The Annals of Probability
1987, Vol. 15, No. 4, 1600–1611

A CHARACTERIZATION OF ADAPTED DISTRIBUTION[1]

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The notion of adapted distribution of a stochastic process was introduced in a recent paper of Hoover and Keisler. Here we give a simple characterization of this notion in terms of filtration embeddability. This characterization allows us to show that for a local martingale M for which some ordinary stochastic differential equation $X_t = \int_0^t f(s, X_s)\, dM_s$ admits sufficient non-uniqueness in law of the solutions X, the class of possible joint laws of (M, X) determines the adapted law of M.

0. Introduction. By a stochastic process we mean a process in the "general theory of processes," that is, a stochastic process together with its filtration [see Definition 1.2 and Meyer (1976)]. For processes so construed, the notion of distribution is not an adequate notion of equivalence, since it does not take into account the filtration. Two stronger notions of equivalence for stochastic processes have recently been introduced: that of synonymity [Aldous (1980) and Definition 1.4.4 below], and that of adapted distribution [Hoover and Keisler (1984) and Definition 1.4 below]. Hoover and Keisler gave reasons to believe that almost any interesting property of stochastic processes is preserved under equality of adapted distribution, at least among processes living on saturated adapted probability spaces [Hoover and Keisler (1984) and Definition 1.5 below]. One would like to know which properties of processes are in fact preserved under synonymity, since that is a much weaker relation [Hoover and Keisler (1984), Section 3]. Some examples of such properties are given in Aldous (1980) and Hoover (1984). The main motivation for this paper was the desire to find out whether in the following result about stochastic differential equations the hypothesis "have the same adapted distribution" can be weakened to "are synonymous."

If M and N are semimartingales with the same adapted distribution, then $M \hookrightarrow^{\mathrm{SDE}} N$. That is to say: Whenever $f(t, x)$ is a measurable function $R^+ \times R \to R$, and the processes $X^1, \dots, X^n$ satisfy

$$X_t^i = X_0^i + \int_0^t f(s, X_{s-}^i)\, dM_s,$$

then there are the processes $Y^1, \dots, Y^n$ satisfying

$$Y_t^i = Y_0^i + \int_0^t f(s, Y_{s-}^i)\, dN_s,$$

such that $(M, X^1, \dots, X^n)$ and $(N, Y^1, \dots, Y^n)$ have the same distribution.

Received September 1985; revised August 1986.
[1]This research has been supported by the Waterman Award (thanks to Harvey Friedman) and by an NSERC Operating Grant. The author was an NSERC University Research Fellow.
AMS 1980 *subject classifications.* Primary 60G07; secondary 60H10.
Key words and phrases. Adapted distribution, synonymity, stochastic differential equations.

This is to say, semimartingales with the same adapted distribution have the same distributions of families of solutions of stochastic differential equations. This is a consequence of Theorem 7.8 in Hoover and Keisler (1984).

In this paper we present a characterization of adapted distribution which can be used to show that, for certain classes of local martingales, if $M \hookrightarrow^{\mathrm{SDE}} N$ and $N \hookrightarrow^{\mathrm{SDE}} M$, then M and N have the same adapted distribution. It follows that synonymity is not sufficient to imply $M \hookrightarrow^{\mathrm{SDE}} N$.

1. Definitions and notation. Let $\mathbf{B}$ denote the Borel sets on $R^+ = [0, \infty)$.

1.1. An adapted probability space (or simply "adapted space") is a structure $\mathbf{\Omega} = (\Omega, \mathbf{F}, P, (\mathbf{F}_t)_{t \in R^+})$, where $(\Omega, \mathbf{F}, P)$ is a probability space and $(\mathbf{F}_t)_{t \in R^+}$ is a filtration (increasing family of sub-σ-algebras of $\mathbf{F}$) which satisfies the "usual conditions" (for all t, $\bigcap_{t < s} \mathbf{F}_s = \mathbf{F}_t$; $\mathbf{F}_0$ contains all null sets of P).

1.2. A stochastic process is a structure $\mathbf{X} = (\mathbf{\Omega}, X) = (\Omega, \mathbf{F}, \mathbf{P}, \mathbf{F}_t, X)_{t \in R^+}$, where X is a $(\mathbf{B} \times \mathbf{F})$-measurable mapping $R^+ \times \Omega \to M$, M a Polish (i.e., complete separable metric) topological space. The adapted space $\mathbf{\Omega}$ appearing in the definition of $\mathbf{X}$ will be denoted $\mathbf{\Omega}(\mathbf{X})$, and $\mathbf{F}(\mathbf{X})$ will denote the filtration of $\mathbf{\Omega}(\mathbf{X})$. $\sigma(X)$ denotes the subfield of $\mathbf{F}$ generated by X. When an underlying process X appears with several different filtrations, we will write $(X, (\mathbf{F}_t))$ instead of $\mathbf{X}$, to keep the notation clearer.

