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ALMOST SURE CONTINUITY OF STABLE MOVING AVERAGE PROCESSES WITH INDEX LESS THAN ONE

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Rootzén (1978) gives a sufficient condition for sample continuity of moving average processes with respect to stable motion with index α less than two. We provide a simple proof of this criterion for $\alpha < 1$ and show that the condition is then also necessary for continuity of the process. The same result holds for the moving-maximum process. A description of the local behaviour of the sample functions of such processes is given.

1. Introduction. Consider a Poisson point process on $\mathbb{R} \times \mathbb{R}_+$ with mean measure $dt\, \alpha x^{-1-\alpha}\, dx$, $\alpha > 0$. Let (T_k, X_k), $k = 1, 2, \dots$ be an enumeration of its points. Let $f: \mathbb{R} \to \mathbb{R}_+$ be Borel measurable such that $\int_{-\infty}^{\infty} f^{\alpha}(s)\, ds$ is finite. We define two stationary processes based on this point process and the function f. The stochastic process Z^f defined by

$$Z_t^f := \sup_k X_k f(T_k + t), \qquad t \in \mathbb{R},$$

is finite a.s. for fixed t [cf. de Haan (1984)]. It is called a moving-maximum process. It is the supremum of a sequence of functions f_k, where f_k is obtained from the given function f by a random shift over T_k to the left, and a random multiplication by the factor X_k in the vertical direction. The ensuing process Z^f is stationary and max-stable: For any time points $t_1, \dots, t_n \in \mathbb{R}$ and any positive constants $c_1, \dots, c_n$ the random variable $U := \max(c_1 Z_{t_1}^f, \dots, c_n Z_{t_n}^f)$ has an extreme value law as distribution function:

$$P\{U \le r\} = \exp(-Mr^{-\alpha}),$$

with

$$M = \int_{-\infty}^{\infty} \max(c_1 f(x + t_1), \dots, c_n f(x + t_n))^{\alpha}\, dx.$$

[The event $\{U \le r\} = \{Z_{t_1}^f \le r/c_1, \dots, Z_{t_n}^f \le r/c_n\}$ has the form "no point (T_k, X_k) of the Poisson point process lies above the graph of the function

$$g(x) = \max\left(\frac{r}{c_1 f(x + t_1)}, \dots, \frac{r}{c_n f(x + t_n)}\right)"$$

and hence has probability $\exp\{-\int_{-\infty}^{\infty}\int_{g(s)}^{\infty} \alpha x^{-1-\alpha}\, dx\, ds\}$.]

For $0 < \alpha < 1$ the stochastic process S^f defined by

$$S_t^f := \sum_k X_k f(T_k + t), \qquad t \in \mathbb{R},$$

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is finite a.s. for fixed t [cf. Schilder (1970)]. This also holds for real-valued f if $|f|^\alpha$ is integrable. It is called a stable moving average process since it can be written as

$$\int_{-\infty}^{\infty} f(u+t)R_\alpha(du),$$

where $\{R_\alpha(t)\}_t = \{\Sigma_k X_k(1_{\{0 \le T_k < t\}} - 1_{\{t \le T_k \le 0\}})\}_t$ is asymmetric stable motion.

We shall prove the following two theorems.

THEOREM 1.1. *Let f: $\mathbb{R} \to \mathbb{R}$ be measurable and let $\alpha \in (0,1)$. The sum process S^f above has a.s. continuous sample paths if and only if f is continuous and*

$$\int_{-\infty}^{\infty} \sup_{0<t<1} |f(t+x)|^\alpha \, dx < \infty. \tag{1.1}$$

Indeed, we shall see that the series $\Sigma_k X_k f(T_k + t)$ which defines the process S^f a.s. converges absolutely and uniformly on bounded time intervals if (1.1) holds (the corollary to Proposition 3.2) and that the sequence $\{X_k f(T_k + t)\}$ is a.s. unbounded in a dense set of (random) time points t if (1.1) fails to hold (Proposition 5.1).

