-k_n,n}$, when properly centered and normalized, converges in distribution to a standard normal random variable. In this paper, we completely characterize such sequences $(k_n)_{n \geq 1}$ for which the corresponding law of the iterated logarithm holds.

1. Introduction and statements of results. Let F be a distribution function with $F(0-) = 0$ and with regularly varying upper tail, i.e., assume

$$1 - F(x) = x^{-a}\tilde{L}(x), \qquad x > 0, \tag{1}$$

for some $0 < a < \infty$ and a function $\tilde{L}$ which is slowly varying at infinity. Let

$$Q(s) = \inf\{x \in \mathbb{R}: F(x) \geq s\}, \qquad 0 < s \leq 1,$$

$Q(0) = Q(0+)$ denote the corresponding quantile function. Then (1) is equivalent to

$$Q(1-s) = s^{-1/a}L(s), \qquad 0 < s < 1, \tag{2}$$

where L is a function which is slowly varying at zero; cf. de Haan (1975), Corollary 1.2.1, 5., or Seneta (1976), Lemma 1.8.

Consider a fixed $0 < a < \infty$ and slowly varying function L. Let $X_1, X_2, \ldots,$ be independent and identically distributed random variables with a common distribution F, and for each $n \geq 1$, let $X_{1,n} \leq \cdots \leq X_{n,n}$ denote the order statistics based on $X_1, \ldots, X_n$. S. Csörgő and Mason (1986) have shown that if $2 \leq a < \infty$, then for any sequence $(k_n)_{n \geq 1}$ of positive integers such that $k_n \to \infty$ and $k_n/n \to 0$ one has

$$A_n(a, k_n)^{-1}\left\{\sum_{j=1}^{k_n} X_{n+1-j,n} - n\mu(k_n/n)\right\} \to_{\mathscr{D}} N(0,1), \tag{3}$$

where

$$\mu(k_n/n) = \int_{1-k_n/n}^{1} Q(s)\, ds,$$

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and

$$A_n(a, k_n) = \begin{cases} N_a n^{1/2}(k_n/n)^{1/2-1/a}L(k_n/n) \\ \quad \text{and } N_a = \left(\dfrac{2}{(a-2)(a-1)}\right)^{1/2}, & \text{for } a > 2, \\ \left(n\displaystyle\int_{1/n}^{k_n/n} s^{-1}L^2(s)\,ds\right)^{1/2}, & \text{for } a = 2, \end{cases}$$

whereas for $0 < a < 2$ the sum of the upper k_n extreme values when properly centered and normalized converges in distribution to a stable law with index a. (Here and throughout all limits are to be understood as n tends to infinity if not stated otherwise.)

Given the asymptotic normality of the sum of the upper k_n extreme values in the case $a \geq 2$, it is only natural to suppose that with perhaps further restrictions on the sequence $(k_n)_{n\geq 1}$ the law of the iterated logarithm should also hold. We will show that this is indeed the case when $a > 2$. Our main result is as follows:

We introduce the normalizing constants

$$\begin{aligned} a_n(k_n) &= N_a(2n\log\log n)^{1/2}(k_n/n)^{1/2-1/a}L(k_n/n) \\ &= N_a(2k_n \log\log n)^{1/2}Q(1 - k_n/n) \end{aligned}$$

and impose the following monotonicity restrictions on k_n and k_n/n:

$$k_n \sim \alpha_n \uparrow \infty \tag{4}$$

and

$$k_n/n \sim \beta_n \downarrow 0, \tag{5}$$

for some sequences $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ of positive numbers. We cannot assume that the sequence of positive integers $(k_n)_{n\geq 1}$ satisfies $k_n \uparrow \infty$ and $k_n/n \downarrow 0$, since such a sequence does not exist.

THEOREM 1. *Under* (1) *for any $a > 2$ and sequence of positive integers $(k_n)_{n\geq 1}$ satisfying* (4) *and* (5) *the following three statements are equivalent for any fixed integer $k \geq 1$:*

$$\sum_{n=1}^{\infty} n^{k-1}(1 - F(a_n(k_n)))^k < \infty, \tag{6}$$

$$a_n(k_n)^{-1}X_{n+1-k,n} \to 0 \quad a.s., \tag{7}$$

and

$$\limsup_{n\to\infty} a_n(k_n)^{-1}\left\{\sum_{j=k}^{k_n} X_{n+1-j,n} - n\mu(k_n/n)\right\} = 1 \quad a.s.; \tag{8}$$

also whenever (6), (7), *or* (8) *holds*

$$\liminf_{n\to\infty} a_n(k_n)^{-1}\left\{\sum_{j=k}^{k_n} X_{n+1-j,n} - n\mu(k_n/n)\right\} = -1 \quad a.s. \tag{9}$$

In addition, the following three statements are equivalent for any fixed integer $k \geq 1$:

$$\sum_{n=1}^{\infty} n^{k-1}\big(1 - F(a_n(k_n))\big)^k = \infty, \tag{10}$$

$$\limsup_{n\to\infty} a_n(k_n)^{-1} X_{n+1-k,n} = \infty \quad a.s., \tag{11}$$

and

$$\limsup_{n\to\infty} a_n(k_n)^{-1}\left\{\sum_{j=k}^{k_n} X_{n+1-j,n} - n\mu(k_n/n)\right\} = \infty \quad a.s. \tag{12}$$

The case $a = 2$ requires a separate analysis and will be considered elsewhere. No law of the iterated logarithm exists in the case $0 < a < 2$; see Einmahl, Haeusler, and Mason (1985).

Under the conditions on F given in Theorem 1, if k_n is chosen to be the integer part of $n\alpha$ where $0 < \alpha < 1$ and $1 - \alpha$ is a continuity point of Q, then an application of Theorem 4 of Wellner (1977) yields a law of the iterated logarithm for $X_{n,n} + \cdots + X_{n-k_n,n}$. The assumptions of his theorem fail when k_n satisfies (4) and (5). For related work on the law of the iterated logarithm for various types of trimmed sums the reader is referred to Griffin (1985), Haeusler and Mason (1987), Hahn and Kuelbs (1985), and Kuelbs and Ledoux (1984, 1987). Also for results on the law of the iterated logarithm for $X_{n-k_n,n}$, see Hall (1979b).

To see the meaning of condition (6), fix $a > 2$ and a sequence $(k_n)_{n\geq 1}$ satisfying (4) and (5). In order that there exist an integer $k \geq 1$ so that (6) holds it is necessary and sufficient that

$$\eta = \liminf_{n\to\infty} (\log k_n)/\log\log n > 0. \tag{13}$$

If (13) is satisfied, then (6) holds for all $k > 2/((a-2)\eta)$. This follows by the same method of proof as given in Lemma 6 in the next section. Consequently, if $(k_n)_{n\geq 1}$ is such that

$$(\log k_n)/\log\log n \to 0 \tag{14}$$

holds, then (12) is true for all integers $k \geq 1$. In this situation one has a stability result for the sums $X_{n+1-k,n} + \cdots + X_{n+1-k_n,n}$. For this, replace in $a_n(k_n)$ the constants $N_a(2\log\log n)^{1/2}$ by any other sequence of constants $b_n \uparrow \infty$, i.e., set

$$a_n(k_n, b_n) = b_n k_n^{1/2} Q(1 - k_n/n).$$

THEOREM 2. *Under* (1) *for any* $a > 2$, *any sequence of positive integers* $(k_n)_{n\geq 1}$ *satisfying* (4), (5) *and* (14) *and any sequence of constants* $b_n \uparrow \infty$ *the following three statements are equivalent for any fixed integer* $k \geq 1$:

$$\sum_{n=1}^{\infty} n^{k-1}\big(1 - F(a_n(k_n, b_n))\big)^k < \infty, \tag{15}$$

$$a_n(k_n, b_n)^{-1} X_{n+1-k,n} \to 0 \quad a.s., \tag{16}$$

and there exists a sequence $(c_n)_{n\geq 1}$ *of centering constants such that*

$$a_n(k_n, b_n)^{-1}\left\{\sum_{j=k}^{k_n} X_{n+1-j,n} - c_n\right\} \to 0 \quad a.s. \tag{17}$$

If (17) *is true, then one can choose* $c_n = n\mu(k_n/n)$.