The distinction between the stochastic process $\mathbf{X}$ and the associated measurable function X will be used to emphasize which properties of a stochastic process depend on the filtration. For example, we would say,

$$\mathbf{X} \text{ is a martingale,}$$

but

$$X \text{ and } Y \text{ have the same distribution.}$$

1.3. An *adapted set* is an $\mathbf{F} \times \mathbf{B}$-measurable set $F \times [t, \infty)$ where $F \in \mathbf{F}_t$. We say that $F \times [t, \infty)$ is *adapted at t*. We say the same of F itself. An *adapted random variable* is a stochastic process X of the form $X^0 \cdot 1([t, \infty))$ for some $\mathbf{F}_t$-measurable random variable X^0. We say that X is *adapted at t*.

An adapted r.v. $\mathbf{X}$ is *simple* if $\sigma(X)$ is finite.

We see that if $\mathbf{X}$ is a random variable adapted at t, then $\sigma(X)$ consists entirely of sets adapted at t. Usually we will abuse notation and speak of X^0 as the adapted random variable.

In Hoover and Keisler (1984) the notion of adapted distribution was introduced, which contains the probabilistic information about a process with filtration, as the distribution does that of a random variable. We repeat the main definitions here, partly for convenience, and partly because we have made a slight modification.

1.4. Let $\mathbf{X}$ and $\mathbf{Y}$ be stochastic processes with values in the Polish space M.

1.4.1. The class AF of *adapted functions* is defined inductively as follows:

(AF1) If Φ: $M \to [0,1]$ is continuous and $t \in R^+$, then $(\Phi, t) \in \text{AF}$.

(AF2) If $f_1, \dots, f_n \in \text{AF}$ and φ: $[0,1]^n \to [0,1]$ is continuous, then $(\varphi, (f_1, \dots, f_n)) \in \text{AF}$.

(AF3) If $g \in \text{AF}$ and $t \in R^+$, then $E(g|t) \in \text{AF}$.

1.4.2. The *value* $f(\mathbf{X})$ of $f \in \text{AF}$ on $\mathbf{X}$ is the random variable defined inductively as follows:

(V1) $(\Phi, t)(\mathbf{X}) = \Phi(X_t)$.

(V2) $(\varphi, (f_1, \dots, f_n))(\mathbf{X}) = \varphi(f_1(\mathbf{X}), \dots, f_n(\mathbf{X}))$.

(V3) $E(g|t)(\mathbf{X}) = E[g(\mathbf{X})|\mathbf{F}(\mathbf{X})_t]$.

1.4.3. A *subfunction* of an adapted function f is any adapted function g which appears as a subexpression of f. Precisely the class of subfunctions of f is the smallest class $\text{sub}(f)$ of adapted functions containing f and having the following closure properties:

(S1) If $(\varphi, (f_1, \dots, f_n)) \in \text{sub}(f)$, then $f_1, \dots, f_n \in \text{sub}(f)$.

(S2) If $E(g|t) \in \text{sub}(f)$, then $g \in \text{sub}(f)$.

1.4.4. (i) $\mathbf{X}$ and $\mathbf{Y}$ have the same adapted distribution, $\mathbf{X} \equiv \mathbf{Y}$, if for every $f \in \text{AF}$,

$$E[f(\mathbf{X})] = E[f(\mathbf{Y})]. \tag{1.4.4}$$

(ii) We write $\mathbf{X} \equiv_0 \mathbf{Y}$ to mean that X and Y have the same distribution.

(iii) $\mathbf{X} \equiv_1 \mathbf{Y}$ means that $\mathbf{X}$ and $\mathbf{Y}$ are synonymous in the sense of Aldous (1980) i.e., for any $n \in N$, $t_1, \dots, t_n, u_1, \dots, u_n \in R^+$ and $\varphi_1, \dots, \varphi_n$ bounded continuous functions $R^n \to R$,

$$\begin{aligned}&\left(E\left[\varphi_1(X_{u_1}, \dots, X_{u_n})|\mathbf{F}_{t_1}\right], \dots, E\left[\varphi_n(X_{u_1}, \dots, X_{u_n})|\mathbf{F}_{t_n}\right]\right)\\ &\quad \equiv_0 \left(E\left[\varphi_1(Y_{u_1}, \dots, Y_{u_n})|\mathbf{G}_{t_1}\right], \dots, E\left[\varphi_n(Y_{u_1}, \dots, Y_{u_n})|\mathbf{G}_{t_n}\right]\right).\end{aligned}$$

1.5. An adapted space Ω is *saturated* if for any process $\mathbf{X}^1$ on Ω and processes $\mathbf{Y}^1$ and $\mathbf{Y}^2$ on another adapted space Ω', if

$$\mathbf{X}^1 \equiv \mathbf{Y}^1,$$

then there is a process $\mathbf{X}^2$ on Ω such that

$$(\mathbf{X}^1, \mathbf{X}^2) \equiv (\mathbf{Y}^1, \mathbf{Y}^2).$$

2. A characterization of adapted distribution. We give now a sufficient condition for equality in adapted distribution. This condition, in terms of partial filtration embeddings, characterizes equality in adapted distribution when the adapted spaces involved are saturated. Although simple, this is really our main result. From it comes the application to stochastic differential equations in Section 4.