THEOREM 1.2. *Let f: $\mathbb{R} \to [0,\infty)$ be measurable and let α be strictly positive. The moving-maximum process Z^f above has a.s. continuous sample paths if and only if f is continuous and* (1.1) *holds.*

Rootzén (1978) has given sufficient conditions for a.s. continuity of S^f for $0 < \alpha < 2$. The condition for $0 < \alpha < 1$ agrees with ours. Our result disproves the conjecture in Remark 4.4 of Rootzén's paper. Continuity of f and condition (1.1) are necessary for sample path continuity of the processes S^f and Z^f. However, it is not difficult to construct a function f which is neither continuous nor satisfies condition (1.1) such that the processes S^f and Z^f have versions with continuous sample functions. (See Section 6.)

Section 2 treats condition (1.1). Section 3 contains a useful probabilistic interpretation of condition (1.1) in terms of the process $Z^{|f|}$. This allows us to obtain a simple proof of Theorem 1.1. Section 4 contains a proof of Theorem 1.2. Section 5 contains a closer analysis of the sample functions of the processes Z^f and S^f. In the case where f is continuous and (1.1) holds the sample functions of Z^f are local maxima of finitely many scaled translates of f. In the case of the sum process S^f the finite linear combinations of translates are only dense (in the metric of uniform convergence on compact intervals). Section 6 treats the problem: For which measurable functions f does there exist a version of S^f (respectively Z^f) with continuous sample functions?

Note that S^f is an example of a stationary stable process [cf. Hardin (1982)] and Z^f is an example of a stationary max-stable process [cf. de Haan and Pickands (1986)]. For related results on continuity of stable processes see Marcus and Pisier (1984).

2. The condition (1.1) for nonnegative functions f. This section contains some alternative formulations of (1.1). These will not be used in the proofs of Theorems 1.1 and 1.2. We begin with some comments.

(1) For any nonnegative function f, the function $x \mapsto \sup\{f(t+x) | 0 < t < 1\}$ is measurable. It is lower semicontinuous as a supremum of the lower semicontinuous step functions $x \mapsto f(a)1_{(a, a+1)}(x)$, $a \in \mathbb{R}$.
(2) Suppose f is nonnegative. If the integral

$$I(a) := \int_{-\infty}^{\infty} \sup_{0<t<a} f(t+x)\, dx$$

is finite for some $a > 0$, it is finite for all $a > 0$, since it is nondecreasing and satisfies $0 \le I(a+b) \le I(a) + I(b)$ for $a, b > 0$.

The Pickands condition (1.1) implies that f is bounded. It will be satisfied if $f \ge 0$ is unimodal and f^{α} is integrable, or if $|f|$ is bounded above by such a function.

Condition (1.1) can be rephrased in a number of ways. Note that the second condition in the next proposition is equivalent to Rooztén's (1978) condition (4.5) except for the continuity.

PROPOSITION 2.1. *Suppose f is measurable and $\phi = |f|^{\alpha}$, $\alpha > 0$. The following are equivalent to the Pickands condition* (1.1):

(1) $\int_{-\infty}^{\infty} \sup_{0<t<1} \phi(t+x)\, dx < \infty$.
(2) $\sum_{k=-\infty}^{\infty} \alpha_k < \infty$ *where* $\alpha_k := \sup_{k \le x < k+1} \phi(x)$.
(3) *The function ϕ is bounded above by an integrable function ψ of bounded variation.*
(4) *The function ϕ is bounded above by a bounded strictly positive function g which satisfies $\int_{-\infty}^{\infty} g(x)\, dx < \infty$ and $g(x_n + h_n)/g(x_n) \to 1$ if $|x_n| \to \infty$, $h_n \to 0$ [cf. Widder* (1971), *page* 204].

PROOF. We may and shall assume $\alpha = 1$. Write $\eta(x) := \sup_{0<t<1} \phi(t+x)$, and $\psi_1 := \sum_k \alpha_k 1_{[k, k+1)}$.