In addition, the following three statements are equivalent for any fixed integer $k \geq 1$:

$$\sum_{n=1}^{\infty} n^{k-1}(1 - F(a_n(k_n, b_n)))^k = \infty, \tag{18}$$

$$\limsup_{n\to\infty} a_n(k_n, b_n)^{-1} X_{n+1-k,n} = \infty \quad a.s., \tag{19}$$

and for any sequence $(c_n)_{n\geq 1}$ *of constants*

$$\limsup_{n\to\infty} a_n(k_n, b_n)^{-1}\left|\sum_{j=k}^{k_n} X_{n+1-j,n} - c_n\right| = \infty \quad a.s. \tag{20}$$

REMARK. Whenever Q is such that for some constants $0 < A < \infty$ and $-\infty < a < 0$, $A - Q(1-s) = s^{-1/a}L(s)$ and the sequence k_n satisfies (4), (5) and (13), the same method used in the proof of Theorem 1 shows that (8) holds with $k = 1$. If F is in the domain of attraction of a Gumbel extreme value distribution, (8) also holds with $k = 1$ and appropriate normalizing and centering constants for such sequences k_n. For this, see Deheuvels, Haeusler and Mason (1986).

2. Proofs. The proofs of Theorems 1 and 2 will be split up into a sequence of lemmas. Our first lemma is an immediate consequence of Lemma 3 in Mori (1976) and is contained in Theorem 4 of Hall (1979a). We restate it here for convenience.

LEMMA 1. *Let* $(x_n)_{n\geq 1}$ *be a sequence of positive constants such that* $x_n \uparrow \infty$. *Then for any integer* $k \geq 1$ *and distribution function* F *satisfying* (1)

$$\sum_{n=1}^{\infty} n^{k-1}(1 - F(x_n))^k < \infty \quad \textit{iff} \quad x_n^{-1} X_{n+1-k,n} \to 0 \quad a.s.$$

and

$$\sum_{n=1}^{\infty} n^{k-1}(1 - F(x_n))^k = \infty \quad \textit{iff} \quad \limsup_{n\to\infty} x_n^{-1} X_{n+1-k,n} = \infty \quad a.s.$$

Lemma 1 does not directly prove the equivalence of statements (6) and (7), and statements (10) and (11), since $a_n(k_n)$ need not be nondecreasing. Let

$$a_n = N_a(2\alpha_n \log\log n)^{1/2} Q(1 - \beta_n).$$

Notice that $a_n \uparrow \infty$ and $a_n \sim a_n(k_n)$ by (4) and (5), and hence by (1)

$$1 - F(a_n(k_n)) \sim 1 - F(a_n).$$

Thus, statements (6), (7), (10), and (11) are equivalent to the corresponding statements with $a_n(k_n)$ replaced by a_n. Lemma 1 is then applicable and the equivalence of the forementioned statements is proven. The same reasoning applies to the proof of the equivalence of (15) and (16), and of (18) and (19), respectively.

Next we will show that (6) and (7) imply (8) and (9), which is the major part of the proof of Theorem 1. From now on we will be concerned with the behavior of the quantile function Q and not with that of the distribution function F. Let $U_1, U_2, \dots,$ be a sequence of independent uniform (0, 1) random variables. For any integer $n \geq 1$, let $U_{1,n} \leq \cdots \leq U_{n,n}$ denote the order statistics and G_n denote the right-continuous empirical distribution function based on $U_1, \dots, U_n$. The two sequences $(X_n)_{n \geq 1}$ and $(Q(U_n))_{n \geq 1}$ are equal in law and consequently the two processes $(X_{k,n}: 1 \leq k \leq n,\ n \geq 1)$ and $(Q(U_{k,n}): 1 \leq k \leq n,\ n \geq 1)$ are also equal in law. Therefore, w.l.o.g. we may assume $X_{k,n} = Q(U_{k,n})$ for all $1 \leq k \leq n$ and $n \geq 1$. Then, we can write

$$\begin{aligned} a_n(k_n)^{-1}&\left\{ \sum_{j=k+1}^{k_n} X_{n+1-j,n} - n\mu(k_n/n) \right\} \\ &= a_n(k_n)^{-1}\left\{ n\int_{U_{n-k_n,n}}^{U_{n-k,n}} Q(s)\, dG_n(s) - n\mu(k_n/n) \right\}, \end{aligned}$$

which by two integrations by parts equals

$$\begin{aligned} &a_n(k_n)^{-1} n\int_{1-k_n/n}^{U_{n-k,n}} (s - G_n(s))\, dQ(s) \\ &+ a_n(k_n)^{-1} n\int_{U_{n-k_n,n}}^{1-k_n/n} \left(\frac{n-k_n}{n} - G_n(s) \right) dQ(s) \\ &+ a_n(k_n)^{-1} n\int_{U_{n-k,n}}^{1-k/n} (s-1)\, dQ(s) + a_n(k_n)^{-1} n\int_{U_{n-k,n}}^{1-k/n} \frac{k}{n}\, dQ(s) \\ &- a_n(k_n)^{-1} n\int_{1-k/n}^{1} Q(s)\, ds \\ &\equiv a_n(k_n)^{-1} n\int_{1-k_n/n}^{U_{n-k,n}} (s - G_n(s))\, dQ(s) + \Delta_{4,n} + \Delta_{3,n} + \Delta_{2,n} - \Delta_{1,n}. \end{aligned} \tag{21}$$

In the above integrals and in all subsequent integrals that appear in the proof of the theorem, we use the following integral conventions:

For a right-continuous function r, a left-continuous monotone function l, and $0 \leq a, b \leq 1$ we write

$$\int_a^b r(s)\, dl(s) = \begin{cases} \displaystyle\int_{[a,b)} r(s)\, dl(s), & \text{if } a \leq b, \\ \displaystyle -\int_{[b,a)} r(s)\, dl(s), & \text{if } b < a, \end{cases}$$

and for a right-continuous monotone function r, a left-continuous function l, and $0 \le a, b \le 1$ we write

$$\int_a^b l(s)\,dr(s) = \begin{cases} \int_{(a,b]} l(s)\,dr(s), & \text{if } a \le b, \\ -\int_{(b,a]} l(s)\,dr(s), & \text{if } b < a. \end{cases}$$

Thus if both r and l are monotone functions, the usual integration by parts formula holds:

$$\int_a^b r(s)\,dl(s) = r(b)l(b) - r(a)l(a) - \int_a^b l(s)\,dr(s).$$

We will first show that under (7) for each $1 \le i \le 4$ we have $\Delta_{i,n} \to 0$ a.s. The following two lemmas are simple consequences of the Karamata representation theorem for slowly varying functions.

LEMMA 2. *Let $(x_n)_{n \ge 1}$ and $(y_n)_{n \ge 1}$ be two sequences of positive constants such that $x_n = o(y_n)$ and $y_n \to 0$. Then $x_n^\beta L(x_n) = o(y_n^\beta L(y_n))$ for any $\beta > 0$.*

LEMMA 3. *Let $(x_n)_{n \ge 1}$ be a sequence of positive constants with $x_n \to 0$.*

(i) *For $a < \gamma < \infty$ and $0 < d < 1$ we have for all large n and all $0 < u \le x_n$,*

$$Q(1-u)/Q(1-x_n) \ge d(u/x_n)^{-1/\gamma}.$$

(ii) *For $0 < \gamma < a$ and $1 < d < \infty$ we have for all large n and all $0 < u \le x_n$,*

$$Q(1-u)/Q(1-x_n) \le d(u/x_n)^{-1/\gamma}.$$

LEMMA 4. *We always have $\Delta_{1,n} \to 0$.*

PROOF. Applying Theorem 1.2.1 in de Haan (1975) we obtain

$$\Delta_{1,n} \sim \frac{ak^{1/2}}{(a-1)N_a}(2\log\log n)^{-1/2}\frac{(k/n)^{1/2-1/a}L(k/n)}{(k_n/n)^{1/2-1/a}L(k_n/n)},$$

which converges to zero by Lemma 2 since $k/k_n \to 0$ and $1/2 - 1/a > 0$. □

LEMMA 5. *Whenever (7) holds, $\Delta_{2,n} \to 0$ a.s.*

PROOF. We have

$$|\Delta_{2,n}| \le ka_n(k_n)^{-1}\{Q(1-k/n) + Q(U_{n-k,n})\}.$$

Now

$$\begin{aligned} 0 \le ka_n(k_n)^{-1}Q(U_{n-k,n}) &\le ka_n(k_n)^{-1}Q(U_{n+1-k,n}) \\ &= ka_n(k_n)^{-1}X_{n+1-k,n} \to 0 \quad \text{a.s.} \end{aligned}$$

by (7), and by Lemma 2

$$ka_n(k_n)^{-1}Q(1-k/n) = k^{1/2}N_a^{-1}\frac{(k/n)^{1/2-1/a}L(k/n)}{(k_n/n)^{1/2-1/a}L(k_n/n)}(2\log\log n)^{-1/2} \to 0.$$