DEFINITION 2.1. Let the function $[\cdot]_\delta$: $R \to R$ be defined by

$$[x]_\delta = i\delta, \quad \text{where } i \in Z,\ i\delta \le x < (i+1)\delta.$$

LEMMA 2.2. *Let X and Y be square integrable, R^n-valued random variables on probability spaces $(\Omega_1, \mathbf{F}, P)$ and $(\Omega_2, \mathbf{G}, P)$, respectively. Let $\mathbf{F}_0 \subseteq \mathbf{F}$ and $\mathbf{G}_0 \subseteq \mathbf{G}$ be* sub-σ-*algebras, and let $X_0 = E[X|\mathbf{F}_0]$ and $Y_0 = E[Y|\mathbf{G}_0]$. Let X_0' and Y_0' be $\mathbf{F}_0$ and $\mathbf{G}_0$ measurable random variables, respectively.*

(a) *If*

$$(X, X_0) \equiv_0 (Y, Y_0') \tag{2.2.1}$$

and

$$(X, X_0') \equiv_0 (Y, Y_0), \tag{2.2.2}$$

then $X_0' = X_0$ a.s. and $Y_0' = Y_0$ a.s.

(b) *For any $\varepsilon > 0$, there is a $\delta > 0$ such that if*

$$(X, [X_0]_\delta) \equiv_0 (Y, Y_0')$$

and

$$(X, X_0') \equiv_0 (Y, [Y_0]_\delta),$$

then $\sigma^2(X_0' - X_0) < \varepsilon$ and $\sigma^2(Y_0' - Y_0) < \varepsilon$.

PROOF. (a) X_0 is the projection of X on $L^2(\mathbf{F}_0)$. The analogous property is true of Y, Y_0 and $\mathbf{G}_0$, hence, in particular, Y_0 is square integrable. By (2.2.2), X_0' is in $L^2(\mathbf{F}_0)$. Hence

$$\sigma^2(X_0' - X) = \sigma^2(X_0' - X_0) + \sigma^2(X_0 - X).$$

Likewise,

$$\sigma^2(Y_0' - Y) = \sigma^2(Y_0' - Y_0) + \sigma^2(Y_0 - Y).$$

Hence

$$\begin{aligned}\sigma^2(X_0' - X) + \sigma^2(Y_0' - Y) = {}& \sigma^2(X_0' - X_0) + \sigma^2(X_0 - X)\\ &+\sigma^2(Y_0' - Y_0) + \sigma^2(Y_0 - Y).\end{aligned}$$

By (2.2.1) and (2.2.2),

$$\sigma^2(X_0' - X) + \sigma^2(Y_0' - Y) = \sigma^2(X_0 - X) + \sigma^2(Y_0 - Y).$$

Hence

$$\sigma^2(X_0' - X_0) + \sigma^2(Y_0' - Y_0) = 0.$$

(b) is a straightforward refinement of (a). Followers of nonstandard analysis will observe that if δ is taken infinitesimal and $=$ is changed to $\approx$ (infinitesimally close) in the proof of (a), (b) follows by overspill. □

Hoover and Keisler (1984), Lemma 5.7, and Theorem 2.4 of this paper show that the foregoing equivalence relations are closely related to the following properties for stochastic processes.

DEFINITION 2.3. **X** is finitely embeddable in **Y**, $\mathbf{X} \hookrightarrow \mathbf{Y}$ iff:

(2.3.1) For any $n \in N$ and simple r.v.'s $X^1, \dots, X^n$ on $\Omega(\mathbf{X})$, respectively adapted at times $t_1, \dots, t_n$, there are r.v.'s $Y^1, \dots, Y^n$ on $\Omega(\mathbf{Y})$, respectively adapted at $t_1, \dots, t_n$, such that

$$(X, X^1, \dots, X^n) \equiv_0 (Y, Y^1, \dots, Y^n).$$

Note that the same concept is defined if the X^i's are taken to be adapted sets. $\mathbf{X} \hookrightarrow \mathbf{Y}$ says that any subfiltration **G** of **F(X)** which is finitely generated over $\sigma(X)$ can be embedded in **F(Y)** over the embedding which takes $\sigma(X)$ to $\sigma(Y)$ in the natural way.

THEOREM 2.4. *If* $\mathbf{X} \hookrightarrow \mathbf{Y}$ *and* $\mathbf{Y} \hookrightarrow \mathbf{X}$, *then* $\mathbf{X} \equiv \mathbf{Y}$.

PROOF. Instead of proving the statement given, we will prove a slightly weaker statement. At the end of the proof we will indicate the routine refinements needed to obtain the theorem.