(1) $\Rightarrow$ (2) since $\psi_1(x) \le \max\{\eta(x-1), \eta(x-\frac{1}{2}), \eta(x)\}$ for all x.

(2) $\Rightarrow$ (3) Take $\psi = \psi_1$.

(3) $\Rightarrow$ (2) Set $\beta_k := \sup_{k \le x < k+1} \psi(x)$ and $\gamma_k := \inf_{k \le x < k+1} \psi(x)$. Then $\sum \gamma_k < \infty$ since ψ is integrable and $\sum \beta_k - \gamma_k < \infty$ since ψ is of bounded variation. Hence $\sum \beta_k$ is finite. This implies $\sum \alpha_k < \infty$.

(2) $\Rightarrow$ (4) We assume that ϕ is not identically zero. For each integer k define

$$\begin{aligned} g_k(x) &:= \alpha_k e^{x-k}, & x < k, \\ &:= \alpha_k, & k \le x < k+1, \\ &:= \alpha_k e^{k+1-x}, & x \ge k+1, \end{aligned}$$

and set $\tilde{g} := \sup_k g_k$. Then $\tilde{g}$ is bounded and integrable. It is strictly positive, and $\alpha_k \to 0$ implies that for each $x \in \mathbb{R}$ there exists an index k such that

$\tilde{g}(x) = g_k(x)$. It follows that

$$e^{-|y-x|} \le \tilde{g}(y)/\tilde{g}(x) \le e^{|y-x|}.$$

[Assume $\tilde{g}(y) \le \tilde{g}(x) = g_k(x)$. Then $\tilde{g}(y) \ge g_k(y) \ge e^{-|y-x|}g_k(x)$.]

(4) ⇒ (1) We claim that (4) implies that $g(x_n + b_n)/g(x_n)$ is bounded for any pair of sequences $x_n \to \infty$ and $b_n \in (0,1)$. [Otherwise there would exist such sequences for which $g(x_n + b_n)/g(x_n) > e^n$ for all n. Now write this quotient as $\prod_{i=1}^n g(x_n + ib_n/n)/g(x_n + (i-1)b_n/n)$, and one has $g(y_n)/g(y_n - b_n/n) > e$ for all n on setting $y_n = x_n + i_n b_n/n$ for some appropriate $i_n \in \{1, \dots, n\}$. This contradicts (4).] It follows that $k(x) := \sup_{0<t<1} g(x+t) = O(g(x))$ for $|x| \to \infty$. Hence k is integrable. This gives (1). □

3. Proof of Theorem 1.1 (continuity of S^f). Our starting point is the following proposition from de Haan and Pickands (1986) on the moving-maximum process Z^f (rephrased for our purpose).

PROPOSITION 3.1. *If* (1.1) *holds and f is nonnegative, the process Z^f is a.s. bounded on every finite interval. If* (1.1) *does not hold, the process Z^f is a.s. unbounded in every finite interval.*

For continuous f we can use Baire's theorem to obtain the following improvement on the second statement above.

PROPOSITION 3.2. *If the function $f \ge 0$ is continuous and* (1.1) *does not hold, then for almost every sample function of the process Z^f the set $\{Z^f = \infty\}$ is a dense G_δ in $\mathbb{R}$.*

PROOF. The random set $W_n := \{t \in \mathbb{R} | Z_t^f > n\}$ is open since Z^f is the sup of the continuous processes $t \mapsto X_k f(T_k + t)$ by definition. By Proposition 3.1 the set W_n is a.s. dense in $\mathbb{R}$. By Baire's theorem the intersection is a dense G_δ. □

COROLLARY. *If $f: \mathbb{R} \to \mathbb{R}$ is a continuous function which does not satisfy* (1.1), *then for a.e. realization there exists a dense G_δ of time points $t \in \mathbb{R}$ such that the sequence $X_k f(T_k + t)$ is unbounded* [*and hence the series $\sum X_k f(T_k + t)$ cannot converge no matter what order of summation one chooses*].