□

LEMMA 6. *Whenever* (7) *holds,*

$$\sum_{n=1}^{\infty} n^{-1}\alpha_n^{-k(\gamma-2)/2}(\log\log n)^{-\gamma k/2} < \infty$$

for all $a < \gamma < \infty$, *whence*

(22) $$(\log\log n)/\alpha_n \to 0 \quad \textit{and} \quad (\log\log n)/k_n \to 0.$$

PROOF. Fix $a < \gamma < \infty$ and $0 < d < 1$. On the event

$$\left\{d(U_1/\beta_n)^{-1/\gamma} > N_a(2\alpha_n \log\log n)^{1/2}\right\} \equiv A_n,$$

we have

$$0 < U_1 < (N_a/d)^{-\gamma}(2\alpha_n\log\log n)^{-\gamma/2}\beta_n = o(1)\beta_n \le \beta_n$$

for all n large, so that Lemma 3(i) implies

$$\begin{aligned}
P(Q(1-U_1) > a_n)
&= P\big(Q(1-U_1)/Q(1-\beta_n) > N_a(2\alpha_n\log\log n)^{1/2}\big)\\
&\ge P(A_n) = P\big(U_1 < (N_a/d)^{-\gamma}(2\alpha_n\log\log n)^{-\gamma/2}\beta_n\big)\\
&\sim (N_a/d)^{-\gamma}\alpha_n^{1-\gamma/2}n^{-1}(2\log\log n)^{-\gamma/2}.
\end{aligned}$$

Observe that $P(Q(1-U_1) > a_n) = 1 - F(a_n)$. Thus, since $a_n \uparrow \infty$, we see by Lemma 1 that statement (7) is equivalent to

$$\sum_{n=1}^{\infty} n^{k-1}P(Q(1-U_1) > a_n)^k < \infty.$$

This combined with the above estimate implies the first assertion of the lemma. From this, observing that $\alpha_n \uparrow \infty$, we obtain

$$(\log n)/\big(\alpha_n^{k(\gamma-2)/2}(\log\log n)^{\gamma k/2}\big) \to 0,$$

which implies (22). □

LEMMA 7. *Whenever* (7) *holds,* $\Delta_{3,n} \to 0$ *a.s.*

PROOF. According to Theorem 2 in Kiefer (1972) we have

$$\limsup_{n\to\infty} n(1-U_{n-k,n})/\log\log n = 1 \quad \text{a.s.},$$

whence $1 - 2n^{-1}\log\log n \le U_{n-k,n}$ with probability one for all large n and, consequently,

$$|\Delta_{3,n}| \le a_n(k_n)^{-1} n \lim_{\delta \downarrow 0} \int_{1-2n^{-1}\log\log n}^{1-\delta} (1-s)\, dQ(s)$$

$$= a_n(k_n)^{-1} n \lim_{\delta \downarrow 0} \Big\{ \delta Q(1-\delta) - 2n^{-1}\log\log n Q(1 - 2n^{-1}\log\log n)$$

$$- \int_{1-2n^{-1}\log\log n}^{1-\delta} Q(s)\, d(1-s) \Big\}$$

$$\le a_n(k_n)^{-1} n \int_0^{2n^{-1}\log\log n} Q(1-s)\, ds,$$

where the last bound is justified by the fact that (2) holding with $a > 1$ implies $\delta Q(1-\delta) \to 0$ as $\delta \downarrow 0$, and $Q \ge 0$. Applying Theorem 1.2.1 in de Haan (1975) we see that the last expression is

$$\sim a_n(k_n)^{-1}(2n^{-1}\log\log n)^{1-1/a} n L(2n^{-1}\log\log n) a/(a-1)$$

$$= \frac{2^{1/2} a (2n^{-1}\log\log n)^{1/2-1/a} L(2n^{-1}\log\log n)}{N_a(a-1)(k_n/n)^{1/2-1/a} L(k_n/n)}.$$

In view of (22), Lemma 2 implies that this last term converges to zero. □

Lemma 8. *Whenever* (7) *holds*, $\Delta_{4,n} \to 0$ *a.s.*

Proof. Notice that since for s in the closed interval formed by $U_{n-k_n,n}$ and $1 - k_n/n$, $|1 - G_n(s) - k_n/n| \le |1 - G_n(1-k_n/n) - k_n/n|$, we have

$$|\Delta_{4,n}| \le N_a^{-1} \frac{n|G_n(1-k_n/n) - (1-k_n/n)|}{(2k_n \log\log n)^{1/2}} \frac{|Q(1-k_n/n) - Q(U_{n-k_n,n})|}{Q(1-k_n/n)}.$$

By Theorem 3.2 in Csáki (1977) we have

$$\limsup_{n\to\infty} \left(\frac{n}{2\log\log n}\right)^{1/2} \sup_{n^{-1}\log\log n < s < 1 - n^{-1}\log\log n} \frac{|G_n(s) - s|}{(s(1-s))^{1/2}} \tag{23}$$

$$= 2^{1/2} \quad \text{a.s.},$$

hence in view of (22)

$$\limsup_{n\to\infty} \frac{n|G_n(1-k_n/n) - (1-k_n/n)|}{(2k_n \log\log n)^{1/2}} \le 2^{1/2} \quad \text{a.s.}$$

It remains to show $Q(U_{n-k_n,n})/Q(1-k_n/n) \to 1$ a.s. On account of (2) this follows from $1 - U_{n-k_n,n} \sim k_n/n$ a.s.; cf. Theorem 4 in Wellner (1978). □

From (21), (7) and Lemmas 4, 5, 7 and 8 we obtain with probability one

$$a_n(k_n)^{-1}\left\{\sum_{j=k}^{k_n} X_{n+1-j,n} - n\mu_n(k_n/n)\right\}$$
$$= a_n(k_n)^{-1}n\int_{1-k_n/n}^{U_{n-k,n}}(s - G_n(s))\,dQ(s) + o(1).$$

It remains to show

$$\limsup_{n\to\infty} a_n(k_n)^{-1}n\int_{1-k_n/n}^{U_{n-k,n}} \pm (s - G_n(s))\,dQ(s) = 1 \quad \text{a.s.} \tag{24}$$

The first step in the proof of (24) is

LEMMA 9. *There exists a finite constant c_a depending only on $2 < a < \infty$ such that for all $0 < \tau < 1$*

$$H(\tau) \equiv \limsup_{n\to\infty} a_n(k_n)^{-1}n\left|\int_{1-\tau k_n/n}^{U_{n-k,n}} (s - G_n(s))\,dQ(s)\right| \le c_a\tau^{1/2-1/a} \quad a.s.$$

The proof of Lemma 9 will require the following result:

LEMMA 10 [Einmahl and Mason (1988)]. *Let $(x_n)_{n\ge1}$ be a sequence of constants such that $0 < x_n \le n$ for all $n \ge 1$ and $x_n \uparrow \infty$. If for an integer $k \ge 1$ and some $0 < \nu < \frac{1}{2}$*

$$\sum_{n=1}^{\infty} n^{-1}x_n^{-2\nu(k+1)/(1-2\nu)}(\log\log n)^{-(k+1)/(1-2\nu)} < \infty, \tag{25}$$

then

$$\limsup_{n\to\infty} K_n(x_n) \le 2 \quad a.s.,$$

where

$$K_n(x_n) \equiv \left(\frac{n}{x_n}\right)^{\nu}\left(\frac{n}{\log\log n}\right)^{1/2} \sup_{1-x_n/n\le s\le U_{n-k,n}} \frac{|G_n(s) - s|}{(1 - s)^{1/2-\nu}}.$$

