The Annals of Probability
1988, Vol. 16, No. 1, 231–245

A MALLIAVIN-TYPE ANTICIPATIVE STOCHASTIC CALCULUS

By Marc A. Berger

The Weizmann Institute of Science

Two extensions of the Itô integral are developed, and put in the perspective of derivative operators in the Malliavin calculus. The divergence operator, δ, is constructed, and its properties and action on these two extended integrals are described. Discussion of iterated stochastic integrals and the extended stochastic integrals as functions of their upper limits is also included.

1. Introduction. This work is concerned with the problem of extending Itô's stochastic integral, so as to be able to integrate certain processes which anticipate the Brownian paths. The first to set up such an extension was Skorohod [12]. Afterwards Itô [8], Berger and Mizel [2], [4] and Ogawa [10], [11] considered different extensions. Skorohod and Berger and Mizel used the Itô–Wiener homogeneous chaos expansion to construct their extensions, and Ogawa used orthonormal expansions on $L^2(0,1)$.

In order to make such a study of interest, one ought to explain how one is led to this problem. The original motivation of Berger and Mizel was primarily based on resolvent formula considerations for Volterra equations which also contain stochastic integrals ([2]–[4]). When trying to interchange the order of integration in what corresponds to the Neumann series expansion, anticipative integrands pop up. Thus, along with extending the stochastic integral, the main concern was to study Fubini-type results for interchanging the order of integration in a double stochastic integral over a triangular region of integration ([1]–[4]).

In [4] Berger and Mizel introduced an "obscure operator" on stochastic processes, denoted by δ in this work, which was used to define that extension of the stochastic integral used for inverting linear Itô–Volterra equations. It turns out that δ is the divergence operator in the Malliavin stochastic calculus, as explained in Section 2. At the same time Gaveau and Trauber [5] showed that the Skorohod integral corresponds to the dual of the Frechét derivative in the Malliavin calculus. Thus it became clear that the extension problem amounts to the problem of identifying the domains of these differential operators and developing their rules of manipulation. The extension problem, then, became significant for reasons other than those to do with Itô–Volterra equations. It is surprising that the study of extensions of the stochastic *integral*, in fact, amounts to the study of *derivatives* in the Malliavin calculus; yet this is clearly apparent in the nature of the domains of definition D and $\overline{D}$ in Section 2. Recently, Nualart and Zakai [9] carried out a systematic study of the Skorohod

Received January 1986.
AMS 1980 *subject classification*. 60H05.
Key words and phrases. Malliavin calculus, multiple Wiener integral, stochastic integral.

and Ogawa extensions in the light of the Malliavin calculus. It is delightful how this light makes many of the results concerning the extensions more meaningful and apparent.

This work can be viewed as a complement to [4], although it is self-contained. Its purpose is to continue the lines set forth in [4] and to develop the δ-operator. In Section 2 the two extensions are defined, some of their basic properties are derived and they are identified within the Malliavin calculus. The development here is a simpler one than in [4], and carries the analysis further. The chaos expansion is still being used to define the extensions, since I believe this is the best way to understand their domains of definition, and since it dictates a nice framework for stating and proving results. This is an easy way to find the "right hypothesis" for a theorem. The theory throughout is an L^2-theory. In Section 3 iterated stochastic integrals are studied (a unique feature of the Berger and Mizel program [1]–[4]), and the action of the δ-operator on the two extensions of the stochastic integral. Theorem 3.V(b) is related to what was referred to as the "correction formula" in [1]. This formula was studied extensively in [2], [3]. It is also shown that if one wants to consider the extended stochastic integrals as functions of their upper limits, then one is forced to set them up as $\int_0^t \gamma(s, t)\, d\theta(s)$, where $\gamma(s, t)$ is measurable w.r.t. $\sigma(\theta(u)\colon u \le t)$, $0 \le s \le t$. Of course, this is exactly the integral one is led to through resolvent considerations for the Itô–Volterra equation. Finally, in Section 4 the action of δ on certain stochastic processes is studied.

A word about the notation! Although it may not seem so at first glance, the notation used in this work is well defined, and each equation has a well-determined interpretation. It is easy to get wrapped up in cumbersome notation and lots of irrelevant variables when working around the homogeneous chaos, and this has the detrimental effect of hiding the *real* points to be made. Once all the extraneous stuff is removed, the emphasis on these points becomes automatic. Furthermore, the manipulations are then easier to follow. So, for example, most of the time differentials (*dt* or *ds*) are left off from ordinary (as opposed to stochastic) integrals, variables of integration, indices on sums, etc., whenever these can be easily (and uniquely!) inferred from the text of the equation or the hypothesis of the theorem.

2. A review of [4], Section 3, with some new perspective. Let $\Theta = (C_0(J), \mathscr{B}, \mathscr{W})$ be Wiener space, where J is some finite interval $[0, M]$. $L^2(\Theta)$ has the orthogonal decomposition $\oplus_0^\infty Z^{(n)}$, where $Z^{(n)}$ is the subspace of n-fold multiple Wiener integrals:

$$Z^{(0)} := \mathbb{R}, \qquad Z^{(n)} := \left\{ \int_{J^n} f\, d\theta^{(n)}\colon f \in \tilde{L}^2(J^n) \right\}, \qquad n \ge 1.$$

Here $\tilde{L}^2$ denotes the symmetric functions in L^2. When dealing strictly with progressively measurable processes it is usually preferable to deal with multiple Wiener integrals over triangular domains, $\int_{T_n} f\, d\theta^{(n)}$, $T_n := \{0 \le \tau_1 \le \cdots \le \tau_n \le M\}$, since (i) these integrals constitute an isometry $L^2(T_n) \to Z^{(n)}$; and (ii) they are defined by iterated integration. On the other hand, the multiple Wiener integrals over square domains, $\int_{J^n} f\,d\theta^{(n)}$, do not quite form an isometry,

$$\mathbb{E}\int_{J^n} f\,d\theta^{(n)}\int_{J^n} g\,d\theta^{(n)} = n!\int_{J^n} fg;$$

and they are not defined iteratively—rather, they are defined in terms of the integrals over triangles,

$$\int_{J^n} f\,d\theta^{(n)} := n!\int_{T^n} f\,d\theta^{(n)}. \tag{2.1}$$

When dealing with anticipative processes, however, the integrals over squares have definite advantages over the integrals over triangles (not the least of which is the ease of changing the order of integration), as will soon be apparent, and so we are willing to make this choice and live with the two drawbacks mentioned above. It is clear, though that $\tilde{L}^2(J^n)$ and $L^2(T_n)$ are isomorphic, and so these two approaches really are the same. We observe here that if $f\colon J \to \tilde{L}^2(J^n)$ is Bochner integrable, then so is $\int_{J^n} f(\cdot, t)\,d\theta^{(n)}$, and

$$\int_J\left[\int_{J^n} f(\cdot, t)\,d\theta^{(n)}\right] = \int_{J^n}\left[\int_J f(\cdot, t)\right]d\theta^{(n)}. \tag{2.2}$$

For any $f\colon J^n \to \mathbb{R}$ let $\tilde{f}\colon J^n \to \mathbb{R}$ denote the symmetric function $\tilde{f}(\tau_1, \dots, \tau_n) := (1/n!)\Sigma_{\pi \in S_n} f(\tau_{\pi_1}, \dots, \tau_{\pi_n})$, S_n being the permutation group on n letters. Observe that $f \mapsto \tilde{f}$ is the orthgonal projection $L^2(J^n) \to \tilde{L}^2(J^n)$.