PROOF OF THEOREM 1.1. First assume that almost all sample functions of the process S^f are continuous. We choose an arbitrary Borel set $A \subset \mathbb{R} \times \mathbb{R}_+$ with finite mean measure

$$0 < \mu A = \int_A dt\, \alpha x^{1-\alpha}\, dx < \infty,$$

and set $\tilde{S}_t^f := \sum\{X_k f(T_k + t) | (T_k, X_k) \in A\}$. Then $\tilde{S}_t^f \equiv 0$ with probability $e^{-\mu A} > 0$. Hence the processes $\tilde{S}^f$ and $S^f - \tilde{S}^f$ are independent by virtue of the underlying Poisson point process, the process $S^f - \tilde{S}^f$ is a.s. continuous, and hence so is $\tilde{S}^f = S^f - (S^f - \tilde{S}^f)$. With probability $\mu A e^{-\mu A} > 0$ the process $\tilde{S}^f$ is the sum of one term,

$$\tilde{S}_t^f = X_0 f(T_0 + t), \qquad t \in \mathbb{R}.$$

This implies that f is continuous. By the corollary above condition (1.1) holds.
Now assume condition (1.1) is satisfied. The function

$$g^\alpha(s) := \sup_{0<t<1} |f(s+t)|^\alpha$$

is integrable and hence

$$\sum_k X_k \sup_{0<t<1} |f(T_k + t)| = S_0^g < \infty \quad \text{a.s.,}$$

by Schilder (1970). For sample functions which satisfy this inequality Lebesgue's theorem on dominated convergence gives

$$\lim_{t \to t_0} \sum_k X_k f(T_k + t) = \sum X_k f(T_k + t_0),$$

for each point $t_0 \in (0,1)$. This proves that S^f is a.s. continuous on the interval $(0,1)$. By stationarity the process is a.s. continuous on $\mathbb{R}$. □

4. Proof of Theorem 1.2. For the proof we need the following.

LEMMA 4.1. *Let M be a Poisson point process on a space E with mean measure μ. Let g_t: $E \to [0, \infty]$ be a family of measurable functions ($t \in T$). Set $Y_t := \sup_{x \in M} g_t(x)$. (We think of M as a random subset of E with generic point x.) Suppose there exist measurable functions g_* and g^* with $g_* \le g_t \le g^*$ for all $t \in T$, and such that*

(a) $\mu\{x | g_*(x) > 0\} = \infty$.
(b) $\mu\{x | g^*(x) > 1/n\} < \infty$ *for* $n = 1, 2, \dots$.

Then with probability one there exists a finite collection $X_1, \dots, X_K \in M$ such that

$$Y_t = \max(g_t(X_1), \dots, g_t(X_K)),$$

for all $t \in T$.

PROOF. After deleting a null set we may assume that for every realization the sets $\{g^* \ge 1/n\}$, $n = 1, 2, \dots$, contain only finitely many points of the random set M, and that the set $\{g_* > 0\}$ contains at least one point, say $X^{(1)}$. Then

$$Y_t \ge W := g_*(X^{(1)}) > 0$$

and hence

$$Y_t = \max\{g_t(X_1), \dots, g_t(X_K)\},$$

where

$$\{X_1, \dots, X_K\} = M \cap \{g^* \ge W\}.$$

□

PROOF OF THEOREM 1.2. First assume f is continuous and (1.1) holds. The theorem is obvious if $f \equiv 0$. Hence we may assume that f is strictly positive on some interval I. We shall apply the lemma above with M the Poisson point process on $E = \mathbb{R} \times \mathbb{R}_+$ with mean measure $d\mu = dt\,\alpha x^{-1-\alpha}\,dx$, and with the functions $g_t(s, x) := xf(s + t)$, $0 < t < h$. If we choose h sufficiently small these are bounded below by a function $g_*(s, x) := \eta x 1_I(s)$ for some $\eta > 0$. By condition (1.1) the functions g_t, $0 < t < h$, are bounded above by $g^*(s, x) := xg(s)$, where $g(s) := \sup_{0<t<h}|f(s + t)|$ satisfies $\int g^\alpha(s)\,ds < \infty$. [Choose $h \leq 1$ or use comment (2) in Section 2.]