PROOF OF LEMMA 9. Fix $0 < \tau < 1$ arbitrarily. As in the proof of Lemma 7 we have with probability one, $1 - 2n^{-1}\log\log n \le U_{n-k,n}$ for all large n, and, consequently, by (22) almost surely $1 - \tau k_n/n < 1 - 2n^{-1}\log\log n \le U_{n-k,n}$. Thus we have for any $0 < \nu < \frac{1}{2}$ with probability one for all large n

$$\begin{aligned} &a_n(k_n)^{-1}n\left|\int_{1-\tau k_n/n}^{U_{n-k,n}} (s - G_n(s))\,dQ(s)\right| \\ &\le a_n(k_n)^{-1}n \sup_{1-\tau k_n/n\le s\le U_{n-k,n}} \frac{|G_n(s) - s|}{(1 - s)^{1/2-\nu}}\int_{1-\tau k_n/n}^{1} (1 - s)^{1/2-\nu}\,dQ(s). \end{aligned} \tag{26}$$

For $0 < \nu < 1/2 - 1/a$ an integration by parts and Theorem 1.2.1 in de Haan

(1975) yield after some routine manipulations

$$\int_{1-\tau k_n/n}^{1}(1-s)^{1/2-\nu}\,dQ(s) \sim \frac{2}{a-2-2a\nu}\left(\frac{\tau k_n}{n}\right)^{1/2-1/a-\nu}L\left(\frac{\tau k_n}{n}\right).$$

Substituting into the right side of inequality (26) we find with probability one

$$H(\tau) \le \frac{2^{1/2}}{N_a(a-2-2a\nu)}\tau^{1/2-1/a}\limsup_{n\to\infty}K_n(\tau k_n).$$

From (4) we conclude $1-2\tau\alpha_n/n \le 1-\tau k_n/n$ and $(n/\tau k_n)^\nu \le 2(n/\tau\alpha_n)^\nu = 2^{1+\nu}(n/2\tau\alpha_n)^\nu$ for all large n, which implies that

$$\limsup_{n\to\infty}K_n(\tau k_n) \le 2^{1+\nu}\limsup_{n\to\infty}K_n(2\tau\alpha_n). \tag{27}$$

The proof of Lemma 9 will be complete if we show that there exists a $0<\nu<1/2-1/a$ such that (25) holds with $x_n = 2\tau\alpha_n$, since by Lemma 10 this implies that the lim sup on the right side of inequality (27) is less than or equal to 2 with probability one, which with a fixed ν depending only on a gives the desired estimate:

$$H(\tau) \le 2^{5/2+\nu}\tau^{1/2-1/a}/\big(N_a(a-2-2a\nu)\big) \quad \text{a.s.}$$

To see that (25) is in fact true with $x_n = 2\tau\alpha_n$ for an appropriately chosen $0<\nu<1/2-1/a$, observe that

$$\lim_{\nu\uparrow 1/2-1/a}\frac{2\nu(k+1)}{1-2\nu} = \frac{a-2}{2}(k+1) > \frac{a-2}{2}k = \lim_{\gamma\downarrow a}\frac{k(\gamma-2)}{2}$$

and

$$\lim_{\nu\uparrow 1/2-1/a}\frac{k+1}{1-2\nu} = \frac{a}{2}(k+1) > \frac{a}{2}k = \lim_{\gamma\downarrow a}\frac{\gamma}{2}k.$$

So for $\gamma > a$ sufficiently close to a and $\nu < 1/2-1/a$ sufficiently close to $1/2-1/a$ we have $2\nu(k+1)/(1-2\nu) > k(\gamma-2)/2$ and $(k+1)/(1-2\nu) > \gamma k/2$, which for large n implies

$$n^{-1}\alpha_n^{-2\nu(k+1)/(1-2\nu)}(\log\log n)^{-(k+1)/(1-2\nu)} \le n^{-1}\alpha_n^{-k(\gamma-2)/2}(\log\log n)^{-\gamma k/2}.$$

We see that (25) now follows from Lemma 6. □

Next we prove for all $0<\tau<1$ that

$$\begin{aligned} M^{\pm}(\tau) &\equiv \limsup_{n\to\infty} a_n(k_n)^{-1} n\int_{1-k_n/n}^{1-\tau k_n/n} \pm\,(s-G_n(s))\,dQ(s) \\ &= M_\tau^{1/2} \quad \text{a.s.,} \end{aligned} \tag{28}$$

where $M_\tau \equiv 1-(a-1)\tau^{1-2/a}+(a-2)\tau^{1-1/a} > 0$.

To prove (28), let $0 < \tau < 1$ be fixed. For $\lambda > 1$ and $l \geq 1$ put $m_l = [\lambda^l]$ where $[x]$ denotes the integer part of x. Obviously, $m_{l-1} < m_l$ for all large l, and $m_l \to \infty$ as $l \to \infty$. For large enough n, let the integer l' be defined by $m_{l'-1} < n \leq m_{l'}$, where for notational convenience the prime is used to indicate the dependence on n. Then we have

$$\begin{aligned}
&a_n(k_n)^{-1} n \int_{1-k_n/n}^{1-\tau k_n/n} \pm (s - G_n(s))\, dQ(s) \\
&\quad = a_n(k_n)^{-1} n \int_{1-k_n/n}^{1-k_{m_{l'-1}}/m_{l'-1}} \pm (s - G_n(s))\, dQ(s) \\
&\qquad + a_n(k_n)^{-1} n \int_{1-k_{m_{l'-1}}/m_{l'-1}}^{1-\tau k_{m_{l'-1}}/m_{l'-1}} \pm (s - G_n(s))\, dQ(s) \\
&\qquad + a_n(k_n)^{-1} n \int_{1-\tau k_{m_{l'-1}}/m_{l'-1}}^{1-\tau k_n/n} \pm (s - G_n(s))\, dQ(s) \\
&\quad \equiv (\pm R_{1,n}(\lambda)) + (\pm A_n(\lambda)) + (\pm R_{2,n}(\lambda)).
\end{aligned}$$

At first we show

LEMMA 11. *For all* $\lambda > 1$ *we have*

$$\limsup_{n\to\infty} |R_{1,n}(\lambda) + R_{2,n}(\lambda)| \leq 2^{3/2} N_a^{-1} \lambda^{1/2-1/a}(\lambda^{1/a} - 1) \quad a.s.$$

PROOF. From $k_n/n \to 0$ and $k_n/\log\log n \to \infty$ [cf. (22)] we obtain for any $0 < \tilde{\tau} \leq 1$ and for all large n the inequality $n^{-1}\log\log n < 1 - \tilde{\tau} k_n/n < 1 - n^{-1}\log\log n$. Analogously, from $k_{m_{l'-1}}/m_{l'-1} \to 0$, $k_{m_{l'-1}}/m_{l'-1} \sim \beta_{m_{l'-1}} \geq \beta_n \sim \alpha_n/n$, and $\alpha_n/\log\log n \to \infty$ it follows for any $0 < \tilde{\tau} \leq 1$ and all large n that $n^{-1}\log\log n < 1 - \tilde{\tau} k_{m_{l'-1}}/m_{l'-1} < 1 - n^{-1}\log\log n$. With $\tilde{\tau} = \tau$ or $\tilde{\tau} = 1$ this means that the regions of integration occurring in $R_{1,n}(\lambda)$ and $R_{2,n}(\lambda)$ are contained in the interval $(n^{-1}\log\log n, 1 - n^{-1}\log\log n)$ for all large n. Furthermore, we have for all $0 < \tilde{\tau} \leq 1$, $0 < \eta < 1$ and all large n the inequalities

$$\begin{aligned}
1 - \tilde{\tau}\beta_{m_{l'-1}}(1+\eta) \leq 1 - \tilde{\tau}\beta_n(1+\eta) < 1 - \tilde{\tau} k_n/n \\
< 1 - \tilde{\tau}\beta_n(1-\eta) \leq 1 - \tilde{\tau}\beta_{m_{l'}}(1-\eta)
\end{aligned}$$

and

$$1 - \tilde{\tau}\beta_{m_{l'-1}}(1+\eta) < 1 - \tilde{\tau} k_{m_{l'-1}}/m_{l'-1} < 1 - \tilde{\tau}\beta_{m_{l'}}(1-\eta).$$