Let $\alpha\colon J \to L^2(\Theta)$ be a stochastic process on $(\Theta, \mathscr{B}, \mathscr{W})$. Then it has a unique expansion

$$\alpha(t) = f_0(t) + \sum\int_{J^n} f_n(\cdot, t)\,d\theta^{(n)}, \tag{2.3}$$

where $f_n\colon J \to \tilde{L}^2(J^n)$, $\forall\ n$, and $\Sigma n!\|f_n(\cdot, t)\|^2_{\tilde{L}^2(J^n)} < \infty$. Many measurability properties of α translate directly into support properties for the kernels f_n. Thus α is progressively measurable if and only if $f_n(\cdot, t) = f_n(\cdot, t)I_{[0, t]^n}$, $\forall\ n, t$, if and only if

$$\tilde{f}_n = n!\,\widetilde{f_n I_{T_{n+1}}}, \qquad \forall\ n, \tag{2.4}$$

where I denotes the indicator function. [N.B. Although $f_n(\cdot, t)$ is a symmetric function of n variables, $\forall\ t$, in general f_n is not symmetric in all $n + 1$ variables.] Similarly, $\alpha(t)$ is measurable w.r.t. $\sigma(\theta(u - \theta(t)\colon M \geq u \geq t)$, $\forall\ t$, if and only if $f_n(\cdot, t) = f_n(\cdot, t)I_{[t, M]^n}$, $\forall\ n, t$, if and only if

$$\tilde{f}_n = n!\,\widetilde{f_n I_{\overline{T}_{n+1}}}, \qquad \forall\ n, \tag{2.5}$$

where $\overline{T}_{n+1} = \{M \geq \tau_1 \geq \cdots \geq \tau_{n+1} \geq 0\}$. We also see that $\alpha \in L^2(J \times \Theta)$ if and only if $f_n \in L^2(J^{n+1})$, $\forall\ n$, and $\Sigma n!\|f_n\|^2_{L^2(J^{n+1})} < \infty$.

Based on (2.3) we introduce a domain

$$D := \left\{\alpha\colon J \to L^2(\Theta)\colon f_n \in L^2(J^{n+1}), \forall\ n, \text{and } \sum(n+1)!\|\tilde{f}_n\|^2_{L^2(J^{n+1})} < \infty\right\}.$$

For $\alpha \in D$ we can define

$$\int \alpha \, d\theta := \sum \int_{J^{n+1}} \tilde{f}_n \, d\theta^{(n+1)}. \tag{2.6}$$

If either (2.4) or (2.5) holds, then it is easy to see that

$$\| \tilde{f}_n \|_{L^2(J^{n+1})} = \frac{1}{\sqrt{n+1}} \| f_n \|_{L^2(J^{n+1})}, \qquad \forall \, n. \tag{2.7}$$

Thus $\|\alpha\|^2_{L^2(J \times \Theta)} = \Sigma (n+1)! \| \tilde{f}_n \|^2_{L^2(J^{n+1})}$, so that in particular D contains all progressively measurable $\alpha \in L^2(J \times \Theta)$. Furthermore it follows from (2.1) and (2.4) that for such α the integral defined above in (2.6) agrees with the classical Itô integral. In other words (2.6) is a *genuine* extension of the Itô integral—"genuine" meaning that if we restrict to progressively measurable α, then its domain is precisely $L^2(J \times \Theta)$, and it coincides with the Itô integral.

On the other hand, it is clear (e.g., by taking all the f_n symmetric) that there are processes in $L^2(J \times \Theta)$ which are not in D. In fact, neither of our extensions of the stochastic integral are defined on all of $L^2(J \times \Theta)$. One reason why we cannot expect to extend the stochastic integral to all of $L^2(J \times \Theta)$ is discussed below. This extension (2.6) is originally due to Skorohod [12], and will be referred to as the Sk-integral in what follows.

We define now a derivative type operator, δ, on stochastic processes α: $J \to L^2(\Theta)$, in terms of the expansion (2.3), by

$$\delta\alpha(t) := f_1(t+, t) + \sum n \int_{J^{n-1}} f_n(\cdot, t+, t) \, d\theta^{(n-1)}. \tag{2.8}$$

The domain of definition for δ is

$D(\delta) = \{\alpha$: $J \to L^2(\Theta)$: $f_n(\cdot, t)$ has a trace belonging to $\tilde{L}_2(J^{n-1})$ on each $(n-1)$-dimensional hyperplane $\tau_n =$ const., $t <$ const. $< t + \varepsilon$, and $f_n(\cdot, \tau_n^j, t) \xrightarrow{L^2(J^{n-1})} f_n(\cdot, t+, t)$ as $j \to \infty$ whenever $\tau_n^j \downarrow t$, $\forall \, n$. Furthermore, $\Sigma n n! \| f_n(\cdot, t+, t) \|^2_{L^2(J^{n-1})} < \infty.\}$.

Observe that if α is progressively measurable, then $\alpha \in D(\delta)$ and $\delta\alpha \equiv 0$. We next define a new extension of the stochastic integral by

$$\overline{\int} \alpha \, d\theta := \int \alpha \, d\theta + \int_J \delta\alpha. \tag{2.9}$$

Again in terms of (2.3) it follows from (2.2) that the domain of definition for this extension is $\overline{D} = \{\alpha \in D \cap D(\delta)$: $f_n(\cdot, t+, t)$ is Bochner integrable, $\forall \, n$, and $\Sigma n n! \| \int f_n(\cdot, t+, t) \|^2 < \infty\}$. This integral originally appeared in [2]. We remind the reader of the following two examples from [4].

EXAMPLE I. $\alpha(t) \equiv \theta(T)$:

$$\int_0^T \alpha \, d\theta = \theta^2(T) - T; \qquad \overline{\int_0^T} \alpha \, d\theta = \theta^2(T).$$

EXAMPLE II. $\alpha(t) = \theta(t)\theta(T)$:

$$\int_0^T \alpha\, d\theta = \tfrac{1}{2}\theta^3(T) - \tfrac{1}{2}T\theta(T) - \int_0^T \theta(s)\, ds,$$

$$\overline{\int_0^T} \alpha\, d\theta = \tfrac{1}{2}\theta^3(T) - \tfrac{1}{2}T\theta(T).$$

We move now to the abstract Wiener space set-up $\Theta^* \subset H \subset \Theta$, where H is the Hilbert space

$$H := \left\{\theta \in \Theta: \theta \text{ is absolutely continuous, and } \int \dot{\theta}^2 < \infty\right\},$$

with $\langle \theta_1, \theta_2\rangle := \int \dot{\theta}_1\dot{\theta}_2$. It is clear that the inclusion i: $H \to \Theta$ is continuous. Let $\mathscr{D}$: $W^1(\Theta) \to L^2(\Theta, H)$ be the H-Frechét derivative. Then $\mathscr{D}^*$: $\mathrm{dom}(\mathscr{D}^*) \to L^2(\Theta)$ is given by $\mathscr{D}^* = -\mathrm{div} + \langle \theta, \cdot\rangle$. This is essentially the identity

$$\begin{aligned}(2.10)\qquad &\int_{\mathbb{R}^n} \langle \nabla\phi(x), \psi(x)\rangle e^{-|x|^2/2}\, dx \\ &\quad = \int_{\mathbb{R}^n} \phi(x)[-\mathrm{div}\,\psi(x) + \langle x, \psi(x)\rangle]\, e^{-|x|^2/2}\, dx,\end{aligned}$$

