The Annals of Probability
1988, Vol. 16, No. 1, 300–304

THE CENTRAL LIMIT THEOREM AND POINCARÉ-TYPE INEQUALITIES[1]

By Louis H. Y. Chen

National University of Singapore and Mathematics Research Center, Madison, Wisconsin

We use Poincaré-type inequalities to prove the sufficiency and necessity of the Lindeberg condition in the central limit theorem.

Borovkov and Utev (1983) defined the functional

$$U(X) = \sup_g \frac{\mathrm{Var}[g(X)]}{\sigma^2 E\left[g'(X)^2\right]},$$

for any random variable X with finite variance σ^2, where the supremum is taken over the class of absolutely continuous functions g such that $0 < \mathrm{Var}[g(X)] < \infty$. They proved that

(1) $U(X) \geq 1$ and if $U(X) = 1$, then X has a normal distribution.

Using this result they further proved that

(2) if $X_1, X_2, \ldots$ is a sequence of random variables such that $U(X_n) \to 1$, then the moment generating function of $(X_n - EX_n)/[\mathrm{Var}(X_n)]^{1/2}$ exists and converges to that of the standard normal random variable in a neighborhood of zero.

It is natural to ask if (1) can also be applied to prove the central limit theorem under the Lindeberg condition. This question was in fact raised by Kotani (1985) and motivated the present work.

The existence of the moment generating function of X_n in (2) is due to the finiteness of $U(X_n)$ (see Borovkov and Utev, Theorem 2). Since the central limit theorem does not require such a strong condition, arguments different from those of Borovkov and Utev would have to be used. It turns out that a Poincaré-type inequality for sums of independent random variables proved along the same line as in Chen (1985), Section 2, is the key to the solution of this problem. Using this inequality we can prove not only the sufficiency of the Lindeberg condition but also its necessity.

In order to facilitate application, we restate (1) in a different form. Let $C_B^1(\mathbb{R})$ be the class of functions g such that g and g' are bounded and continuous.

Proposition 1. *Let X be a random variable with finite variance $\sigma^2 > 0$. If $\mathrm{Var}[g(X)] \leq \sigma^2 E[g'(X)^2]$ for $g \in C_B^1(\mathbb{R})$, then X has a normal distribution.*

Received April 1986.

[1]Research partially sponsored by the United States Army under Contract No. DAAG29-80-C-0041. Part of this work was done when the author was visiting the Institute for Mathematics and Its Applications at the University of Minnesota.

AMS 1980 *subject classifications*. Primary 60F05, 60E15; secondary 26D10.

Key words and phrases. Central limit theorem, Poincaré-type inequalities, characterization of the normal distribution.

To see that (1) and Proposition 1 are equivalent, we define

$$U_0(X) = \sup_{\substack{g \in C_0^\infty(\mathbb{R}) \\ \mathrm{Var}[g(X)] > 0}} \frac{\mathrm{Var}[g(X)]}{\sigma^2 E\left[g'(X)^2\right]}$$

and

$$U_B(X) = \sup_{\substack{g \in C_B^1(\mathbb{R}) \\ \mathrm{Var}[g(X)] > 0}} \frac{\mathrm{Var}[g(X)]}{\sigma^2 E\left[g'(X)^2\right]},$$

where $C_0^\infty(\mathbb{R})$ is the class of C^∞ functions on $\mathbb{R}$ with compact support. Then we observe that by definition, $U(X) \geq 1$ and that by Theorem 2(i) of Borovkov and Utev, $U_0(X) = U_B(X) = U(X)$.

Since we are interested in both an application of Poincaré-type inequalities and a new proof of the central limit theorem, it is fitting to examine the arguments which lead to (1) and hence Proposition 1. There are three different proofs of (1). All begin with a variational argument. After that the first uses the method of moments (see Borovkov and Utev, Theorem 3). The second uses the characteristic function [see Chen and Lou (1987), Theorem 2.1 and Corollary 2.1]. The third uses differential equations (see Chen and Lou, Lemmas 4.1 and 4.2).