Now $\mu\{g_* > 0\} = \int_{I\times(0,\infty)} dt\,\alpha x^{-1-\alpha} = \infty$ and $\mu\{g^* > \varepsilon\} = \mu\{x > \varepsilon/g\} = \int(g(s)/\varepsilon)^\alpha\,ds < \infty$. By Lemma 4.1 almost every realization of the process Z_t^f, $0 < t < h$, is the maximum of a finite number of continuous functions $X_k f(T_k + t)$, $k = 1, \ldots, K$. Hence Z^f is a.s. continuous on the interval $(0, h)$. By stationarity the process Z^f is a.s. continuous on $\mathbb{R}$. Conversely, let Z^f have continuous sample functions. Then (1.1) holds by Proposition 3.1. It remains to prove that f is continuous.

Let $c > 0$ be arbitrary. Write $Z^f = \max(Z', Z'')$, where Z' is the sup over all points (T, x) in the vertical strip $(-c, c) \times [1, \infty)$ and Z'' is the sup over the remaining points. The processes Z' and Z'' are independent, and so are the events

$$E' = \text{"there is exactly one point, say } (T_0, X_0), \text{ in } (-c, c) \times [1, \infty)\text{,"}$$

$$E'' = \{Z_t'' \leq 1 \text{ on } (-c, c)\}.$$

Note that $PE'' > 0$ since $\sup_{|t|<c} Z_t^f = Z_0^g$, where $g^\alpha(s) = \sup_{|t|<c} f^\alpha(s + t)$ is integrable by (1.1).

Conditional on $E' \cap E''$ we have

$$\max(1, Z_t) = \max(1, X_0 f(T_0 + t)), \qquad |t| < c.$$

Since (T_0, X_0) has a positive density on $(-c, c) \times [1, \infty)$, it follows that f is continuous on the interval $(-2c, 2c)$. □

5. Sample path behaviour of the processes S^f and Z^f. Theorems 1.1 and 1.2 would seem to suggest that for nonnegative functions f which satisfy condition (1.1) the processes Z^f and S^f behave very much the same. There is, however, an essential difference between maxima and sums. If f is piecewise linear, say $f(t) = (1 - |t|)_+$, then Z^f is piecewise linear (on any bounded interval), but almost every sample function of S^f is nonlinear on every interval. If f is a step function, say $f = 1_{[0,1]}$, then Z^f is an (usc) step function locally, but again the sample functions of the process S^f are a.s. nonconstant on every interval.

PROPOSITION 5.1. *Suppose f: $\mathbb{R} \to \mathbb{R}$ is measurable and satisfies condition (1.1) with $0 < \alpha < 1$. Then the series $\Sigma X_k f(T_k + t)$ a.s. converges uniformly on bounded intervals. If f is not a null function, then for each t the series a.s. contains infinitely many nonzero terms.*

PROOF. By Proposition 2.1 we can choose a continuous $g \geq |f|$ which satisfies (1.1). The series $\Sigma X_k g(T_k + t)$ converges uniformly on $[-c, c]$ by Dini's theorem. This then also holds for the series $\Sigma X_k f(T_k + t)$.

If f is not a null function, the set $A = \{f \neq 0\}$ has positive Lebesgue measure and the strip $A \times (0, \infty)$ a.s. contains infinitely many points (T_k, X_k). □

Let f be continuous and satisfy (1.1). On any bounded interval $[-c, c] \subset \mathbb{R}$ the sample functions of S^f are uniform limits of positive linear combinations of translates of f. For the sample functions of the process Z^f one has a stronger result.