$\psi = (\psi_1, \dots, \psi_n)$ for $\phi, \psi_1, \dots, \psi_n \in C_0^1(\mathbb{R}^n)$. Given $\alpha \in L^2(J \times \Theta)$ set $\beta(t) = \int_0^t \alpha$. Then $\beta \in L^2(\Theta, H)$. If α is progressively measurable, then $\beta \in \mathrm{dom}(\mathscr{D}^*)$ and $\mathscr{D}^*\beta = \int \alpha\, d\theta$. To see this we use simple processes. Let $\alpha(t) = \Sigma\alpha(t_i)I_{[t_i, t_{i+1})}$. Then $\beta(t) = \Sigma\sqrt{t_{i+1} - t_i}\,\alpha(t_i)e_i(t)$, where $e_i(t) = (1/\sqrt{t_{i+1} - t_i})\int_0^t I_{[t_i, t_{i+1})}$. Observe that the e_i form an orthonormal system in H. Thus

$$\mathrm{div}\beta = \sum\sqrt{t_{i+1} - t_i}\ \partial_{e_i}\alpha(t_i),$$

$$\langle \theta, \beta\rangle = \sum\alpha(t_i)[\theta(t_{i+1}) - \theta(t_i)] = \int \alpha\, d\theta,$$

where ∂_{e_i} denotes the directional derivative in the direction e_i.

Since $\alpha(t_i)$ is measurable w.r.t. $\sigma(\theta(u)$: $0 \le u \le t_i)$ and $e_i(t) = 0$, $0 \le t \le t_i$, it follows that $\alpha(t_i, \theta + he_i) = \alpha(t_i, \theta)$, $\theta \in \Theta$, $h \in \mathbb{R}$. Thus

$$\partial_{e_i}\alpha(t_i) = 0, \qquad \forall\, i,$$

and $\mathrm{div}\beta = 0$. The analogue to (2.10) here is

$$(2.11)\qquad \mathrm{div}(\text{const.}, \psi_2(x_1), \psi_3(x_1, x_2), \dots, \psi_n(x_1, \dots, x_{n-1})) = 0,$$

since $\partial/\partial x_i$ acts on a function which only depends on $x_1, \dots, x_{i-1}$.

In Example I we have $\beta(t) = \theta(T)\sqrt{T}\,e(t)$, where $e(t) = (1/\sqrt{T})\int_0^t I_{[0, T)}$. Thus

$$\mathrm{div}\beta = \sqrt{T}\ \partial_e\theta(T) = \sqrt{T}\,e(T) = T,$$

$$\langle \theta, \beta\rangle = \theta^2(T).$$

In Example II we approximate by a step process $\alpha(t) \approx \Sigma\theta(t_i)\theta(T)I_{[t_i, t_{i+1})}$, where $0 = t_0 \le \cdots \le t_m = T$ is a partition. Then

$$\beta(t) \approx \Sigma\sqrt{t_{i+1} - t_i}\,\theta(t_i)\theta(T)e_i(t),$$

where e_i is as above. Thus

$$\begin{aligned}\operatorname{div}\beta &\approx \sum\sqrt{t_{i+1}-t_i}\ \partial_{e_i}\theta(t_i)\theta(T)\\ &= \sum\sqrt{t_{i+1}-t_i}\,\theta(t_i)e_i(T) = \sum\theta(t_i)(t_{i+1}-t_i),\\ \langle\theta,\beta\rangle &\approx \sum\theta(t_i)\theta(T)[\theta(t_{i+1})-\theta(t_i)]\\ &= \theta(T)\Sigma\theta(t_i)[\theta(t_{i+1})-\theta(t_i)].\end{aligned}$$

Here $\approx$ signifies equality in the limit as the mesh of the partition tends to zero.

In general, if α: $J \to Z^{(n)}$, $\alpha \in \overline{D}$, is a simple process, $\alpha(t) = \int_{J^n} f(\cdot, t)\, d\theta^{(n)}$, $f(\cdot, t) = \Sigma f(\cdot, t_i) I_{[t_i, t_{i+1})}$, then

$$\partial_{e_i}\alpha(t_i) = \frac{n}{\sqrt{t_{i+1}-t_i}} \int_{J^{n-1}} \left[\int_{t_i}^{t_{i+1}} f(\cdot, \tau, t_i)\right] d\theta^{(n-1)},$$

where e_i is as above. Thus

$$\operatorname{div}\beta = n\int_{J^{n-1}} G\, d\theta^{(n-1)} = \int \delta\alpha, \tag{2.12}$$

where $G = \Sigma\int_{t_i}^{t_{i+1}} f(\cdot, \tau, t_i) = \int f(\cdot, t+, t)$. Furthermore,

$$\langle\theta,\beta\rangle = \sum\alpha(t_i)[\theta(t_{i+1})-\theta(t_i)]. \tag{2.13}$$

We use Itô's Lemma 2.2.III in [7] which asserts that

$$\int_J \phi\, d\theta \int_{J^n} g\, d\theta^{(n)} = \int_{J^{n+1}} \widetilde{\phi g}\, d\theta^{(n+1)} + n\int_{J^{n-1}}\left[\int_J \phi(\tau) g(\cdot, \tau)\right] d\theta^{(n-1)}.$$

Thus

$$\begin{aligned}\alpha(t_i)[\theta(t_{i+1})-\theta(t_i)] &= \int_{J^{n+1}} \widetilde{f(\cdot, t_i) I_{[t_i, t_i+1)}}\, d\theta^{(n+1)}\\ &\quad + n\int_{J^{n-1}}\left[\int_{t_i}^{t_{i+1}} f(\cdot, \tau, t_i)\right] d\theta^{(n-1)}.\end{aligned}$$

Substituting back into (2.11) and (2.12) we see that

$$\mathscr{D}^*\beta = \int_{J^{n+1}} \tilde{F}\, d\theta^{(n+1)},$$

where $F = \Sigma f(\cdot, t_i) I_{[t_i, t_{i+1})} = f$. To summarize, then, if $\alpha \in \overline{D}$ is a simple process $\alpha(t) = \Sigma\alpha(t_i) I_{[t_i, t_{i+1})}$, then

$$\int \alpha\, d\theta = \mathscr{D}^*\beta, \tag{2.14}$$

$$\overline{\int}\alpha\, d\theta = \langle\theta,\beta\rangle = \sum\alpha(t_i)[\theta(t_{i+1})-\theta(t_i)]. \tag{2.15}$$

We see then that the Sk-integral corresponds to $\mathscr{D}^*$, a fact which Gaveau and Trauber [5] discovered. Furthermore, $\overline{\int}\alpha\, d\theta$ corresponds to the Ogawa integral described in Nualart and Zakai [9], Section 6.