We now prove the sufficiency of the Lindeberg condition. Let $X_{n1}, \dots, X_{nr_n}$, $n \geq 1$, be a triangular array of row-wise independent random variables with zero means and finite variances $\sigma_{n1}^2, \dots, \sigma_{nr_n}^2$ such that $\sum_{i=1}^{r_n} \sigma_{ni}^2 = 1$. Let $W_n = \sum_{i=1}^{r_n} X_{ni}$ and $W_n^{(i)} = W_n - X_{ni}$. By Theorem 2.1 of Chen (1985),

$$\mathrm{Var}[g(W_n)] \leq \sum_{i=1}^{r_n} E\left[\mathrm{Var}^{W_n^{(i)}} g(W_n)\right],$$

for $g \in C^1(\mathbb{R})$ such that $|g(x)| \leq C(1 + |x|)$ for some constant C. Now the right-hand side of the inequality satisfies

$$\begin{aligned}
&\sum_{i=1}^{r_n} E E^{W_n^{(i)}}\left\{\left[g(W_n) - g(W_n^{(i)}) - E^{W_n^{(i)}}\left(g(W_n) - g(W_n^{(i)})\right)\right]^2\right\} \\
&\quad \leq \sum_{i=1}^{r_n} E\left\{\left[g(W_n) - g(W_n^{(i)})\right]^2\right\} \\
&\quad = \sum_{i=1}^{r_n} E\left\{\left[\int_0^{X_{ni}} g'(W_n^{(i)} + t)\, dt\right]^2\right\} \\
&\quad \leq \sum_{i=1}^{r_n} E\left\{X_{ni} \int_0^{X_{ni}} g'(W_n^{(i)} + t)^2\, dt\right\} \\
&\quad = \sum_{i=1}^{r_n} E\left\{X_{ni} \int_{-\infty}^{\infty} \left[I(X_{ni} > t > 0) - I(X_{ni} < t < 0)\right]\left[g'(W_n^{(i)} + t)^2\right] dt\right\} \\
&\quad = \sum_{i=1}^{r_n} \int_{-\infty}^{\infty} E\left[g'(W_n^{(i)} + t)^2\right] K_n^{(i)}(t)\, dt,
\end{aligned}$$

where $K_n^{(i)}(t) = EX_{ni}[I(X_{ni} > t > 0) - I(X_{ni} < t < 0)] \geq 0$. Define the probability measure ν_n on $\mathscr{B}(\mathbb{R}^2)$ by

$$\int_{\mathbb{R}^2} f\, d\nu_n = \sum_{i=1}^{r_n} \int_{-\infty}^{\infty} Ef\left(W_n^{(i)}, t\right) K_n^{(i)}(t)\, dt,$$

for bounded and continuous functions f on $\mathbb{R}^2$. Also define ψ: $\mathbb{R}^2 \to \mathbb{R}$ by $\psi(x, y) = x + y$. Then, combining the above inequalities, we have the following Poincaré-type inequality for W_n: For $g \in C^1(\mathbb{R})$ such that $|g(x)| \leq C(1 + |x|)$ for some constant C,

$$\text{(3)} \qquad \operatorname{Var}[g(W_n)] \leq \int_{\mathbb{R}} (g')^2\, d\nu_n \circ \psi^{-1}.$$

Let $\nu_n^{(1)}(A) = \nu_n(A \times \mathbb{R})$ and $\nu_n^{(2)}(A) = \nu_n(\mathbb{R} \times A)$ for $A \in \mathscr{B}(\mathbb{R})$. Since

$$\begin{aligned} \nu_n^{(2)}(|y| > \varepsilon) &= \sum_{i=1}^{r_n} \int_{|t|>\varepsilon} K_n^{(i)}(t)\, dt \\ &= \sum_{i=1}^{r_n} E|X_{ni}|(|X_{ni}| - \varepsilon)^+ \\ &\leq \sum_{i=1}^{r_n} EX_{ni}^2 I(|X_{ni}| > \varepsilon), \end{aligned}$$

the Lindeberg condition implies that $\nu_n^{(1)} \Rightarrow \varepsilon_0$, the Dirac measure at 0, as $n \to \infty$. Now $EW_n^{(i)2} \leq EW_n^2 = 1$ for each n and i implies that $\{\nu_n^{(1)}\}$ is tight and hence relatively compact. Let $\{\nu_{n'}^{(1)}\}$ be a weakly convergent subsequence and let $\nu_{n'}^{(1)} \Rightarrow \mathscr{L}(Z)$. Then $\nu_{n'} \Rightarrow \mathscr{L}\binom{Z}{0}$ and hence $\nu_{n'} \circ \psi^{-1} \Rightarrow \mathscr{L}(Z)$. It follows from (3) that for $g \in C_B^1(\mathbb{R})$,