Say that $\mathbf{X} \hookrightarrow^+ \mathbf{Y}$ iff (2.3) holds for any adapted r.v.'s $X^1, \dots, X^n$. We will show that

(2.4.1) if $\mathbf{X} \hookrightarrow^+ \mathbf{Y}$ and $\mathbf{Y} \hookrightarrow^+ \mathbf{X}$, then $\mathbf{X} \equiv \mathbf{Y}$.

Assume, then, that $\mathbf{X} \hookrightarrow^+ \mathbf{Y}$ and $\mathbf{Y} \hookrightarrow^+ \mathbf{X}$, and let $f \in \mathrm{AF}$. We will show that $E[f(\mathbf{X})] = E[f(\mathbf{Y})]$. By definition of $\mathbf{X} \hookrightarrow^+ \mathbf{Y}$, there exist adapted r.v.'s X^g on $\Omega(\mathbf{X})$ and Y^g on $\Omega(\mathbf{Y})$, $g \in \mathrm{sub}(f)$, such that

$$(X, X^g)_{g \in \mathrm{sub}(f)} \equiv_0 (Y, g(\mathbf{Y}))_{g \in \mathrm{sub}(f)},$$

$$(X, g(\mathbf{X}))_{g \in \mathrm{sub}(f)} \equiv_0 (Y, Y^g)_{g \in \mathrm{sub}(f)},$$

and whenever g is of the form $E(h|t)$, X^g is $\mathbf{F}_t$-adapted and Y^g is $\mathbf{G}_t$-adapted.

We will show by induction on formation of $g \in \mathrm{sub}(f)$ that

$$g(\mathbf{X}) = X^g \quad \text{a.s.}$$

and

$$g(\mathbf{Y}) = Y^g \quad \text{a.s.}$$

The induction is trivial for g formed by AF1 or AF2. If g is obtained by clause AF3, $g = E(h|t)$, then by the inductive hypothesis,

$$g(\mathbf{X}) = E[h(\mathbf{X})|\mathbf{F}_t] = E[X^h|\mathbf{F}_t] \quad \text{a.s.},$$

$$g(\mathbf{Y}) = E[h(\mathbf{Y})|\mathbf{G}_t] = E[Y^h|\mathbf{G}_t] \quad \text{a.s.}$$

Applying Lemma 2.2 with $X = h(\mathbf{X})$, $X' = X^g$, etc., we find that $X^g = g(\mathbf{X})$ a.s. and $Y^g = g(\mathbf{Y})$ a.s. This proves (2.4.1). To prove the statement of the theorem, observe that for each $g \in \mathrm{AF}$, $g(\mathbf{X})$ and $g(\mathbf{Y})$ take values in $[0,1]$, $[g(\mathbf{X})]_\delta$ and $[g(\mathbf{Y})]_\delta$ are finite-valued for each $i \leq m$. Choose Y^g and X^g, $g \in \mathrm{sub}(f)$, such that

$$(X, [g(\mathbf{X})]_\delta)_{g \in \mathrm{sub}(f)} \equiv_0 (Y, Y^g)_{g \in \mathrm{sub}(f)}$$

and

$$(Y, [g(\mathbf{Y})]_\delta)_{g \in \text{sub}(f)} \equiv_0 (X, X^g)_{g \in \text{sub}(f)}$$

and X^g and Y^g are adapted at t where $g = E(h|t)$.

It suffices now to transform the proof of (2.4.1) into a proof by induction on $g \in \text{sub}(f)$ that for any $\varepsilon > 0$, if δ is sufficiently small, then

$$\sigma^2(g(\mathbf{X}) - X^g) < \varepsilon$$

and

$$\sigma^2(g(\mathbf{Y}) - Y^g) < \varepsilon.$$

This is easily done by using (b) of Lemma 2.2, and observing that the mappings

$$Z \to E[Z|\mathbf{G}], \quad \mathbf{G} \text{ a } \sigma\text{-algebra},$$

and

$$Z \to \varphi(Z), \qquad \varphi\colon [0,1]^k \to [0,1] \ \ \text{(uniformly) continuous},$$

of square-integrable $[0,1]^k$-valued random variables, are L^2-continuous. □

COROLLARY 2.5. *Let* **X** *and* **Y** *be processes with* $\Omega(\mathbf{X})$ *and* $\Omega(\mathbf{Y})$ *saturated. Then*

(2.5.1) $$\mathbf{X} \equiv \mathbf{Y} \quad \textit{iff} \quad \mathbf{X} \hookrightarrow \mathbf{Y} \textit{ and } \mathbf{Y} \hookrightarrow \mathbf{X}.$$

PROOF. Theorem 2.4 is one direction, and the other is by definition of saturation. □

QUESTION 2.6. Does $\mathbf{X} \hookrightarrow \mathbf{Y}$ imply $\mathbf{X} \hookrightarrow^+ \mathbf{Y}$?