For $\tilde{\tau} = 1$ or $\tilde{\tau} = \tau$ this means that the regions of integration occurring in $R_{1,n}(\lambda)$ and $R_{2,n}(\lambda)$ are also contained in the interval $(1 - \tilde{\tau}\beta_{m_{l'-1}}(1+\eta),$

$1 - \tilde{\tau}\beta_{m_{l'}}(1-\eta))$ for all large n. Thus for all large n we get the estimate

$$\begin{aligned}
&|R_{1,n}(\lambda) + R_{2,n}(\lambda)| \\
&\quad \le a_n(k_n)^{-1} n \sup_{n^{-1}\log\log n < s < 1 - n^{-1}\log\log n} \frac{|G_n(s) - s|}{(s(1-s))^{1/2}} \\
&\quad\quad \times \left\{ \int_{1-\beta_{m_{l'-1}}(1+\eta)}^{1-\beta_{m_{l'}}(1-\eta)} (1-s)^{1/2}\, dQ(s) \right. \\
&\quad\quad\quad \left. + \int_{1-\tau\beta_{m_{l'-1}}(1+\eta)}^{1-\tau\beta_{m_{l'}}(1-\eta)} (1-s)^{1/2}\, dQ(s) \right\} \\
&\quad \sim N_a^{-1} \left(\frac{n}{2\log\log n} \right)^{1/2} \sup_{n^{-1}\log\log n < s < 1 - n^{-1}\log\log n} \frac{|G_n(s) - s|}{(s(1-s))^{1/2}} \\
&\quad\quad \times \frac{1}{\beta_n^{1/2-1/a} L(\beta_n)} \left\{ \int_{1-\beta_{m_{l'-1}}(1+\eta)}^{1-\beta_{m_{l'}}(1-\eta)} (1-s)^{1/2}\, dQ(s) \right. \\
&\quad\quad\quad \left. + \int_{1-\tau\beta_{m_{l'-1}}(1+\eta)}^{1-\tau\beta_{m_{l'}}(1-\eta)} (1-s)^{1/2}\, dQ(s) \right\}
\end{aligned}$$

by definition of $a_n(k_n)$ and (4) and (5). For $0 < \tilde{\tau} \le 1$, $0 < \eta < 1$, and all large n we have

$$\begin{aligned}
&\frac{1}{\beta_n^{1/2-1/a} L(\beta_n)} \int_{1-\tilde{\tau}\beta_{m_{l'-1}}(1+\eta)}^{1-\tilde{\tau}\beta_{m_{l'}}(1-\eta)} (1-s)^{1/2}\, dQ(s) \\
&\quad \le \tilde{\tau}^{1/2-1/a}(1+\eta)^{1/2} \frac{\beta_{m_{l'-1}}^{1/2-1/a}}{\beta_n^{1/2-1/a}} \left\{ (1-\eta)^{-1/a} \left(\frac{\beta_{m_{l'}}}{\beta_{m_{l'-1}}} \right)^{-1/a} \frac{L(\tilde{\tau}(1-\eta)\beta_{m_{l'}})}{L(\beta_n)} \right. \\
&\quad\quad \left. - (1+\eta)^{-1/a} \frac{L(\tilde{\tau}\beta_{m_{l'-1}}(1+\eta))}{L(\beta_n)} \right\} \equiv Z_n.
\end{aligned}$$

From (4) and (5) it follows that

$$1 \le \frac{\beta_{m_{l'-1}}}{\beta_n} \sim \frac{k_{m_{l'-1}}}{m_{l'-1}} \frac{n}{k_n} \sim \frac{\alpha_{m_{l'-1}}}{m_{l'-1}} \frac{n}{\alpha_n} \le \frac{m_{l'}}{m_{l'-1}} \to \lambda$$

and, for n replaced by $m_{l'}$, $1 \le \beta_{m_{l'-1}}/\beta_{m_{l'}} \le (1+o(1))m_{l'}/m_{l'-1} \to \lambda$. Analogously, $1 \ge \beta_{m_{l'}}/\beta_n \ge (1+o(1))m_{l'-1}/m_{l'} \to \lambda^{-1}$. Consequently, since L is slowly varying at zero,

$$L(\tilde{\tau}(1-\eta)\beta_{m_{l'}})/L(\beta_n) \to 1 \quad \text{and} \quad L(\tilde{\tau}\beta_{m_{l'-1}}(1+\eta))/L(\beta_n) \to 1,$$

so that we arrive at

$$\limsup_{n\to\infty} Z_n \le \tilde{\tau}^{1/2-1/a}(1+\eta)^{1/2}\lambda^{1/2-1/a}\left\{(1-\eta)^{-1/a}\lambda^{1/a} - (1+\eta)^{-1/a}\right\}.$$

Using this estimate for $\tilde{\tau} = \tau$ and $\tilde{\tau} = 1$ and combining it with (23) we obtain

$$\limsup_{n\to\infty} |R_{1,n}(\lambda) + R_{2,n}(\lambda)|$$
$$\leq 2^{3/2} N_a^{-1}(1+\eta)^{1/2}\lambda^{1/2-1/a}\left\{(1-\eta)^{-1/a}\lambda^{1/a} - (1+\eta)^{-1/a}\right\} \quad \text{a.s.}$$

Since $0 < \eta < 1$ is arbitrary, this implies the desired result. □

It remains to deal with $\pm A_n(\lambda)$. We will show

LEMMA 12. *For any $\varepsilon > 0$ there exists a $\lambda_\varepsilon > 1$ such that for all $1 < \lambda < \lambda_\varepsilon$*

$$\limsup_{n\to\infty}(\pm A_n(\lambda)) \leq M_\tau^{1/2} + 4\varepsilon \quad a.s.$$

PROOF. For all large n we write

$$\pm A_n(\lambda) = a_n(k_n)^{-1}\sum_{i=1}^{n}(\pm\xi(i, l')),$$

where for $i, l \geq 1$

$$\pm\xi(i,l) = \int_{1-k_{m_{l-1}}/m_{l-1}}^{1-\tau k_{m_{l-1}}/m_{l-1}} \pm\left(s - 1_{[0,s]}(U_i)\right) dQ(s);$$

here we suppress the fixed parameters $\lambda > 1$ and $0 < \tau < 1$ in $\pm\xi(i,l)$ for notational convenience, and for further notational convenience $\xi(i,l)$ will denote either $+\xi(i,l)$ or $-\xi(i,l)$. For fixed l, the random variables $\xi(i,l)$, $i \geq 1$, are independent and identically distributed with mean zero, $|\xi(i,l)| \leq Q(1-\tau k_{m_{l-1}}/m_{l-1})$ and

$$\text{(29)} \quad \operatorname{Var}(\xi(i,l)) = \int_{1-k_{m_{l-1}}/m_{l-1}}^{1-\tau k_{m_{l-1}}/m_{l-1}}\int_{1-k_{m_{l-1}}/m_{l-1}}^{1-\tau k_{m_{l-1}}/m_{l-1}}(s\wedge t - st)\, dQ(s)\, dQ(t).$$

We need to know the asymptotic behavior of $\operatorname{Var}(\xi(i,l))$ as l gets large. For $0 < u < 1$ we set

$$\sigma^2(u) = \int_{1-u}^{1-\tau u}\int_{1-u}^{1-\tau u}(s\wedge t - st)\, dQ(s)\, dQ(t).$$

Integrating this out by parts gives

$$\sigma^2(u) = \int_{1-u}^{1-\tau u} Q(t)^2\, dt - 2Q(1-u)\int_{1-u}^{1-\tau u}(1-s)\, dQ(s)$$
$$+\tau u Q(1-\tau u)^2 - uQ(1-u)^2 - \left(\int_{1-u}^{1-\tau u}(1-s)\, dQ(s)\right)^2.$$

Analyzing the behavior of the summands on the r.h.s. as $u \downarrow 0$ by means of Theorem 1.2.1 in de Haan (1975) leads to

$$\text{(30)} \qquad \sigma^2(u) \sim N_a^2 M_\tau u^{1-2/a} L(u)^2 \quad \text{as } u\downarrow 0.$$