$$\text{(4)} \qquad \operatorname{Var}[g(Z)] \leq E\left[g'(Z)^2\right].$$

If we show that Z has a standard normal distribution, then $\mathscr{L}(W_n) \Rightarrow N(0,1)$ is proved. By Proposition 1 it remains to prove that $\operatorname{Var}(Z) = 1$. First, we observe that by virtue of $EW_n^2 = 1$, Z is square integrable. Let $\varphi_a \in C_B^1(\mathbb{R})$ be increasing such that $|\varphi_a'| \leqq 1$ and

$$\varphi_a(x) = \begin{cases} x, & \text{if } |x| \leq a, \\ a + 1, & \text{if } x \geq a + 2, \\ -a - 1, & \text{if } x \leq -a - 2. \end{cases}$$

By (3),

$$\operatorname{Var}[W_{n'} - \varphi_a(W_{n'})] \leq \int_{\mathbb{R}} (1 - \varphi_a')^2\, d\nu_{n'} \circ \psi^{-1}.$$

Since $\nu_{n'} \circ \psi^{-1}$ converges weakly, $\{\nu_{n'} \circ \psi^{-1}\}$ is tight. Therefore for $\varepsilon > 0$,

$$\operatorname{Var}[W_{n'} - \varphi_a(W_{n'})] \leq \int_{\mathbb{R}} (1 - \varphi_a')^2\, d\nu_{n'} \circ \psi^{-1} \leq \varepsilon,$$

for sufficiently large a. Now $(\mathrm{Var})^{1/2}$ is a seminorm and so

$$\begin{aligned}\left|1 - [\mathrm{Var}(\varphi_a(W_{n'}))]^{1/2}\right| &= \left|[\mathrm{Var}(W_{n'})]^{1/2} - [\mathrm{Var}(\varphi_a(W_{n'}))]^{1/2}\right| \\ &\le [\mathrm{Var}(W_{n'} - \varphi_a(W_{n'}))]^{1/2} \\ &\le \varepsilon^{1/2}.\end{aligned}$$

By letting $n' \to \infty$ and then $a \to \infty$, we obtain

$$\left|1 - [\mathrm{Var}(Z)]^{1/2}\right| \le \varepsilon^{1/2}.$$

This implies that $\mathrm{Var}(Z) = 1$ and hence $\mathscr{L}(W_n) \Rightarrow N(0,1)$.

We now prove the necessity of the Lindeberg condition. First, we need two simple propositions.

PROPOSITION 2. *The Lindeberg condition holds if and only if* $\nu_n^{(2)} \Rightarrow \varepsilon_0$.

PROOF. The "only if" part has been proved above. The "if" part follows from the inequalities

$$\begin{aligned}\sum_{i=1}^{r_n} E|X_{ni}|(|X_{ni}| - \varepsilon)^+ &\ge \sum_{i=1}^{r_n} E|X_{ni}|(|X_{ni}| - \varepsilon)I(|X_{ni}| > 2\varepsilon) \\ &\ge \tfrac{1}{2}\sum_{i=1}^{r_n} EX_{ni}^2 I(|X_{ni}| > 2\varepsilon).\end{aligned}$$ □

For the next proposition let $C_U^2(\mathbb{R})$ be the class of functions g on $\mathbb{R}$ such that g, g' and g'' are uniformly continuous, $|g(x)| \le C(1 + |x|)$ for some constant C and g' and g'' are bounded.

PROPOSITION 3. *Let Z and T be independent random variables such that Z has the normal distribution with mean* 0 *and variance* $\sigma^2 > 0$. *If* $\mathrm{Var}[g(Z)] \le \sigma^2 E[g'(Z + T)^2]$ *for* $g \in C_U^2(\mathbb{R})$, *then* $T = 0$ *w.p.*1.