We think that the answer to this question is probably "yes," but it appears that the proof will require a fairly thorough measure theoretic analysis of $\hookrightarrow$. Certainly, if $\Omega(\mathbf{Y})$ is saturated, the answer is "yes."

Corollary 2.5 suggests the following question.

QUESTION 2.7. Is there a class U of adapted functions such that, for $\Omega(\mathbf{Y})$ saturated, $\mathbf{X} \hookrightarrow \mathbf{Y}$ iff

$$E[f(\mathbf{X})] \leq E[f(\mathbf{Y})], \quad \text{for } f \in S?$$

3. Generalization of a theorem of Engelbert and Schmidt. A recent paper of Engelbert and Schmidt (1985) described solutions of a general class of one-dimensional, driftless Brownian stochastic differential equations. In this section we will generalize one of the main results of that paper to one-dimensional SDE's driven by a local martingale. This will be a source of examples for the next section.

We assume that the reader is familiar with the basic facts about time change contained in the first part of Chapter 10 of Jacod (1978).

LEMMA 3.1. *Suppose that*

$$(\mathbf{X},\mathbf{T}) \equiv (\mathbf{X}',\mathbf{T}'),$$

where T and T' are time changes. Then

$$(\mathbf{X}_T) \equiv (\mathbf{X}'_{T'}).$$

PROOF. This follows because, for any $f \in \mathrm{AF}$, there is a Borel function φ such that $f(\mathbf{X}_T) = \varphi(g(\mathbf{X},\mathbf{T})\colon g \in \mathrm{AF})$ and $f(\mathbf{X}'_{T'}) = \varphi(g(\mathbf{X}',\mathbf{T}')\colon g \in \mathrm{AF})$. □

Let AF_0 be a countable subset of AF such that for any process $\mathbf{X}$, the values $E[f(\mathbf{X})]$, $f \in \mathrm{AF}_0$, determine the adapted law of $\mathbf{X}$ [cf. CP_0 in Hoover and Keisler (1984)]. $m(\mathbf{X})$ is the R^∞-valued martingale

$$m(\mathbf{X})_t = \big(E[f(\mathbf{X})|\mathbf{F}(\mathbf{X})_t];\ f \in \mathrm{AF}_0\big),$$

whose (ordinary) law determines the adapted law of $\mathbf{X}$.

LEMMA 3.2. *Suppose that $(\mathbf{F}_t)$ and $(\mathbf{G}_t)$ are filtrations such that $\mathbf{F}_t \subseteq \mathbf{G}_t$ for each $t \in R^+$. If $m(X,(\mathbf{G}_t))$ is $(\mathbf{F}_t)$-adapted, then*

$$(X,(\mathbf{F}_t)) \equiv (X,(\mathbf{G}_t)).$$

PROOF. Since $m(X,(\mathbf{G}_t))$ is $(\mathbf{F}_t)$-adapted, one proves easily by induction that for each $f \in \mathrm{AF}_0$,

$$f(X,(\mathbf{G}_t)) = f(X,(\mathbf{F}_t)) \quad \text{a.s.}$$

Hence $m(X,(\mathbf{G}_t)) = m(X,(\mathbf{F}_t))$ a.s., and the lemma follows. □

THEOREM 3.3. *Suppose that $\mathbf{M}$ is a one-dimensional continuous local martingale on a saturated space and that $1/h^2(x)$ is locally integrable. Then there is a solution X of*

$$X_t = \int_0^t h(X_s)\,dM_s, \tag{3.3.1}$$

such that

$$E\left[\int_0^\infty 1(h(X_s) = 0)\,d\langle M\rangle_s\right] = 0. \tag{3.3.2}$$

The law of (X, M) is unique.

PROOF. If $\mathbf{M}$ is a Brownian motion, the result follows by Engelbert and Schmidt (1985), Theorem (5.4), and Hoover and Keisler (1984), Theorem 7.8.

In the general case, let T be the time change

$$T(t) = \inf\{s\colon \langle M\rangle_s > t\}.$$

Then the time changed process $\mathbf{B} = \mathbf{M}_T$ is essentially a Brownian motion stopped at $\langle M\rangle_\infty$. If $\Omega(\mathbf{B})$ were saturated, we could find an appropriate solution for $\mathbf{B}$, apply the inverse time change to T (which is just $\langle M\rangle$) and we would get a solution for $\mathbf{M}$. Unfortunately, $\Omega(\mathbf{B})$ need not be saturated, so we will have to do something a little more complicated. Let $Z = m(\mathbf{M})$. On some saturated space, find processes B', T', Z' such that

$$(B, T, Z) \equiv (B', T', Z'). \tag{3.3.3}$$

Since $\Omega(\mathbf{B}')$ is saturated, it has a Brownian motion W independent of B', so the Brownian motion

$$\begin{aligned} B_t'' &= B_t', & t \le \langle M \rangle_\infty, \\ &= B'_{\langle M \rangle_\infty} + \left(W_t - W_{\langle M \rangle_\infty}\right), & \text{otherwise}, \end{aligned}$$

is a Brownian motion extending B'. We can apply the Brownian motion case of this theorem to get a solution Y' of (3.3.1) with B' replacing M. Let S' be the inverse time change of T',

$$S'(t) = \inf\{s \colon T'(s) > t\}$$

and let $M' = B'_{T'}$, $X' = Y'_{S'}$. Then by Lemma 3.1,

$$\left(M, Z, \left((\mathbf{F}(\mathbf{M})_T)_{\langle M \rangle}\right)_t\right) \equiv \left(M', Z', \left((\mathbf{F}(\mathbf{B}'))_{S'}\right)_t\right).$$