From (4) and (5) we get

$$a_n(k_n) \sim a_n = N_a(2\alpha_n \log\log n)^{1/2} Q(1 - \beta_n)$$
$$\geq N_a\big(2\alpha_{m_{l'-1}} \log\log m_{l'-1}\big)^{1/2} Q\big(1 - \beta_{m_{l'-1}}\big) \tag{31}$$
$$= a_{m_{l'-1}} \sim a_{m_{l'-1}}\big(k_{m_{l'-1}}\big).$$

Fix now $\varepsilon > 0$ and $\lambda > 1$. Then

$$P\big(\pm A_n(\lambda) \geq M_\tau^{1/2} + 4\varepsilon \text{ i.o. in } n\big)$$
$$= P\left(\sum_{i=1}^{n} \xi(i, l') \geq \big(M_\tau^{1/2} + 4\varepsilon\big) a_n(k_n) \text{ i.o. in } n\right),$$

which by (31) is

$$\leq P\left(\max_{m_{l'-1} < p \leq m_{l'}} \sum_{i=1}^{p} \xi(i, l') \geq \big(M_\tau^{1/2} + 3\varepsilon\big) a_{m_{l'-1}}\big(k_{m_{l'-1}}\big) \text{ i.o. in } n\right)$$
$$\leq P\left(\max_{1 \leq p \leq m_l} \sum_{i=1}^{p} \xi(i, l) \geq \big(M_\tau^{1/2} + 3\varepsilon\big) a_{m_{l-1}}\big(k_{m_{l-1}}\big) \text{ i.o. in } l\right).$$

Of course, our aim is to apply the Borel–Cantelli lemma so that we have to bound

$$P_l \equiv P\left(\max_{1 \leq p \leq m_l} \sum_{i=1}^{p} \xi(i, l) \geq \big(M_\tau^{1/2} + 3\varepsilon\big) a_{m_{l-1}}\big(k_{m_{l-1}}\big)\right)$$

for all large l by the summands of a finite series. Using (29) and (30) we find for all large l

$$B_l \equiv \big(M_\tau^{1/2} + 3\varepsilon\big) a_{m_{l-1}}\big(k_{m_{l-1}}\big) - \left(2 \operatorname{Var}\left(\sum_{i=1}^{m_l} \xi(i, l)\right)\right)^{1/2}$$
$$= N_a(2 m_{l-1} \log\log m_{l-1})^{1/2} \left(\frac{k_{m_{l-1}}}{m_{l-1}}\right)^{1/2 - 1/a} L\left(\frac{k_{m_{l-1}}}{m_{l-1}}\right)$$
$$\times \left\{M_\tau^{1/2} + 3\varepsilon - M_\tau^{1/2} \left(\frac{m_l}{m_{l-1}}\right)^{1/2} (\log\log m_{l-1})^{-1/2} (1 + o(1))\right\},$$

which since $m_l / m_{l-1} \to \lambda$ and $\log\log m_{l-1} \to \infty$ is

$$\sim N_a\big(M_\tau^{1/2} + 3\varepsilon\big)(2 m_{l-1} \log\log m_{l-1})^{1/2} \left(\frac{k_{m_{l-1}}}{m_{l-1}}\right)^{1/2 - 1/a} L\left(\frac{k_{m_{l-1}}}{m_{l-1}}\right).$$

Consequently, for all large l we have

$$B_l \geq N_a\big(M_\tau^{1/2} + 2\varepsilon\big)(2 m_{l-1} \log\log m_{l-1})^{1/2} \left(\frac{k_{m_{l-1}}}{m_{l-1}}\right)^{1/2 - 1/a} L\left(\frac{k_{m_{l-1}}}{m_{l-1}}\right) \equiv c(l, \varepsilon).$$

Therefore, Lévy's inequality, cf., for example, Loève (1977), pages 259–260, together with Bernstein's inequality, cf. inequality (8) in Bennett (1962), imply for all large l

$$P_l \le 2P\left(\sum_{i=1}^{m_l} \xi(i, l) \ge c(l, \varepsilon)\right) \le 2\exp(-\rho(l, \varepsilon)),$$

where

$$\rho(l, \varepsilon) = c(l, \varepsilon)^2 / \left\{2m_l\sigma^2\left(k_{m_{l-1}}/m_{l-1}\right) + \tfrac{2}{3}c(l, \varepsilon)Q\left(1 - \tau k_{m_{l-1}}/m_{l-1}\right)\right\}.$$

From (22) and (30) we get as $l \to \infty$

$$c(l, \varepsilon)Q\left(1 - \tau k_{m_{l-1}}/m_{l-1}\right)/\left(m_l\sigma^2\left(k_{m_{l-1}}/m_{l-1}\right)\right) \to 0$$

and

$$c(l, \varepsilon)^2/\left(2m_l\sigma^2\left(k_{m_{l-1}}/m_{l-1}\right)\right) \sim \left(1 + 2\varepsilon/M_\tau^{1/2}\right)^2\lambda^{-1}\log\log m_{l-1},$$

whence

$$\rho(l, \varepsilon) \sim \left(1 + 2\varepsilon/M_\tau^{1/2}\right)^2\lambda^{-1}\log\log m_{l-1} \sim \left(1 + 2\varepsilon/M_\tau^{1/2}\right)^2\lambda^{-1}\log(l - 1).$$

This implies for all large l

$$P_l \le 2(l - 1)^{-(1+\varepsilon/M_\tau^{1/2})^2\lambda^{-1}}.$$

For given $\varepsilon > 0$ and $\lambda > 1$ close enough to 1 one has $(1 + \varepsilon/M_\tau^{1/2})^2\lambda^{-1} > 1$, so that the sum over the bounds on the r.h.s. is finite. This concludes the proof of Lemma 12. □

For $\varepsilon > 0$ fix $\lambda_\varepsilon > 1$ according to Lemma 12. Then for all $1 < \lambda < \lambda_\varepsilon$ by Lemmas 11 and 12 with probability one

$$\begin{aligned} M^{\pm}(\tau) &\le \limsup_{n\to\infty} |R_{1,n}(\lambda) + R_{2,n}(\lambda)| + \limsup_{n\to\infty} (\pm A_n(\lambda)) \\ &\le 2^{3/2}N_a^{-1}\lambda^{1/2-1/a}(\lambda^{1/a} - 1) + M_\tau^{1/2} + 4\varepsilon. \end{aligned}$$

Since the l.h.s. of this inequality is independent of $\varepsilon > 0$ and $\lambda > 1$ we get by letting λ tend to one first and then letting ε tend to zero that almost surely $M^{\pm}(\tau) \le M_\tau^{1/2}$. Thus we have proven the $\le$ -part of (28).

We now turn to the $\ge$ -part and show for any $0 < \varepsilon < M_\tau^{1/2}/2$ with probability one $M^{\pm}(\tau) \ge M_\tau^{1/2} - 2\varepsilon$. For this, fix $0 < \varepsilon < M_\tau^{1/2}/2$ and $\lambda > 1$ and put $m_l = [\lambda^l]$ for $l \ge 1$ as before. Writing

$$\pm\eta(i, l) = \int_{1-k_{m_l}/m_l}^{1-\tau k_{m_l}/m_l} \pm\left(s - 1_{[0,s]}(U_i)\right) dQ(s),$$

$$\pm S_{l-1} = \sum_{i=1}^{m_{l-1}} (\pm\eta(i, l))$$

and

$$\pm T_l = \sum_{i=m_{l-1}+1}^{m_l} (\pm\eta(i, l)),$$

we have

$$a_{m_l}(k_{m_l})^{-1} m_l \int_{1-k_{m_l}/m_l}^{1-\tau k_{m_l}/m_l} \pm \left(s - G_{m_l}(s)\right) dQ(s)$$

$$= a_{m_l}(k_{m_l})^{-1}(\pm S_{l-1}) + a_{m_l}(k_{m_l})^{-1}(\pm T_l).$$

For notational convenience we will write $\eta(i, l) = \pm\eta(i, l)$, $S_{l-1} = \pm S_{l-1}$, and $T_l = \pm T_l$. For fixed l, the random variables $\eta(i, l)$, $i \geq 1$, are independent and identically distributed with mean zero, $|\eta(i, l)| \leq Q(1 - \tau k_{m_l}/m_l)$ and $\mathrm{Var}(\eta(i, l)) = \sigma^2(k_{m_l}/m_l)$, where σ^2 is defined as in the proof of Lemma 12. At first we show