The fact that $\operatorname{div}\beta = 0$ whenever $\alpha \in L^2(J \times \Theta)$ is progressively measurable explains why we cannot expect to extend the stochastic integral to all of $L^2(J \times \Theta)$. Somewhere there must be a condition that $\beta \in \operatorname{dom}(\mathscr{D}^*)$. If α is progressively measurable, then on account of the analogy to (2.11), the differentiability condition is overlooked. Indeed, any function $\psi(x) = (\text{const.}, \psi_2(x_1), \psi_3(x_1, x_2), \dots, \psi_n(x_1, \dots, \psi_{n-1}))$ is in the domain of the divergence operator, regardless of differentiability of the functions $\psi_2, \dots, \psi_n$. On the other hand, when we leave the class of progressively measurable processes, then it is clear that a differentiability condition on α must necessarily arise. In the framework of the expansion (2.3) this is precisely the condition

$$\sum (n+1)! \|\tilde{f}_n\|^2_{L^2(J^{n+1})} < \infty.$$

The result (2.15) reminds us of the classical "backward" approximating Riemann sum for Itô integrals, and leads us to consider *sample path representations* for $\overline{\int} \alpha d\theta$. By a "sample path representation" we mean a measurable mapping Q: $\Theta \times \mathbb{R}^J \to \mathbb{R}$ such that $\overline{\int} \alpha\, d\theta = Q(\theta, \alpha(\cdot, \theta))$ a.s. $[\mathscr{W}]$ for some class of $\alpha \in \overline{D}$. For any α: $J \to Z^{(n)}$, $\alpha \in \overline{D}$, say $\alpha(t) = \int_{J^n} f(\cdot, t)\, d\theta^{(n)}$, it follows from (2.9) and (2.15) that

$$\mathbb{E}\left\{ \sum \alpha(t_i)[\theta(t_{i+1}) - \theta(t_i)] - \overline{\int} \alpha\, d\theta \right\}^2 \tag{2.16}$$
$$= (n+1)! \int_{J^{n+1}} (\tilde{F} - \tilde{f}\,)^2 + nn! \int_{J^{n-1}} \left[G - \int_J f(\cdot, t+, t) \right]^2,$$

where $F = \sum f(\cdot, t_i) I_{[t_i, t_{i+1})}$, $G = \sum \int_{t_i}^{t_{i+1}} f(\cdot, \tau, t_i)$. Since $\alpha \in D(\delta)$ it follows that $G \to \int_J f(\cdot, t+, t)$ in $L^2(J^{n-1})$ as we take finer and finer partitions $0 = t_0 \leq \cdots \leq t_m = M$ with $\max(t_{i+1} - t_i) \to 0$. Furthermore, if α is right mean-square continuous, then $F \to f$ in $L^2(J^{n+1})$ in this same fashion. Thus, in general, if $\alpha \in \overline{D}$ is right mean-square continuous, then we have the sample path representation

$$\overline{\int} \alpha\, d\theta = \lim_k \sum_i \alpha\big(t_i^{(k)}\big)\big[\theta\big(t_{i+1}^{(k)}\big) - \theta\big(t_i^{(k)}\big)\big], \tag{2.17}$$

in $L^2(\Theta)$, where $0 = t_0^{(k)} \leq \cdots \leq t_{m_k}^{(k)} = M$ and $\lim_k \max_i (t_{i+1}^{(k)} - t_i^{(k)}) = 0$. This is [4], Theorem 3D.

The problem of sample path representation is more complicated for the Sk-integral. Since it extends the Itô integral it is clear that (2.17) holds for the Sk-integral as well, whenever $\alpha \in L^2(J \times \Theta)$ is progressively measurable. On the other hand, the mapping $\theta \to \theta_*$, where $\theta_*(t) = \theta(M) - \theta(M - t)$, is measure preserving on $(\Theta, \mathscr{B}, \mathscr{W})$. For any f: $J^n \to \mathbb{R}$ let $R_n f$: $J^n \to \mathbb{R}$ denote the function

$$R_n f(\tau_1, \dots, \tau_n) := f(M - \tau_1, \dots, M - \tau_n).$$

It follows from (2.1) that for $f \in \tilde{L}^2(J^n)$,

$$\int_{J^n} f\, d\theta^{(n)} = \int_{J^n} R_n f\, d\theta_*^{(n)}. \tag{2.18}$$

Thus if α: $J \to L^2(\Theta)$ has the expansion (2.3), then

$$\alpha(t) = f_0(t) + \sum \int_{J^n} R_n f_n(\cdot, t)\, d\theta_*^{(n)}.$$

Since

$$R_{n+1} \tilde{f}_n(\cdot, t) = \widetilde{R_n f_n}(\cdot, M - t), \tag{2.19}$$

it follows that

$$\int \alpha(t)\, d\theta(t) = \int \alpha(M - t)\, d\theta_*(t).$$

Now if $\alpha(t)$ is measurable w.r.t. $\sigma(\theta(u) - \theta(t)\colon M \geq u \geq t)$, $\forall\ t$, then $\alpha(M - t)$ is measurable w.r.t. $\sigma(\theta_*(u)\colon 0 \leq u \leq t)$, $\forall\ t$. Thus $\int \alpha(M - t)\, d\theta_*(t)$ is a classical Itô integral, and

$$\begin{aligned} \int \alpha\, d\theta &= \lim_k \sum \alpha\big(M - t_i^{(k)}\big)\big[\theta_*\big(t_{i+1}^{(k)}\big) - \theta_*\big(t_i^{(k)}\big)\big] \\ &= \lim_k \sum \alpha\big(t_{i+1}^{(k)}\big)\big[\theta\big(t_{i+1}^{(k)}\big) - \theta\big(t_i^{(k)}\big)\big], \end{aligned} \tag{2.20}$$

in $L^2(\Theta)$. This is [4], Theorem 3.G. The Sk-integral, then, is sometimes given by a limit of "backward" sums (2.17), and sometimes given by a limit of "forward" sums (2.20). This explains why the sample path representation problem is more delicate for the Sk-integral.

Finally, I point out that (2.17) leads to a nice formula for evaluating $\bar{\int} \alpha\, d\theta$ in terms of a classical Itô integral, when α is of the form $\alpha(t) = \gamma(t, X)$. If γ: $\mathbb{R} \to L^2(J \times \Theta)$ is such that $\gamma(x)$ is progressively measurable, $\forall\ x$, and if X is a random variable measurable w.r.t. $\mathscr{B}$, and if $\phi(X) \in L^2(\Theta)$, where $\phi(x) := \int \gamma(x)\, d\theta$, then $\gamma(X) \in \overline{D}$ and $\bar{\int} \gamma(X)\, d\theta = \phi(X)$. This is [4], Theorem 3.A.

3. Anticipative calculus. We begin our development of the integration theory described above by proving Fubini-type theorems in this framework.

THEOREM I. *Let* γ: $J \to D(\delta)$ *be Bochner integrable,* $\int_J \gamma \in D(\delta)$. *Then* $\delta \int_J \gamma = \int_J \delta\gamma$.