Now we would like to use saturation of $\mathbf{M}$ to transfer X' back to $\Omega(\mathbf{M})$. Unfortunately, though, unless $\langle M \rangle$ is strictly increasing, $((\mathbf{F}(\mathbf{M})_T)_{\langle M \rangle})_t$ may be larger than $\mathbf{F}(\mathbf{M})_t$. Thus we must resort to one more trick. Since $\langle M \rangle$ is continuous, T is strictly increasing. According to Exercise 10.5(c) in Jacod (1978) $\mathbf{G}_t = (\mathbf{F}(\mathbf{M})_T)_{\langle M \rangle (t)^-} = \mathbf{F}(\mathbf{M})_{T(\langle M \rangle (t)^-)^-}$. Since

$$\begin{aligned} T(\langle M \rangle_{t-}) &= \sup\{s \colon \langle M \rangle_s < \langle M \rangle_t\} \\ &\le t, \end{aligned}$$

$\mathbf{G}_t \subseteq \mathbf{F}(\mathbf{M})_t$. Hence, if $(\mathbf{H}_t)$ is the right-continuous filtration generated by $(\mathbf{G}_t)$ and the natural filtration of $m(\mathbf{M})$, then

$$(M, (\mathbf{F}(\mathbf{M})_t)) \equiv (M, (\mathbf{H}_t)).$$

On the other hand, if $\mathbf{G}_t' = (\mathbf{F}(\mathbf{B}')_{S'(t)^-}$, and $(\mathbf{H}_t')$ is the right-continuous filtration generated by $(\mathbf{G}_t')$ and the natural filtration of Z', then it follows by a proof similar to that of 3.1 that

$$(M, (\mathbf{H}_t)) \equiv (M', (\mathbf{H}_t')).$$

By Jacod (1978), (10.18), X' is a solution of (3.3.1) for $(M', ((\mathbf{F}(\mathbf{B})_{S'})_t)$. Since Y' is continuous, it is $(\mathbf{F}(\mathbf{B}')_{t^-})$-adapted, hence X' is $(\mathbf{G}_t')$-adapted, and so $(\mathbf{H}_t')$-adapted, hence a solution of (3.3.1) for $(M', (\mathbf{H}_t'))$. Since (3.3.2) is preserved by time change and by $\equiv$, X satisfies (3.3.2). If uniqueness in law failed, then a time change argument would show that it also failed for Brownian motion, contradicting Engelbert and Schmidt (1985), (5.4)(ii). □

4. Application to stochastic differential equations. In this section we will give an application of Theorem 2.4, that continuous local martingales which have strictly increasing quadratic variation, and have the same laws of solutions of a given stochastic differential equation must have the same adapted distribution.

In the sequel, **M** and **N** are continuous d-dimensional local martingales, **U** and **W** are other stochastic processes on the same space as **M** and **N**, respectively, and $f\colon R^+ \times R^d \to R$ is a Borel function.

DEFINITION 4.1. (i) $(\mathbf{Y},\mathbf{M}) \hookrightarrow^{\mathrm{SDE}(f)} (\mathbf{W},\mathbf{N})$ if for every n and adapted $R^{d\times n}$-valued processes $\mathbf{X}^1,\dots,\mathbf{X}^n$ satisfying

$$X_t^i = \int_0^t f(s, X_s^i)\, dM_s, \qquad 1 \le i \le n, \tag{4.1.1}$$

there are adapted $\mathbf{Y}^1,\dots,\mathbf{Y}^n$ satisfying

$$Y_t^i = \int_0^t f(s, Y_s^i)\, d\mathbf{N}_s, \qquad 1 \le i \le n, \tag{4.1.2}$$

such that

$$(U, M, X) \equiv_0 (W, N, Y).$$

(ii) $(\mathbf{U},\mathbf{M}) \hookrightarrow^{\mathrm{SDE}} (\mathbf{W},\mathbf{N})$ if every bounded Borel f, $(\mathbf{U},\mathbf{M}) \hookrightarrow^{\mathrm{SDE}(f)} (\mathbf{W},\mathbf{N})$.