LEMMA 13. *For all $\varepsilon > 0$, there exists a $\lambda_\varepsilon < \infty$ such that for all $\lambda > \lambda_\varepsilon$*

$$P\left(S_{l-1} < -\varepsilon a_{m_l}(k_{m_l}) \text{ i.o. in } l\right) = 0.$$

PROOF. Applying Bernstein's inequality to $-S_{l-1}$ we obtain for all large l

$$P\left(S_{l-1} < -\varepsilon a_{m_l}(k_{m_l})\right)$$
$$\leq \exp\left\{-\varepsilon^2 a_{m_l}(k_{m_l})^2 / \left(2m_{l-1}\sigma^2(k_{m_l}/m_l) + \tfrac{2}{3}\varepsilon a_{m_l}(k_{m_l}) Q\left(1 - \tau k_{m_l}/m_l\right)\right)\right\}.$$

As in the proof of Lemma 12 we get that the argument of the exponential function on the r.h.s. behaves like

$$-\left(\lambda\varepsilon^2/M_\tau\right)\log\log m_l \sim -\left(\lambda\varepsilon^2/M_\tau\right)\log l$$

as $l \to \infty$. Since $\lambda\varepsilon^2/M_\tau > 1$ for big enough λ we get

$$\sum_{l=1}^{\infty} P\left(S_{l-1} < -\varepsilon a_{m_l}(k_{m_l})\right) < \infty$$

for all such λ, and the assertion of the lemma follows from the Borel–Cantelli lemma. □

Next we show

LEMMA 14. *For all $0 < \varepsilon < M_\tau^{1/2}/2$, there exists a $\lambda_\varepsilon < \infty$ such that for all $\lambda > \lambda_\varepsilon$*

$$P\left(T_l > \left(M_\tau^{1/2} - \varepsilon\right) a_{m_l}(k_{m_l}) \text{ i.o. in } l\right) = 1.$$

PROOF. Since

$$T_l = \sum_{i=m_{l-1}+1}^{m_l} \int_{1-k_{m_l}/m_l}^{1-\tau k_{m_l}/m_l} \pm \left(s - 1_{[0,s]}(U_i)\right) dQ(s)$$

is a function of $U_{m_{l-1}+1}, \ldots, U_{m_l}$, the variables T_l, $l \geq 1$, are independent. Hence the assertion of the lemma follows from the second Borel–Cantelli lemma provided that we show

$$\sum_{l=1}^{\infty} P\left(T_l > \left(M_\tau^{1/2} - \varepsilon\right) a_{m_l}(k_{m_l})\right) = \infty \tag{32}$$

for all large enough λ. For this, we shall employ one of Kolmogorov's exponential inequalities which for our purposes reads as follows: Let $X_1, \dots, X_n$ be independent random variables with mean zero,

$$s_n^2 = \sum_{i=1}^{n} \operatorname{Var}(X_i) > 0 \quad \text{and} \quad |X_i| \le c s_n, \quad \text{for } i = 1, \dots n.$$

For any $\gamma > 0$ there exist constants $u_\gamma < \infty$ and $\pi_\gamma > 0$ depending only on γ such that if $u \ge u_\gamma$ and $uc \le \pi_\gamma$, then

$$P\left(\sum_{i=1}^{n} X_i > u s_n\right) > \exp(-u^2(1+\gamma)/2). \tag{33}$$

For a proof the reader is referred to Burrill (1972), Theorem 13-7C or Stout (1974), Theorem 5.2.2.

We can apply (33) for all large l to the independent and centered random variables $\eta(m_{l-1}+1, l), \dots, \eta(m_l, l)$ and to

$$u = u(l) \equiv \left(M_\tau^{1/2} - \varepsilon\right) a_{m_l}(k_{m_l}) \left(\sum_{i=m_{l-1}+1}^{m_l} \operatorname{Var}(\eta(i,l))\right)^{-1/2},$$

since for

$$c = c(l) \equiv Q(1 - \tau k_{m_l}/m_l)(m_l - m_{l-1})^{-1/2} \sigma(k_{m_l}/m_l)^{-1}$$

we have

$$|\eta(i,l)| \le c(l) \left(\sum_{i=m_{l-1}+1}^{m_l} \operatorname{Var}(\eta(i,l))\right)^{1/2},$$

and furthermore by elementary computations using (30)

$$u(l) \sim 2^{1/2}(1 - \varepsilon/M_\tau^{1/2})(\lambda/(\lambda-1))^{1/2}(\log l)^{1/2} \to \infty, \tag{34}$$

and by (22), $u(l)c(l) \to 0$ as $l \to \infty$. Therefore, for any $\gamma > 0$ being fixed, we have for all large l

$$\begin{aligned} &P\left(T_l > \left(M_\tau^{1/2} - \varepsilon\right) a_{m_l}(k_{m_l})\right) \\ &\quad > \exp\left(-u(l)^2(1+\gamma)/2\right) > l^{-(1-\varepsilon/M_\tau^{1/2})^2(\lambda/(\lambda-1))(1+2\gamma)} \end{aligned}$$

in view of (34). For $0 < \varepsilon < M_\tau^{1/2}/2$ we can choose a $\gamma > 0$ such that $(1 - \varepsilon/M_\tau^{1/2})^2(1+2\gamma) < 1$. Then we have $(1 - \varepsilon/M_\tau^{1/2})^2(1+2\gamma) \times (\lambda/(\lambda-1)) < 1$ for all large λ which implies (32) and concludes the proof of Lemma 14. □

We have for $0 < \varepsilon < M_\tau^{1/2}/2$ and $\lambda > 1$

$$\begin{aligned} &P\left(M^{\pm}(\tau) \ge M_\tau^{1/2} - 2\varepsilon\right) \\ &\quad \ge P\left(S_{l-1} + T_l \ge \left(M_\tau^{1/2} - 2\varepsilon\right) a_{m_l}(k_{m_l}) \text{ i.o. in } l\right) \\ &\quad \ge P\Big(\left\{T_l \ge \left(M_\tau^{1/2} - \varepsilon\right) a_{m_l}(k_{m_l}) \text{ i.o. in } l\right\} \\ &\qquad \cap \left\{S_{l-1} \ge -\varepsilon a_{m_l}(k_{m_l}) \text{ eventually in } l\right\}\Big). \end{aligned}$$

Lemmas 13 and 14 show that for fixed ε and large enough λ the probability on the r.h.s. equals one, yielding the $\geq$ -part of (28). Thus (28) is proven.

Since the r.h.s. of the estimate given in Lemma 9 converges to zero as $\tau \downarrow 0$ and since $M_\tau^{1/2} \to 1$ as $\tau \downarrow 0$, (24) follows from Lemma 9 and (28) in an obvious fashion. Thus we have shown that (7) implies (24). Therefore, (7) also implies (8) and (9).

The following lemma will show that (8) implies (7), and (11) implies (12).

LEMMA 15. *For any integer* $k \geq 1$

$$\limsup_{n\to\infty} a_n(k_n)^{-1}\left\{\sum_{j=k}^{k_n} X_{n+1-j,n} - n\mu(k_n/n)\right\} < \infty \quad a.s. \tag{35}$$

implies (7).