For any f: $\mathbb{R} \to [0, \infty)$ define Γ^f to be the smallest set of functions g: $\mathbb{R} \to [0, \infty)$ such that

(1) $f \in \Gamma^f$,
(2) $g, h \in \Gamma^f \Rightarrow \max(g, h) \in \Gamma^f$,
(3) $g \in \Gamma^f,\ c > 0 \Rightarrow cg \in \Gamma^f$,
(4) $g \in \Gamma^f,\ t \in \mathbb{R} \Rightarrow \tau_t g \in \Gamma^f$, where $(\tau_t g)(s) = g(s + t)$.

It is easily seen that Γ^f is the class of all functions of the form

$$g = \max(c_1 \tau_{t_1} f, \dots, c_m \tau_{t_m} f),$$

with $m \geq 1$, $t_i \in \mathbb{R}$ and $c_i > 0$ for $i = 1, \dots, m$.

If f is continuous, satisfies (1.1) for some $\alpha > 0$, and does not vanish identically, then almost all sample functions of the process Z^f have the following properties: The restriction to any bounded interval is bounded away from zero and agrees with some element of Γ^f on this interval. This is an immediate consequence of the following result.

PROPOSITION 5.2. *Suppose* f: $\mathbb{R} \to [0, \infty)$ *is measurable, satisfies* (1.1) *for some* $\alpha > 0$, *and is bounded away from zero on some interval. Then for any* $c > 0$ *there exists an a.s. strictly positive random variable* V *and an a.s. finite index* N *such that*

$$Z_t^f = \max(X_1 f(T_1 + t), \dots, X_N f(T_N + t)) \geq V, \qquad -c \leq t \leq c.$$

PROOF. In the proof of Theorem 1.2 we saw that Z^f restricted to a sufficiently small time interval $[0, h]$ can be expressed as the maximum of finitely many processes of the form $X_k f(T_k + t)$. In order to have a similar expression over a longer time interval, we need an extension of Lemma 4.1 in which the assumption (b) is replaced by

(b′) There exists a finite set of functions $g_*^{(1)}, \dots, g_*^{(m)}$ such that each function g_t is bounded below by one of these functions and such that $\mu[g_*^{(j)} > 0\} = \infty$ for $j = 1, \dots, m$.

In this case we can a.s. choose points $X^{(j)}$ such that $g_*^{(j)}(X^{(j)}) > 0$. Now set $W := \min_j g_*^{(j)}(X^{(j)}) > 0$. Then for each t,

$$\sup\{g_t(x) | x \in M\} = \sup\{g_t(x) | x \in M,\ g^*(x) \geq W\},$$

and hence there is an a.s. finite set $\{X_1, \dots, X_k\} \subset M$ such that for all $t \in T$,

$$\sup\{g_t(x)|x \in M\} = \max(g_t(X_1), \dots, g_t(X_k)).$$

The remainder of the proof is similar to the proof of Theorem 1.2. □

EXAMPLE. Let $f = 1_{\{0\}}$. Then almost every realization of Z^f is strictly positive on a dense subset of $\mathbb{R}$. Elements of Γ^f have finite support.

6. Continuous versions of moving average and moving-maximum processes. The function $f = 1_{\mathbb{Z}}$ does not satisfy condition (1.1) for any α, and is not continuous. The corresponding process S^f is a.s. unbounded on all intervals (a, b), $a < b$. For fixed $t \in \mathbb{R}$ the random variable S_t^f vanishes with probability 1. Hence there exists a version of the process S^f which has continuous sample functions. (This is the zero process.)

Now suppose f is a Borel function, $\alpha \in (0,1)$ and the process S^f has a version S with continuous sample functions. (This means that for each $t \in \mathbb{R}$ we have $P\{S_t \neq S_t^f\} = 0$.) Does this imply that there exists a continuous function ϕ such that the processes S and S^ϕ are indistinguishable [i.e., $S(\omega) \equiv S^\phi(\omega)$ outside a null set in Ω]? This is still an open problem.

We can prove that the existence of a continuous version entails that f is Lebesgue a.e. equal to some continuous function ψ (at least if f is locally integrable). Now if ψ were to satisfy condition (1.1), then S^ψ would be continuous, and since S^ψ is a version of S^f, it would be indistinguishable from S. We are only able to prove that ψ satisfies condition (1.1) for nonnegative locally integrable functions f.