(iii) Suppose that $\langle M\rangle$ is strictly increasing. f is **M**-*flexible* if for every $u \in R^+$ and $F \in \mathbf{F}_u$, $P(F) > 0$, there is an adapted process **X** satisfying

$$X_t = \int_0^t f(s, X_s)\, dM_s, \tag{4.1.3}$$

such that $\{\langle X\rangle > 0\} = F \times (u,\infty)$ a.e. on $\Omega(\mathbf{X}) \times R^+$.

(iv) f is flexible for a class S of local martingales (with strictly increasing quadratic variation) if f is **M**-flexible for each $\mathbf{M} \in S$.

THEOREM 4.2. *If* **M** *is a continuous local martingale such that* $\langle M\rangle$ *is strictly increasing, then* $(\mathbf{U},\mathbf{M}) \hookrightarrow^{\mathrm{SDE}} (\mathbf{W},\mathbf{N})$ *and* $(\mathbf{W},\mathbf{N}) \hookrightarrow^{\mathrm{SDE}} (\mathbf{U},\mathbf{M})$ *implies* $(\mathbf{U},\mathbf{M}) \equiv (\mathbf{W},\mathbf{N})$.

Theorem 4.2 will follow immediately from the following theorem, once we show that there exist flexible functions.

THEOREM 4.3. (*i*) *Suppose* f *is* **M**-*flexible and* $(\mathbf{U},\mathbf{M}) \hookrightarrow^{\mathrm{SDE}(f)} (\mathbf{W},\mathbf{N})$. *Then* $(\mathbf{U},\mathbf{M}) \hookrightarrow (\mathbf{W},\mathbf{N})$.

(ii) *If* f *is* **M**-*flexible and* **N**-*flexible and* $(\mathbf{U},\mathbf{M}) \hookrightarrow^{\mathrm{SDE}(f)} (\mathbf{W},\mathbf{N})$ *and* $(\mathbf{W},\mathbf{N}) \hookrightarrow^{\mathrm{SDE}(f)} (\mathbf{U},\mathbf{M})$, *then* $(\mathbf{U},\mathbf{M}) \equiv (\mathbf{W},\mathbf{N})$.

PROOF. (ii) follows from (i) by Theorem 2.4.

(i) Let $F^i \times [u^i,\infty)$, $1 \le i \le n$, be adapted sets, and let $Z^1,\dots,Z^n$ be their indicator functions. Since f is **M**-flexible, there are solutions $\mathbf{X}^1,\dots,\mathbf{X}^n$ of (4.1.3) such that

$$F^i \times (u^i,\infty) = \{\langle X^i\rangle > 0\} \quad \text{a.e. on } \Omega \times R^+.$$

By $(U, M) \hookrightarrow^{\mathrm{SDE}(f)} (W, N)$, choose solutions $Y^1, \ldots, Y^n$ of (4.1.3) such that

$$(U, M, X^1, \ldots, X^n) \equiv_0 (W, N, Y^1, \ldots, Y^n).$$

By Theorem 3 of Chapter IV of Meyer (1976), the quadratic variation $[\mathbf{Z}]$ of a semimartingale Z can be obtained as a measurable function of Z. As X and Y are both continuous local martingales, $[X^i] = \langle X^i \rangle$ and $[Y^i] = \langle Y^i \rangle$ for $1 \le i \le n$, so

$$\langle X^i \rangle \equiv_0 \langle Y^i \rangle, \qquad 1 \le i \le n.$$

Thus, for each i, there is an adapted indicator function W^i which is a.e. equal to $1(\langle Y^i \rangle > 0)$. Then

$$(U, M, Z^1, \ldots, Z^n) \equiv_0 (W, N, W^1, \ldots, W^n).$$

As n is arbitrary, it follows that $(\mathbf{U}, \mathbf{M}) \hookrightarrow (\mathbf{W}, \mathbf{N})$. □

To prove the theorem we need to find, for each dimension d, one function f which is flexible for all d-dimensional local martingales $\mathbf{M}$ which have $\langle M \rangle$ strictly increasing. Of these we have one trivial example. First we give a useful sufficient condition for flexibility.

Proposition 4.4. *Let* $\mathbf{M}$ *be a continuous local martingale with strictly increasing quadratic variation. Suppose that* $f(s,0) = 0$ *and for any local martingale* $\mathbf{M}'$ *on* $\Omega(\mathbf{M})$ *there is a solution* X *of*

$$X_t = \int_0^t f(s, X_s)\, dM_s', \tag{4.4.1}$$

such that for $s < t$, $\langle M' \rangle_s < \langle M' \rangle_t$ *implies* $\langle X \rangle_s < \langle X \rangle_t$. *Then* f *is* $\mathbf{M}$-*flexible*.