PROOF. For $n \geq 1$ set $a_n' = N_a \alpha_n (2\log\log n)^{1/2} Q(1-\beta_n)$. Then $a_n(k_n)/a_n' \sim \alpha_n^{-1/2} \to 0$ so that from (35)

$$\limsup_{n\to\infty} a_n'^{-1}\left\{\sum_{j=k}^{k_n} X_{n+1-j,n} - n\mu(k_n/n)\right\} \leq 0 \quad \text{a.s.}$$

Furthermore, by Theorem 1.2.1 in de Haan (1975) we obtain

$$a_n'^{-1} n\mu(k_n/n) \sim \frac{a}{N_a(a-1)}(2\log\log n)^{-1/2} \to 0.$$

Combining the last two facts and taking $X_{i,n} \geq 0$ into account we obtain

$$a_n'^{-1}\sum_{j=k}^{k_n} X_{n+1-j,n} \to 0 \quad \text{a.s.},$$

in particular,

$$a_n'^{-1} X_{n+1-k,n} \to 0 \quad \text{a.s.}$$

Since the sequence a_n' is nondecreasing, we can apply Lemma 1 to obtain

$$\sum_{n=1}^{\infty} n^{k-1} P(Q(1-U_1) > a_n')^k < \infty,$$

and from this it is easy to see that

$$\sum_{n=1}^{\infty} n^{-1}\alpha_n^{-k(\gamma-1)}(\log\log n)^{-k\gamma/2} < \infty \tag{36}$$

for all $a < \gamma < \infty$ using the arguments as in the proof of Lemma 6. Fix $2 < \delta < a$ and $1 < d < \infty$. On the event

$$\{Q(1-U_1) > a_n\} = \left\{Q(1-U_1) > N_a(2\alpha_n \log\log n)^{1/2} Q(1-\beta_n)\right\},$$

we have $Q(1-U_1) > Q(1-\beta_n)$ for all large n, and, consequently, $0 < U_1 \leq \beta_n$.

Lemma 3(ii) therefore implies

$$\begin{aligned} P(Q(1-U_1) > a_n) &= P\big(Q(1-U_1)/Q(1-\beta_n) > N_a(2\alpha_n \log\log n)^{1/2}\big) \\ &\le P\big(d(U_1/\beta_n)^{-1/\delta} > N_a(2\alpha_n \log\log n)^{1/2}\big) \\ &= 2^{-\delta/2}(N_a/d)^{-\delta}\beta_n(\alpha_n \log\log n)^{-\delta/2} \\ &\sim 2^{-\delta/2}(N_a/d)^{-\delta} n^{-1}\alpha_n^{-\delta/2+1}(\log\log n)^{-\delta/2}, \end{aligned}$$

whence for any $m \ge 1$ and all large n

$$n^{m-1}P(Q(1-U_1) > a_n)^m \le K n^{-1}\alpha_n^{-m(\delta/2-1)}(\log\log n)^{-m\delta/2},$$

for some finite constant K being independent of n. For fixed γ and δ and all large enough m the r.h.s. of this inequality is dominated for all large n by the summands of the series in (36), hence

$$\sum_{n=1}^{\infty} n^{m-1}P(Q(1-U_1) > a_n)^m < \infty$$

for all large m. Since the sequence a_n is nondecreasing, Lemma 1 yields

$$a_n^{-1}X_{n+1-m,n} \to 0 \quad \text{a.s.}$$

for all large m. Since $a_n \sim a_n(k_n)$, this means that (7) is true for all large m. Fix such an integer $m > k$. Then

$$\begin{aligned} &a_n(k_n)^{-1}\left\{\sum_{j=k}^{k_n} X_{n+1-j,n} - n\mu(k_n/n)\right\} \\ &\quad \ge a_n(k_n)^{-1}\left\{\sum_{j=m}^{k_n} X_{n+1-j,n} - n\mu(k_n/n)\right\} + a_n(k_n)^{-1}X_{n+1-k,n} \end{aligned}$$

since $X_{i,n} \ge 0$. We have already shown that (7) implies (8) and (9). Using this result for m instead of k we see that the first summands on the right side of this inequality are bounded with probability one. Consequently, if we were to assume (11), we would obtain

$$\limsup_{n\to\infty} a_n(k_n)^{-1}\left\{\sum_{j=k}^{k_n} X_{n+1-j,n} - n\mu(k_n/n)\right\} = \infty \quad \text{a.s.},$$

which is in contradiction to (35). Thus the lim sup in (11) must be finite almost surely. But since (6) and (7) are equivalent, and (10) and (11) are also equivalent, we then must have (7), as desired. □

Lemma 15 finishes the proof of Theorem 1.

Next we will complete the proof of Theorem 2. For this, assume first that (15) holds for some integer $k \ge 1$. Replacing $N_a(2\log\log n)^{1/2}$ by b_n in the proof of Lemma 6 yields

$$\sum_{n=1}^{\infty} n^{-1}k_n^{-k(\gamma-2)/2}b_n^{-k\gamma} < \infty$$

for all $a < \gamma < \infty$, hence $k_n^{-k(\gamma-2)/2}b_n^{-k\gamma}\log n \to 0$, so that because of (14) we must have

(37) $$k_n^2/b_n \to 0.$$

To verify (17) with $c_n = n\mu(k_n/n)$ it is enough to show that there exists an integer $l \geq k$ with

$$a_n(k_n, b_n)^{-1}\left\{\sum_{j=l}^{k_n} X_{n+1-j,n} - n\mu(k_n/n)\right\} \to 0 \quad \text{a.s.},$$

since (16) implies

$$a_n(k_n, b_n)^{-1} \sum_{j=k}^{l-1} X_{n+1-j,n} \to 0 \qquad \text{a.s.}$$

Applying Theorem 1.2.1 in de Haan (1975) we obtain

$$a_n(k_n, b_n)^{-1} n\mu(k_n/n) \sim a k_n^{1/2} b_n^{-1}/(a-1) \to 0$$

because of (37). Furthermore

$$a_n(k_n, b_n)^{-1} \sum_{j=l}^{k_n} X_{n+1-j,n} \leq \frac{k_n}{b_n^{1/2}} \frac{X_{n+1-l,n}}{b_n^{1/2} k_n^{1/2} Q(1-k_n/n)}.$$

Now by (37) we have $k_n b_n^{-1/2} \to 0$, whereas by an application of Lemma 1

$$X_{n+1-l,n}/\left(b_n^{1/2} k_n^{1/2} Q(1-k_n/n)\right) \to 0 \quad \text{a.s.}$$

for all large enough l. Thus (17) is established. It remains to demonstrate that

(38) $$\limsup_{n\to\infty} a_n(k_n, b_n)^{-1}\left|\sum_{j=k}^{k_n} X_{n+1-j,n} - c_n\right| < \infty \quad \text{a.s.}$$

for some sequence of constants $(c_n)_{n\geq 1}$ implies (15). Our first step will be to show that the constants c_n can be replaced by $n\mu(k_n/n)$. Since $b_n \uparrow \infty$, the central limit theorem (3) implies

$$a_n(k_n, b_n)^{-1}\left\{\sum_{j=1}^{k_n} X_{n+1-j,n} - n\mu(k_n/n)\right\} \to_P 0.$$

As remarked in S. Csörgő and Mason (1986) the variables

$$Q(1-1/n)^{-1} \sum_{j=1}^{k-1} X_{n+1-j,n}, \qquad n = 1, 2, \dots,$$

converge in distribution, so that necessarily

$$a_n(k_n, b_n)^{-1} \sum_{j=1}^{k-1} X_{n+1-j,n} \to_P 0$$

because by $k_n \to \infty$ and Lemma 2

$$\frac{Q(1-1/n)}{a_n(k_n, b_n)} = \frac{1}{b_n} \frac{(1/n)^{1/2-1/a} L(1/n)}{(k_n/n)^{1/2-1/a} L(k_n/n)} \to 0.$$

Thus

$$a_n(k_n, b_n)^{-1}\left\{ \sum_{j=k}^{k_n} X_{n+1-j,n} - n\mu(k_n/n) \right\} \to_P 0,$$

which combined with (38) implies that the sequence

$$\left(a_n(k_n, b_n)^{-1}|n\mu(k_n/n) - c_n|\right)_{n\geq 1}$$

is bounded. Hence in view of (38) we have

$$\limsup_{n\to\infty} a_n(k_n, b_n)^{-1}\left| \sum_{j=k}^{k_n} X_{n+1-j,n} - n\mu(k_n/n) \right| < \infty \quad \text{a.s.} \tag{39}$$

Now repeating the proof of Lemma 15 with $a_n' = N_a\alpha_n(2\log\log n)^{1/2}Q(1-\beta_n)$ replaced by $a_n' = \alpha_n b_n Q(1-\beta_n)$ shows that (39) implies (15) which concludes the proof of Theorem 2.

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