PROOF. For $\gamma_t(s) = \int_{J^n} f(\cdot, s, t)\, d\theta^{(n)}$, $f(\cdot, s, t) \in \tilde{L}^2(J^n)$, $\forall\ s, t$, this follows directly from (2.2). In general, write

$$\gamma_t(s) = f_0(s, t) + \sum \int_{J^n} f_n(\cdot, s, t)\, d\theta^{(n)}, \tag{3.1}$$

where $f_n(\cdot, s, t) \in \tilde{L}^2(J^n)$, $\forall\ s, t$. By assumption

$$\sum n n! \| \textstyle\int f_n(\cdot, s + \ , s, t)\, dt \|^2_{L^2(J^{n-1})} < \infty.$$

Since

$$\left\| \delta\left\{ \int_J \left[\sum_N^\infty \int_{J^n} f_n(\cdot, s, t)\, d\theta^{(n)} \right] dt \right\} \right\|^2_{L^2(\Theta)}$$

$$= \left\| \int_J \delta_s \left[\sum_N^\infty \int_{J^n} f_n(\cdot, s, t)\, d\theta^{(n)} \right] dt \right\|^2_{L^2(\Theta)}$$

$$= \sum_N^\infty n n! \left\| \int \hat{f}_n(\cdot, s + , s, t)\, dt \right\|^2_{L^2(J^{n-1})} \to 0,$$

the desired conclusion follows. □

Theorem II. (a) *Let γ: $J \to D$ be Bochner integrable, $\int_J \gamma \in D$. Then*

$$\int \left(\int_J \gamma \right) d\theta = \int_J \left(\int \gamma\, d\theta \right). \tag{3.2}$$

(b) *Let γ: $J \to \overline{D}$ be Bochner integrable, $\int_J \gamma \in \overline{D}$. Then*

$$\overline{\int} \left(\int_J \gamma \right) d\theta = \int_J \left(\overline{\int} \gamma\, d\theta \right). \tag{3.3}$$

Proof. (a) For $\gamma_t(s) = \int_{J^n} f(\cdot, s, t)\, d\theta^{(n)}$, $f(\cdot, s, t) \in \tilde{L}^2(J^n)$, $\forall$ s, t, (3.2) follows directly from (2.2) and the fact that symmetrization commutes with Bochner integration. In general, in terms of the expansion (3.1),

$$\Sigma(n + 1)! \| \int_J \widetilde{f_n(\cdot, t)}\, dt \|^2_{L^2(J^{n+1})} < \infty.$$

Since

$$\left\| \int \left[\int_J \sum_N^\infty \int_{J^n} f_n(\cdot, s, t)\, d\theta^{(n)}\, dt \right] d\theta(s) \right\|^2_{L^2(\Theta)}$$

$$= \left\| \int_J \left[\int \sum_N^\infty \int_{J^n} f_n(\cdot, s, t)\, d\theta^{(n)}\, d\theta(s) \right] dt \right\|^2_{L^2(\Theta)}$$

$$= \sum_N^\infty (n + 1)! \left\| \int_J \widetilde{f_n(\cdot, t)}\, dt \right\|^2_{L^2(J^{n+1})} \to 0,$$

the desired conclusion follows.

(b) Having proved (a) it suffices now, by (2.9), to show that

$$\int_J \delta \left(\int_J \gamma \right) = \int_J \int_J \delta\gamma;$$

and this follows at once from Theorem I. □

REMARK. If $\gamma: J^2 \to L^2(\Theta)$ is such that $\gamma(s, t)$ is Bochner integrable in t, for each fixed s, then $\int_J \gamma(\cdot, t) \in D$ (resp. $\overline{D}$) implies that $\gamma(\cdot, t) \in D$ (resp. $\overline{D}$) for a.e. t. This is a consequence of the inequality

$$\left\| \int_J f \right\|^2_{L^2(J^n)} \leq M \int_J \| f \|^2_{L^2(J^n)},$$

for $f: J \to L^2(J^n)$ Bochner integrable.

There is an interesting consequence of Theorem II(b) in the special case $\gamma_t = \beta(t)\gamma_t'$, where $\gamma': J \to L^2(J \times \Theta)$ is such that γ_t' is progressively measurable, $\forall\ t$, and $\beta: J \times \Theta \to \mathbb{R}$ is a stochastic process. Then, on account of the remark at the end of Section 2, (3.3) becomes

$$\overline{\int}\left(\int_J \gamma\right) d\theta = \int_J \beta(t)\left[\int \gamma_t'\, d\theta\right].$$

Observe that the stochastic integral on the right is the classical Itô integral.

THEOREM III. *Let $\gamma: J^2 \to L^2(\Theta)$ be such that $\gamma(\cdot, t) \in D$, $\forall\ t$, and $\gamma(s, \cdot) \in D(\delta)$, $\forall\ s$. If the process $s \mapsto \delta_t \gamma(s, t)$ is in D, $\forall\ t$, and $\gamma(t+, t)$ exists in $L^2(\Theta)$, $\forall\ t$, then the process $t \mapsto \int \gamma(s, t)\, d\theta(s)$ is in $D(\delta)$, and*

$$\delta_t \int \gamma(s, t)\, d\theta(s) = \int \delta_t \gamma(s, t)\, d\theta(s) + \gamma(t+, t), \qquad \forall\ t. \tag{3.4}$$

PROOF. For $\gamma(s, t) = \int_{J^n} f(\cdot, s, t)\, d\theta^{(n)}$, $f(\cdot, s, t) \in \tilde{L}^2(J^n)$, $\forall\ s, t$, (3.4) follows from (2.6) and (2.8). The rest follows as in the proof of Theorem II, using the expansion (3.1). □

THEOREM IV. *Let $\gamma: J^2 \to L^2(\Theta)$ be such that $\gamma(\cdot, t) \in D(\delta)$, $\forall\ t$, and $\gamma(s, \cdot) \in D(\delta)$, $\forall\ s$. If the process $s \mapsto \delta_t \gamma(s, t)$ is in $D(\delta)$, $\forall\ t$, then the process $t \mapsto \delta_s \gamma(s, t)$ is in $D(\delta)$, $\forall\ s$, and $\delta_t \delta_s \gamma(s, t) = \delta_s \delta_t \gamma(s, t)$.*

PROOF. For $\gamma(s, t) = \int_{J^n} f(\cdot, s, t)\, d\theta^{(n)}$, $f(\cdot, s, t) \in \tilde{L}^2(J^n)$, $\forall\ s, t$, this follows from (2.8) and the symmetry of f. The rest is as above. □

REMARK. It follows from Theorems I, III and IV that (3.4) holds for $\overline{\int}$ as well. That is, if we replace D by $\overline{D}$ in the hypothesis of Theorem III, then

$$\delta_t \overline{\int} \gamma(s, t)\, d\theta(s) = \overline{\int} \delta_t \gamma(s, t)\, d\theta(s) + \gamma(t+, t), \qquad \forall\ t. \tag{3.5}$$