Proof. Given an adapted set $F \times [t, \infty)$, let

$$M_s' = (M_s - M_t) 1(F \times [t, \infty)),$$

and let X be the solution of (4.4.1) guaranteed by the hypothesis. Then X is the solution of (4.1.3) required by 4.1(iii). □

Proposition 4.5. *The function* $f(t, x) = 1\,(x \ne 0)$ *is flexible for any continuous local martingale with strictly increasing quadratic variation.*

Proof. By 4.4, it suffices to show that for any continuous local martingale $\mathbf{M}$,

$$M_t = \int_0^t 1(M_s \ne 0)\, dM_s. \tag{4.5.1}$$

First assume that $\mathbf{M}$ is a one dimensional Brownian motion $\mathbf{B}$. Clearly,

$$B_t = \int_0^t 1(B_s \ne 0)\, dB_s + \int_0^t 1(B_s = 0)\, dB_s.$$

But the second term is 0, since its quadratic variation, $\int_0^t 1(B_s = 0)\, ds$, is 0. For general one-dimensional continuous local martingales, (4.5.1) follows by a time change argument. If **M** is d-dimensional, $M = (M^1, \dots, M^d)$,

$$\begin{aligned}\left\langle \int_0^t 1(M_s = 0)\, dM_s \right\rangle &= \int_0^t 1(M_s = 0)\, d\langle M_s \rangle \\ &\le \sum_{1 \le i \le d} \int_0^t 1\big(M_s^i = 0\big)\, d\langle M_s^i \rangle \\ &= \sum_{1 \le i \le d} \left\langle \int_0^t 1\big(M_s^i = 0\big)\, dM_s^i \right\rangle \\ &= 0,\end{aligned}$$

the last equality by the one-dimensional case. Hence (4.5.1) holds for **M**. □

In light of Theorem 4.3, we think it interesting to find more examples of flexibility. Theorem 3.3 gives us a large class of functions f which are flexible for all one-dimensional continuous increasing quadratic variation local martingales on a saturated space. If in the definitions of flexible and $\hookrightarrow^{\text{SDE}}$ we had allowed the underlying adapted spaces to be enlarged (without changing the adapted distributions of the processes one started with) in order to accommodate solutions of the SDE in question, we would not need the hypothesis of saturation. We have not followed such an approach only because the details of enlarging spaces in the appropriate way have not been published.

PROPOSITION 4.6. *Let* **M** *satisfy the hypotheses of* 4.4, *and let* **M** *be one dimensional,* $\langle M \rangle$ *strictly increasing and* $\Omega(\mathbf{M})$ *saturated. Let* $f(t, x) = h(x)$, h *a Borel function. If* $h(0) = 0$ *and* $1/h(x)^2$ *is locally integrable, then* f *is* **M**-*flexible.*

PROOF. This follows by Proposition 4.4 from Theorem 3.3, since (3.3.2) implies that for any $s, t \in R^+$, $\langle M \rangle_s < \langle M \rangle_t$ implies $\langle X \rangle_s < \langle X \rangle_t$ a.s. □

REMARK 4.7. We cannot guarantee **M**-flexibility without the requirement that $\langle M \rangle$ be increasing. For instance, if $M \equiv_0 N$ and M and N each consist of a single jump at a deterministic time t, then for any f, $\mathbf{M} \hookrightarrow^{\text{SDE}(f)} \mathbf{N}$ and vice versa. But Example 3.1 in Hoover and Keisler (1984) shows that such processes need not even be synonymous. It is not clear, though, whether **M** need be continuous or need be a local martingale rather than a general semimartingale.

Hoover and Keisler (1984), Theorem 7.8, implies that for all f, $(\mathbf{U}, \mathbf{M}) \equiv (\mathbf{W}, \mathbf{N})$ implies $(\mathbf{U}, \mathbf{M}) \hookrightarrow^{\text{SDE}(f)} (\mathbf{W}, \mathbf{N})$. Theorem 4.2 is a partial converse to this. It is still open, though, whether in restricted situations, adapted distribution can be replaced by synonymity in the premise of this implication.

QUESTION 4.8. Let **M** and **N** be semimartingales on saturated adapted spaces.

(i) Suppose that the solution of (4.1.3) is unique in distribution. Does $\mathbf{M} \equiv_1 \mathbf{N}$ imply that

$$\mathbf{M} \hookrightarrow^{\mathrm{SDE}(f)} \mathbf{N}?$$

(ii) If $\mathbf{M} \equiv_1 \mathbf{N}$ and solutions of (4.1.3) satisfy uniqueness in distribution for $\mathbf{M}$, must (4.1.3) satisfy uniqueness in distribution for $\mathbf{N}$?

(iii) If $\mathbf{M} \equiv_1 \mathbf{N}$, does that imply that (4.1.3) has a solution for Y for N such that there is some solution X for M such that $(X, M) \equiv_0 (Y, N)$?

It seems to us that the work of Engelbert and Schmidt (1985) on fundamental solutions of SDE's can probably be extended so as to answer these questions affirmatively for many cases.

Acknowledgments. The author would like to thank Ed Perkins and Neil Falkner for answering many questions and Harvey Friedman for making his stay at OSU possible. The author also thanks the University of British Columbia probability seminar for catalyzing a great simplification in the exposition.

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