THEOREM 6.1. *Let f: $\mathbb{R} \to \mathbb{R}$ be locally integrable and* $\alpha \in (0,1)$. *Suppose there exists a continuous process S such that for each $t \in \mathbb{R}$,*

$$S_t = \sum_k X_k f(T_k + t) \quad a.s. \tag{6.1}$$

Then there exists a continuous function ϕ such that $f = \phi$ a.e. dx. If the function f is nonnegative, then ϕ satisfies condition (1.1) *and S^ϕ and S are indistinguishable.*

PROOF. The first part proceeds as in the proof of Theorem 1.1. except that it is convenient to work with the continuous processes $J_t = \int_0^t S_s^f \, ds$ and $\tilde{J}_t = \int_0^t \tilde{S}_s^f \, ds$, $t \in \mathbb{R}$, in order to avoid measurability problems. [The set $\{(t, \omega)|S_t(\omega) \neq S_t^f(\omega)\}$ is a $\lambda \times P$-null subset of $\mathbb{R} \times \Omega$, where λ denotes Lebesgue measure, hence for P a.e. ω the realizations $S(\omega)$ and $S^f(\omega)$ agree λ a.e. on $\mathbb{R}$. It follows that the integral processes $\int_0^t S_s^f \, ds$ and $\int_0^t S_s \, ds$ are indistinguishable.] Let C' be the subset of all continuously differentiable functions in the space of all continuous functions. Then C' is a measurable subset (with respect to the usual Borel σ-algebra on C) and one uses the independence of $\tilde{J}$ and $J - \tilde{J}$ to conclude that $\tilde{J} \in C'$ a.s. If the set A contains only one point

(T_0, X_0), then $\tilde{J}_t = X_0 \int_0^t f(T_0 + s)\, ds$ which proves that f a.s. agrees with a continuous function ϕ (since $\tilde{J}$ is a.s. C^1).

For the second part observe that if ϕ did not satisfy (1.1), then $\{Z^\phi > n\}$ and a fortiori $\{S^\phi > n\}$ would contain an open dense subset of $\mathbb{R}$ for each n. Since almost all sample functions of S^ϕ and S agree a.e. on $\mathbb{R}$, it would follow that S is a.s. unbounded on all nonempty open intervals. This would contradict the continuity of S. □

For the moving-maximum process Z^f we have a more complete result.

THEOREM 6.2. *Let $f: \mathbb{R} \to [0, \infty)$ be measurable and let α be strictly positive. Suppose there exists a right-continuous process Z such that for each $t \in \mathbb{R}$,*

$$Z_t = \sup_k X_k f(T_k + t) \quad a.s. \tag{6.2}$$

Then there exists a right-continuous function ϕ which satisfies (1.1) *such that*

$$f = \phi \quad a.e.\ dx,$$

$$Z^\phi \text{ and } Z \text{ are indistinguishable.}$$

PROOF. Suppose h is measurable and $h = f$ a.e. dx. Then we may replace f by h in (6.2). In particular, if h is right-continuous and satisfies condition (1.1), then Z^h and Z will be indistinguishable.

We first replace f by a function h such that $h = f$ a.e. dx and such that for $a < b$,

$$\operatorname*{ess\,sup}_{a<t<b} f(t) = \sup_{a<t<b} h(t).$$

(This function h will not be right-continuous in general.)