THEOREM V. (a) *Let $\gamma: J^2 \to L^2(\Theta)$ be such that $\gamma(\cdot, t) \in D$, $\forall\ t$, and $\gamma(s, \cdot) \in D$, $\forall\ s$. If the process $s \mapsto \int \gamma(s, t)\, d\theta(t)$ is in D, then so is the process $t \mapsto \int \gamma(s, t)\, d\theta(s)$, and*

$$\int\left[\int \gamma(s, t)\, d\theta(s)\right] d\theta(t) = \int\left[\int \gamma(s, t)\, d\theta(t)\right] d\theta(s). \tag{3.6}$$

(b) *Let* γ: $J^2 \to L^2(\Theta)$ *be such that* $\gamma(\cdot, t) \in \overline{D}$, $\forall$ t, *and* $\gamma(s, \cdot) \in \overline{D}$, $\forall$ s. *If* $\gamma(t+, t)$ *and* $\gamma(t, t+)$ *exist in* $L^2(\Theta)$, *and if the process* $s \mapsto \int \gamma(s, t)\, d\theta(t)$ *is in* $\overline{D}$, *then so is the process* $t \mapsto \int \gamma(s, t)\, d\theta(s)$, *and*

$$\overline{\int}\left[\overline{\int}\gamma(s,t)\,d\theta(s)\right]d\theta(t) + \int_J \gamma(t, t+) = \overline{\int}\left[\overline{\int}\gamma(s,t)\,d\theta(t)\right]d\theta(s) + \int_J \gamma(t+, t). \tag{3.7}$$

Proof. (a) For $\gamma(s, t) = \int_{J^n} f(\cdot, s, t)\, d\theta^{(n)}$, $f(\cdot, s, t) \in \tilde{L}^2(J^n)$, $\forall$ s, t, this follows at once from (2.6). The rest is as above.

(b) Having proved (a) this follows now from (2.8), (3.5) and Theorems II(b) and IV. □

We move on now to consider the stochastic integrals described in Section 2 as functions of their upper limits. If $\alpha \in D$, $\tau \in J$, we would like to define $\int_0^\tau \alpha\, d\theta := \int \alpha I_{[0,\tau]}\, d\theta$, but there is an inherent problem here. In terms of domains of definition the problem is that, in general, $\alpha I_{[0,\tau]}$ need not be in D, even though α is. If α has the expansion (2.3), then $\alpha I_{[0,\tau]}(t)$ has the expansion

$$\alpha(t) I_{[0,\tau]}(t) = f_0(t) I_{[0,\tau]}(t) + \sum \int_{J^n} f_n(\cdot, t) I_{[0,\tau]}(t)\, d\theta^{(n)}.$$

The condition that $\alpha I_{[0,\tau]} \in D$ is then

$$\sum (n+1)! \left\| \widetilde{f_n(\cdot, t) I_{[0,\tau]}(t)} \right\|^2_{L^2(J^{n+1})} < \infty,$$

and this does not follow from the condition $\alpha \in D$. To see this use, for example, kernels f_n satisfying $\tilde{f}_n = 0$, $\forall$ n, but $\widetilde{f_n(\cdot, t) I_{[0,\tau]}(t)} \neq 0$.

More fundamentally, in terms of measurability, the problem is that $\alpha(t) I_{[0,\tau]}(t)$ need not be measurable w.r.t. $\sigma(\theta(u)\colon u \le \tau)$. Let us introduce the notation $\mathscr{M}^\tau$, $\mathscr{M}_\eta$ and $\mathscr{M}_\eta^\tau$ for $\sigma(\theta(u)\colon u \le \tau)$, $\sigma(\theta(u) - \theta(\eta)\colon \eta \le u \le M)$ and $\sigma(\theta(u) - \theta(\eta)\colon \eta \le u \le \tau)$, respectively. The Sk-integral $\int_0^\tau \alpha d\theta$ is really only designed to integrate processes α: $[0, \tau] \to L^2(\Theta)$ for which $\alpha(t)$ is measurable w.r.t. $\mathscr{M}^\tau$, $\forall$ t. In other words, the integrand should not anticipate further than the upper limit of integration. On the other hand, if this is the case, and $\alpha(t)$ is measurable w.r.t. $\mathscr{M}^\tau$, $0 \le t \le \tau$, then the kernels $f_n(\cdot, t)$ above are all supported on $[0, \tau]^n$, $0 \le t \le \tau$. In particular,

$$\widetilde{f_n(\cdot, t) I_{[0,\tau]}(t)} = \widetilde{f_n(\cdot, t)} I_{[0,\tau]^{n+1}}(\cdot, t), \qquad \forall\, n, \tag{3.8}$$

and we see here at once that $\alpha I_{[0,\tau]} \in D$ as long as $\alpha \in D$. Furthermore, since any nonrandom function factors in and out of δ, we have

$$\delta(\alpha I_{[0,\tau]}) = (\delta\alpha) I_{[0,\tau]};$$

so that $\alpha I_{[0,\tau]} \in \overline{D}$ here as long as $\alpha \in \overline{D}$. Even more, there is consistency here, and $\int \alpha I_{[0,\tau]}\, d\theta$ (resp. $\overline{\int} \alpha I_{[0,\tau]}\, d\theta$) coincides with $\int \alpha|_{[0,\tau]}\, d\theta$ (resp. $\overline{\int} \alpha|_{[0,\tau]}\, d\theta$) when we work on Wiener space $(C_0[0, \tau], \mathscr{B}, \mathscr{W})$.

The upshot of this is that if $\alpha \in D$ and if $\alpha(t)$ is measurable w.r.t. $\mathscr{M}^\tau$, $0 \le t \le \tau$, then we can define $\int_0^\tau \alpha\, d\theta$. In particular, if we want to define $\int_0^\tau \alpha\, d\theta$ for *every* $\tau \in J$, then under these conditions α would have to be progressively measurable. To get around this we study two-parameter processes γ: $J^2 \to L^2(\Theta)$ for which $\gamma(s, \tau)$ is measurable w.r.t. $\mathscr{M}^\tau$, $s \le \tau$. Then we can talk about $\int_0^\tau \gamma(s, \tau)\, d\theta(s)$ for every $\tau \in J$. This is precisely the set-up we were led to in [2]–[4] when inverting the Itô–Volterra integral equation.

The general condition on α: $J \to L^2(\Theta)$ for which the Sk-integral $\int \alpha I_{[0,\tau]}\, d\theta$ can be used to define a process $\tau \mapsto \int_0^\tau \alpha\, d\theta$ from $J \to L^2(\Theta)$, is fairly complicated. For example, if we want this process to be an element of D (so that we can further integrate it), then the condition on α [in terms of its expansion (2.3)] is $\Sigma(n+2)!\|\tilde{g}_n\|^2_{L^2(J^{n+2})} < \infty$, where $g_n(\cdot, t, \tau) = f_n(\cdot, t) I_{[0,\tau]}(t)$; and if we want this process to be an element of $L^2(J \times \Theta)$, then the condition on α is $\Sigma(n+1)!\|h_n\|^2_{L^2(J^{n+2})} < \infty$, where $h_n(\cdot, \tau) = \tilde{H}_{n,\tau}$, $H_{n,\tau}(\cdot, t) = f(\cdot, t) I_{[0,\tau]}(t)$. (The g_n are also symmetrized in τ, the h_n are not.) Using the projection property of symmetrization it is easily seen that this last condition is equivalent to

$$\sum (n+1)! \left\| \tilde{f}_n(\cdot, t) f_n(\cdot, t)(M - t) \right\|^2_{L^2(J^{n+1})} < \infty. \tag{3.9}$$

It follows from (3.8) that if $\alpha \in L^2(J \times \Theta)$ is progressively measurable, then $\int_0^\tau \alpha\, d\theta$ is defined for every $\tau \in J$, and the process $\tau \mapsto \int_0^\tau \alpha\, d\theta$ is also progressively measurable. The condition for this process to be in $L^2(J \times \Theta)$ is then simply the condition that the process $t \mapsto \alpha(t)\sqrt{M - t}$ be in $L^2(J \times \Theta)$; i.e.,

$$\sum n! \left\| f_n(\cdot, t)\sqrt{M - t} \right\|^2_{L^2(J^{n+1})} < \infty. \tag{3.10}$$

Using (2.7) one sees that (3.9) and (3.10) are equivalent here.