PROOF. By the variational argument used in Borovkov and Utev (1983) or Chen and Lou (1987), we have

$$EZh(Z) = \sigma^2 Eh'(Z + T),$$

for $h \in C_U^2(\mathbb{R})$. This equation also holds for $h(x) = x^3$ by approximating this function by functions of $C_U^2(\mathbb{R})$. So

$$3\sigma^4 = EZ^4 = 3\sigma^2 E(Z + T)^2.$$

The finiteness of $E(Z + T)^2$ and EZ^2 implies that of ET^2. By expanding $E(Z + T)^2$ we obtain $\sigma^2 = \sigma^2 + ET^2$ which implies that $ET^2 = 0$ and so $T = 0$ w.p.1. □

For the proof of the necessity of the Lindeberg condition we need the usual assumption that for every $\varepsilon > 0$, $\max_{1 \le i \le r_n} P(|X_{ni}| > \varepsilon) \to 0$ as $n \to \infty$. Let $g \in C_U^2(\mathbb{R})$. For every $\varepsilon > 0$, let $\delta > 0$ be such that $|g'(x)^2 - g'(y)^2| \le \varepsilon$ for

$|x - y| \le \delta$. Then we have

$$\begin{aligned} &\left| \sum_{i=1}^{r_n} \int_{-\infty}^{\infty} E\left\{g'(W_n^{(i)} + t)^2 - g'(W_n + t)^2\right\} K_n^{(i)}(t)\, dt \right| \\ &\quad \le C\left\{ \sum_{i=1}^{r_n} P(|X_{ni}| > \delta) \int_{-\infty}^{\infty} K_n^{(i)}(t)\, dt + \varepsilon \sum_{i=1}^{r_n} \int_{-\infty}^{\infty} K_n^{(i)}(t)\, dt \right\} \\ &\quad \le C\left[\max_{1 \le i \le r_n} P(|X_{ni}| > \delta) + \varepsilon \right], \end{aligned} \tag{5}$$

for some constant C. Define the probability measure $\tilde{\nu}_n$ on $\mathscr{B}(\mathbb{R})$ by

$$\int_{\mathbb{R}} f\, d\tilde{\nu}_n = \sum_{i=1}^{r_n} \int_{-\infty}^{\infty} f(t) K_n^{(i)}(t)\, dt,$$

for bounded and continuous functions f on $\mathbb{R}$. In view of (5), the inequality (3) can be written as

$$\operatorname{Var}[g(W_n)] \le \int_{-\infty}^{\infty} E\left[g'(W_n + t)^2\right] \tilde{\nu}_n(dt) + o(1). \tag{6}$$

Suppose $\mathscr{L}(W_n) \Rightarrow N(0,1)$. Let $\{\tilde{\nu}_{n'}\}$ be a subsequence of $\{\tilde{\nu}_n\}$ which converges vaguely to a subprobability measure $\tilde{\nu}$. By an application of the Ascoli-Arzelà theorem, it is not difficult to show that for $g \in C_U^2(\mathbb{R})$ such that g' has compact support,

$$\int_{-\infty}^{\infty} E\left[g'(W_{n'} + t)^2\right] \tilde{\nu}_{n'}(dt) \to \int_{-\infty}^{\infty} E\left[g'(Z + t)^2\right] \tilde{\nu}(dt),$$

where Z is a standard normal random variable. So by (6) we have

$$\operatorname{Var}[g(Z)] \leqq \int_{-\infty}^{\infty} E\left[g'(Z + t)^2\right] \tilde{\nu}(dt). \tag{7}$$

By approximating $C_U^2(\mathbb{R})$ by those functions g of $C_U^2(\mathbb{R})$ such that g' has compact support, (7) holds for $g \in C_U^2(\mathbb{R})$. By letting $g(x) = x$, we get

$$1 \leqq \int_{-\infty}^{\infty} \tilde{\nu}(dt),$$

and so $\tilde{\nu}$ is a probability measure. By Proposition 3, $\tilde{\nu}$ must be ε_0. Hence $\tilde{\nu}_n \Rightarrow \varepsilon_0$. But $\tilde{\nu}_n = \tilde{\nu}_n^{(2)}$. By Proposition 2, the Lindeberg condition holds. This completes the proof.

REFERENCES

BOROVKOV, A. A. and UTEV, S. A. (1983). On an inequality and a related characterization of the normal distribution. *Theory Probab. Appl.* **28** 219–228.

CHEN, L. H. Y. (1985). Poincaré-type inequalities via stochastic integrals. *Z. Wahrsch. verw. Gebiete* **69** 251–277.

CHEN, L. H. Y. and LOU, J. H. (1987). Characterization of probability distributions by Poincaré-type inequalities. *Ann. Inst. H. Poincaré Sect. B* (*N.S.*) **23** 91–110.

KOTANI, S. (1985). Private communication.

DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE
LOWER KENT RIDGE ROAD
SINGAPORE 0511
REPUBLIC OF SINGAPORE