It can be constructed as follows. Let N be a Borel null set in $\mathbb{R}$ such that x is a Lebesgue point of f for $x \notin N$, and define $h = f 1_{N^c}$. Now let $\varepsilon > 0$ be given and define

$$g(x) = \sup_{0<t<\varepsilon} h(x + t) = \operatorname*{ess\,sup}_{0<t<\varepsilon} f(x + t).$$

On any measure space one has the equality $\sup\|\psi_n\|_\infty = \|\sup \psi_n\|_\infty$. If we apply this to the interval $(0, \varepsilon)$ with Lebesgue measure we find that a.s.

$$\sup_{0<t<\varepsilon} Z_t = \operatorname*{ess\,sup}_{0<t<\varepsilon} Z_t^h = \sup_k X_k g(T_k) =: Z_0^g.$$

If ε is small, the left-hand side will be finite with positive probability. (The process Z is right-continuous, and hence $\sup_{0<t<\varepsilon} Z_t \leq Z_0 + 1 < \infty$ with positive probability.) It follows that h is locally bounded. (Otherwise g would be infinite on an interval of length ε and $Z_0^g = \infty$ a.s.) Hence g is locally bounded. The variables

$$U_j = \sup\{X_k g(T_k) | j \leq T_k < j + 1\}, \qquad j \in \mathbb{Z},$$

are a.s. finite and independent, and $Z_0^g = \sup U_j$. By Kolomogorov's 0–1 law the tail event $\{Z_0^g = \infty\}$ has probability 0. Then g^α is integrable [cf. de Haan (1984), page 1199]. Thus h satisfies (1.1) by comment (2) in Section 2.

Now proceed as in the proof of Theorem 1.2 in Section 4. Write $Z^h = \max(Z', Z'')$ and observe that on $E' \cap E''$ a.s.

$$\max(1, Z_t) = \max(1, X_0 h(T_0 + t)) \quad \text{a.e. on } (-c, c).$$

Since (T_0, X_0) has a positive density on $(-c, c) \times [1, \infty)$, there exists a right-continuous function ϕ on $(-2c, 2c)$ which agrees with h a.e. on $(-2c, 2c)$. (Since there is at most one such right-continuous function, it does not depend on ω.)

Now let $c \to \infty$ and observe that ϕ satisfies condition (1.1) since

$$\sup_{a<t<b} \phi(t) = \operatorname*{ess\,sup}_{a<t<b} \phi(t) = \operatorname*{ess\,sup}_{a<t<b} h(t) = \sup_{a<t<b} h(t). \qquad \square$$

We close with an a.e. extension of Proposition 3.1 (the de Haan–Pickands dichotomy).

Proposition 6.3. *Let $f\colon \mathbb{R} \to [0, \infty)$ be measurable and $\alpha > 0$. Define*

$$g(x) := \operatorname*{ess\,sup}_{0<t<1} f(x + t).$$

Then g is measurable. Either $\int_{-\infty}^{\infty} g^\alpha(x)\,dx$ is finite, in which case f is essentially bounded, $\operatorname{ess\,sup}_{a<t<b} X_k f(T_k + t) \to 0$, $k \to \infty$, *and hence a.s.*

$$\operatorname*{ess\,sup}_{a<t<b} Z_t^f < \infty, \quad \textit{for all intervals } (a, b) \subset \mathbb{R},$$

or $\int_{-\infty}^{\infty} g^\alpha(x)\,dx = \infty$ and a.s.

$$\operatorname*{ess\,sup}_{a<t<b} Z_t^f = \infty, \quad \textit{for all intervals } (a, b), \qquad a < b.$$

Proof. Construct h as in the proof of Theorem 6.2. Then $g(x) = \sup_{0<t<1} h(x + t)$ and sample functions of Z^f and Z^h agree a.e. on $\mathbb{R}$. Hence

$$\operatorname*{ess\,sup}_{a<t<b} Z_t^f = \sup X_k \operatorname*{ess\,sup}_{a<t<b} f(T_k + t)$$
$$= \sup X_k \sup_{a<t<b} h(T_k + t) = \sup_{a<t<b} Z_t^h.$$

Now apply Proposition 3.1.

It remains to prove ess local uniform convergence. For $0 < \alpha < 1$ this follows from Theorem 5.1 which states that the series $\sum X_k h(T_k + t)$ a.s. converges uniformly on bounded intervals. The case $\alpha \geq 1$ then follows by a simple transformation. □

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