A similar problem arises when we try to introduce $\int_\tau^M \alpha\, d\theta$, $\alpha \in D$. In terms of the expansion (2.3), we could require that the kernels $f_n(\cdot, t)$ be all supported on $[\tau, M]^n$, $\forall\, t$; and then it would follow that $\alpha I_{[\tau, M]} \in D$. This would effectively be setting up the Sk-integral on Wiener space $(C_0[\tau, M], \mathscr{B}, \mathscr{W})$. However, this is too restrictive, as it would automatically rule out progressively measurable integrands, α: $J \to L^2(\Theta)$. Instead, we can simply define $\int_\tau^M \alpha\, d\theta := \int \alpha\, d\theta - \int_0^\tau \alpha\, d\theta$ whenever α is such that $\alpha(t)$ is measurable w.r.t. $\mathscr{M}^\tau$, $0 \le t \le \tau$. In fact, this is equivalent to defining $\int_\tau^M \alpha\, d\theta := \int \alpha I_{[\tau, M]}\, d\theta$, since in this case, on account of (3.8),

$$\widetilde{f_n(\cdot, t) I_{[\tau, M]}(t)} = \tilde{f}_n(\cdot, t)\left[1 - I_{[0,\tau]^{n+1}}(\cdot, t)\right], \qquad \forall\, n. \tag{3.11}$$

What we conclude from the above discussion is that for $\alpha \in D$ (resp. $\alpha \in \overline{D}$), $\int_\eta^\tau \alpha\, d\theta$ (resp. $\bar{\int}_\eta^\tau \alpha\, d\theta$) can be defined through $\int \alpha I_{[\eta,\tau]} d\theta$ (resp. $\bar{\int} \alpha I_{[\eta,\tau]}\, d\theta$) as long as $\alpha(t)$ is measurable w.r.t. $\mathscr{M}^\tau$, $\eta \le t \le \tau$. Similarly, this definition is valid if $\alpha(t)$ is measurable w.r.t. $\mathscr{M}_\eta$, $\eta \le t \le \tau$, as can be demonstrated by running the above discussion in reverse. If $\alpha(t)$ is actually measurable w.r.t. $\mathscr{M}_\eta^\tau$, $\eta \le t \le \tau$, then, in fact, this definition coincides with $\int \alpha|_{[\eta,\tau]}\, d\theta$ (resp. $\bar{\int} \alpha|_{[\eta,\tau]}\, d\theta$) on Wiener space $(C_0[\eta, \tau], \mathscr{B}, \mathscr{W})$. Of course, the condition that $\alpha(t)$ be measurable w.r.t. $\mathscr{M}^\tau$, $\eta \le t \le \tau$, or measurable w.r.t. $\mathscr{M}_\eta$, $\eta \le t \le \tau$, is not the most general condition for defining $\int_\eta^\tau \alpha\, d\theta$, $\alpha \in D$. What one really wants is

$$\sum (n+1)! \left\| \widetilde{f_n(\cdot, t) I_{[\eta,\tau]}(t)} \right\|^2_{L^2(J^{n+1})} < \infty,$$

in terms of the expansion (2.3).

THEOREM VI. *Let γ: $J^2 \to L^2(\Theta)$ be such that $\gamma(s,t)$ is measurable w.r.t. $\mathscr{M}^t$, $s \leq t$, and $\gamma(\cdot, t) \in D$, $\forall\ t$, and $\gamma(s, \cdot) \in D(\delta)$, $\forall\ s$. If the process $s \mapsto \delta_t \gamma(s,t)$ is in D, then $\int_0^t \gamma(s,t)\, d\theta(s)$ is in $D(\delta)$, and*

$$\delta_t \int_0^t \gamma(s,t)\, d\theta(s) = \int_0^t \delta_t \gamma(s,t)\, d\theta(s). \tag{3.12}$$

PROOF. This follows at once by applying Theorem III to $\gamma(s,t) I_{[0,t]}(s)$. □

REMARK. Again, if we replace D by $\overline{D}$ in the hypotheses of Theorem VI, then, on account of (3.5),

$$\delta_t \overline{\int_0^t \gamma(s,t)\, d\theta(s)} = \overline{\int_0^t} \delta_t \gamma(s,t)\, d\theta(s). \tag{3.13}$$

4. The δ-operator. We recall now the Malliavin $\mathscr{L}$-operator $\mathscr{L}$: dom($\mathscr{L}$) $\subset L^2(\Theta) \to L^2(\Theta)$, defined in terms of the chaos expansion $L^2(\Theta) = \oplus_0^\infty Z^{(n)}$ by $\mathscr{L}X := -(n/2)X$, $\forall\ X \in Z^{(n)}$. (The subspaces $Z^{(n)}$ are defined above in Section 2.) Thus if $Y \in L^2(\Theta)$ has the expansion

$$Y = g_0 + \sum \int_{J^n} g_n\, d\theta^{(n)},$$

then $Y \in$ dom($\mathscr{L}$) if and only if $\sum n^2 n! \|g_n\|^2_{L^2(J^n)} < \infty$; and

$$\mathscr{L}Y := -\tfrac{1}{2} \sum n \int_{J^n} g_n\, d\theta^{(n)}.$$

In terms of the H-Frechét derivative operator $\mathscr{D}$ introduced above in Section 2, $\mathscr{L} = -\frac{1}{2}\mathscr{D}^*\mathscr{D}$. See Ikeda and Watanabe [6] and Stroock [13] for a discussion of this operator.

It follows from (2.6) that if

$$\sum n^3 n! \|\tilde{f}_n\|^2_{L^2(J^{n+1})} < \infty, \tag{4.1}$$

then

$$\mathscr{L} \int \alpha\, d\theta - \int \left(\mathscr{L}\alpha - \tfrac{1}{2}\alpha \right) d\theta = 0 \tag{4.2}$$

(and, of course, both terms above are defined). This is essentially the identity [cf. (2.10)]

$$\begin{aligned} &\int_{\mathbb{R}^n} \langle \nabla \mathscr{G}\phi(x), \psi(x) \rangle e^{-|x|^2/2}\, dx \\ &\quad = \int_{\mathbb{R}^n} \langle \nabla \phi(x), \mathscr{G}\psi(x) - \tfrac{1}{2}\psi(x) \rangle e^{-|x|^2/2}\, dx, \end{aligned} \tag{4.3}$$

$\psi = (\psi_1, \ldots, \psi_n)$ for $\phi \in C_0^3(\mathbb{R}^n)$ and $\psi_1, \ldots, \psi_n \in C_0^2(\mathbb{R}^n)$, where $\mathscr{G}$ is the Ornstein–Uhlenbeck generator

$$\mathscr{G} = \frac{1}{2} \sum \left(\frac{\partial^2}{\partial x_i^2} - x_i \frac{\partial}{\partial x_i} \right), \tag{4.4}$$

and $\mathscr{G}\psi := (\mathscr{G}\psi_1, \ldots, \mathscr{G}\psi_n)$. Similarly, it follows from (2.9) that if, in addition to (4.1),

$$\sum n^3 n! \left\| \int_J f_n(\cdot, t+, t) \right\|^2_{L^2(J^{n-1})} < \infty, \tag{4.5}$$

then

$$\mathscr{L} \overline{\int} \alpha \, d\theta - \overline{\int} \left(\mathscr{L}\alpha - \tfrac{1}{2}\alpha \right) d\theta = \int_J \delta\alpha \tag{4.6}$$

(and again all terms above are defined). This is essentially the identity

$$\mathscr{G}\langle x, \psi(x) \rangle - \langle x, \mathscr{G}\psi(x) - \tfrac{1}{2}\psi(x) \rangle = \operatorname{div} \psi(x), \tag{4.7}$$

for $\psi_1, \ldots, \psi_n \in C^2(\mathbb{R}^n)$. Incidentally, using (2.10) and (4.7) we see that (4.3) really amounts to the identity

$$\mathscr{G} \operatorname{div} = \operatorname{div} \mathscr{G} + \tfrac{1}{2} \operatorname{div}. \tag{4.8}$$

Using the framework of the Malliavin calculus, it is very straightforward to evaluate the action of δ on a wide variety of processes. The basic identity here is

$$\int_J \delta\alpha = \lim \sum \partial_{h_i} \alpha(t_i), \tag{4.9}$$

where $h_i(t) = \int_0^t I_{[t_i, t_{i+1})}$, and the limit is taken in $L^2(\Theta)$ as the mesh size, $\max(t_{i+1} - t_i)$, of the partition $0 = t_0 \leq t_1 \leq \cdots \leq t_m = M$ goes to zero. This was proved above in Section 2 when we established (2.14), (2.15) and (2.17).

Let $\alpha(t) = \phi(\beta_1(t), \ldots, \beta_k(t))$, where $\phi \in C_b^1(\mathbb{R}^k)$ and $\beta_1, \ldots, \beta_k \in D(\delta)$. Then it follows from (4.9) that if each $\delta\beta_i$ is Bochner integrable, $\int_J \delta\beta_i \in L^2(\Theta)$, then

$$\int_J \delta\alpha = \int_J \sum \frac{\partial \phi}{\partial x_i} (\beta_1(t), \ldots, \beta_k(t)) \, \delta\beta_i(t) \, dt.$$

Since nonrandom functions factor in and out of δ, we can replace α with $\alpha\lambda$ for any test function $\lambda \in C_b(J)$; and in this way conclude that

$$\delta\phi(\beta_1(t), \ldots, \beta_k(t)) = \sum \frac{\partial \phi}{\partial x_i} (\beta_1(t), \ldots, \beta_k(t)) \delta\beta_i(t), \tag{4.10}$$

for a.e. $t \in J$. In particular, if $\alpha(t) = \phi(\beta(t), \theta(T_1), \ldots, \theta(T_k))$, where $\phi = \phi(x, y_1, \ldots, y_k) \in C(\mathbb{R}^{k+1})$ is differentiable in $y_1, \ldots, y_k$ with bounded derivatives, β is progressively measurable and $T_1, \ldots, T_k \in J$, then

$$\delta\alpha(t) = \sum \frac{\partial \phi}{\partial y_i} (\beta(t), \theta(T_1), \ldots, \theta(T_k)) I_{[0, T_i]}(t),$$

for a.e. $t \in J$. Still more specifically,

$$\delta\psi(\theta(t), \theta(M) - \theta(t)) = \frac{\partial\psi}{\partial y}(\theta(t), \theta(M) - \theta(t)),$$

for a.e. $t \in J$, a result which was mentioned in [4].

In follow-up papers I will try to further develop these extensions of Itô's integral, concentrating on their distribution, support and structure as semimartingales. I will also examine the action of Malliavin's covariance operator on them, and the calculus for processes

$$\xi(t) = \xi_0 + \int_0^t \alpha\, d\theta + \int_0^t \beta.$$

Acknowledgments. Finally, I would like to acknowledge the very helpful conversations I had with L. Gross and D. Stroock at the IMA in Minneapolis, and with M. Zakai at the Technion.

REFERENCES

[1] BERGER, M. A. and MIZEL, V. J. (1978). A Fubini theorem for iterated stochastic integrals. *Bull. Amer. Math. Soc.* **84** 159–160.

[2] BERGER, M. A. and MIZEL, V. J. (1979). Theorems of Fubini type for iterated stochastic integrals. *Trans. Amer. Math. Soc.* **252** 249–274.

[3] BERGER, M. A. and MIZEL, V. J. (1980). Volterra equations with Itô integrals. I, II. *J. Integral Equations* **2** 187–245, 319–337.

[4] BERGER, M. A. and MIZEL, V. J. (1982). An extension of the stochastic integral. *Ann. Probab.* **10** 435–450.

[5] GAVEAU, B. and TRAUBER, P. (1982). L'integrale stochastique comme opérateur de divergence dans l'espace fonctionnel. *J. Funct. Anal.* **46** 230–238.

[6] IKEDA, N. and WATANABE, S. (1983). An introduction to Malliavin's calculus. In *Stochastic Analysis, Proc. Taniguchi Internat. Symp., 1982.* 1–52. North-Holland, Amsterdam.

[7] ITÔ, K. (1951). Multiple Wiener integral. *J. Math. Soc. Japan* **3** 157–169.

[8] ITÔ, K. (1976). Extension of stochastic integrals. In *Proc. Internat. Symp. SDE, Kyoto.* 95–109.

[9] NUALART, D. and ZAKAI, M. (1986). Generalized stochastic integrals and the Malliavin calculus. *Probab. Theory Related Fields* **73** 255–280.

[10] OGAWA, S. (1979). Sur le produit direct du bruit blanc par lui-même. *C. R. Acad. Sci. Paris Sér. A-B* **288** 359–362.

[11] OGAWA, S. (1984). Une remarque sur l'approximation de l'intégrale stochastique du type noncausal par une suite des intégrales de Stieltjes. *Tôhoku Math. J.* **36** 41–48.

[12] SKOROHOD, A. V. (1975). On a generalization of a stochastic integral. *Theory Probab. Appl.* **20** 219–233.

[13] STROOCK, D. W. (1981). The Malliavin calculus, a functional analytic approach. *J. Funct. Anal.* **44** 212–257.

DEPARTMENT OF MATHEMATICS
THE WEIZMANN INSTITUTE OF SCIENCE
REHOVOT 76100
ISRAEL