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A GENERALIZATION OF KOLMOGOROV'S EXTENSION THEOREM AND AN APPLICATION TO THE CONSTRUCTION OF STOCHASTIC PROCESSES WITH RANDOM TIME DOMAINS

BY K. Y. HU

National University of Singapore

Kolmogorov's extension theorem is generalized so that the time domain is a random set. This is applied to the construction of stochastic processes with random time domains, generalizing certain results of Dynkin and Kuznetsov.

1. Introduction. Let T be a set, $\mathbf{F}(T)$ be the set of all nonempty finite subsets of T, and S_t be a complete separable metric space for each $t \in T$. For any $F \in \mathbf{F}(T)$, let S_F be the product set $\prod\{S_t: t \in F\}$, $\mathscr{B}_F$ be the Borel σ-algebra in S_F, and π_F: $S_T \to S_F$ be the projection map. Kolmogorov's extension theorem ([2], [5]) says that if P_F is a probability measure on $(S_T, \pi_F^{-1}[\mathscr{B}_F])$ for each $F \in \mathbf{F}(T)$ and if $\{P_F: F \in \mathbf{F}(T)\}$ satisfies the consistency condition: $A \in \pi_F^{-1}[\mathscr{B}_F] \cap \pi_{F'}^{-1}[\mathscr{B}_{F'}] \Rightarrow P_F(A) = P_{F'}(A)$, then there is a unique probability measure P on $(S_T, \mathscr{B}_T)$ such that $P|_{\pi_F^{-1}[\mathscr{B}_F]} = P_F$ for each $F \in \mathbf{F}(T)$, where $\mathscr{B}_T$ is the σ-algebra in S_T generated by $\bigcup\{\pi_F^{-1}[\mathscr{B}_F]: F \in \mathbf{F}(T)\}$. In this way Kolmogorov's theorem randomizes the graph of a function in the function space S_T, which consists of functions with the same domain T. The first objective of this paper is to extend Kolmogorov's theorem so that the domain of a function is also randomized. This is done in Sections 2 and 3.

The second objective is to introduce an application of this extension theorem to the construction of stochastic processes with random time domains. We formulate an easily recognizable theorem (Theorem 1′) which generalizes certain results of [1] and [3].

2. Preliminaries and notation. Since we will have to consider σ-algebras of sets of sets, the notation will be somewhat complicated. As far as the index set T is concerned, we shall adopt the following convention: We shall use ordinary letters (like D, E, F, etc.) to denote subsets of T, boldface letters (like $\mathbf{T}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$, etc.) to denote sets of subsets of T and script letters (like $\mathscr{D}$) to denote sets of sets of subsets of T.

Let T be a nonempty set and $\mathbf{T}$ be a set of subsets of T. For each $t \in T$, let $(S_t, \mathscr{B}_t)$ be a standard space (i.e., a measurable space which is isomorphic to a Borel subset of a complete separable metric space). For any $E \subset T$, $\mathbf{F}(E)$ denotes the set of all nonempty finite subsets of E, and S_E denotes the product set $\prod\{S_t: t \in E\}$. If $F_0 \in \mathbf{F}(E)$, we let $\mathbf{F}_{F_0}(E) = \{F \in \mathbf{F}(E): F \supset F_0\}$. For any two subsets $E, E' \subset T$ for which $E \subset E'$, let $\pi_{E', E}$: $S_{E'} \to S_E$ such that

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$(\pi_{E',E}(x))_t = x_t$ for all $t \in E$. Φ denotes $\{(D, x)\colon D \in \mathbf{T},\ x \in S_D\}$. For any $F \in \mathbf{F}(T)$, let $\mathbf{T}_F = \{D \in \mathbf{T}\colon D \supset F\}$, $\Phi_F = \{(D, x) \in \Phi\colon D \in \mathbf{T}_F\}$, $\mathscr{D}_F$ be a σ-algebra in $\mathbf{T}_F$ and $\psi_F\colon \Phi_F \to \mathbf{T}_F \times S_F$ be defined by $\psi_F(D, x) = (D, \pi_{D,F}(x))$. Let $\mathscr{D} = \bigcup\{\mathscr{D}_F\colon F \in \mathbf{F}(T)\}$. For any class $\mathscr{E}$ of subsets of a set $\mathbf{Y}$, let $\mathscr{S}(\mathscr{E})$ be the σ-ring in $\mathbf{Y}$ generated by $\mathscr{E}$. If $\mathscr{E}$ is a σ-ring in $\mathbf{Y}$ and $\mathbf{D} \in \mathscr{E}$, we let $\mathscr{E}|_{\mathbf{D}}$ denote the set $\{\mathbf{E} \cap \mathbf{D}\colon \mathbf{E} \in \mathscr{E}\}$. A nonempty class $\mathscr{E}$ of subsets of a set $\mathbf{Y}$ is called a *pre-ring* if the intersection of any two members of $\mathscr{E}$ is a member and the difference of any two members is a finite pairwise disjoint union of members.

Assume that $\{\mathscr{D}_F\colon F \in \mathbf{F}(T)\}$ satisfies the following condition:

(C_1) For all $F, F' \in \mathbf{F}(T)$, $\mathbf{D} \cap \mathbf{D}' \in \mathscr{D}_F \cap \mathscr{D}_{F'}$ for all $\mathbf{D} \in \mathscr{D}_F$ and $\mathbf{D}' \in \mathscr{D}_{F'}$.

PROPOSITION 1. (C_1) *holds if and only if* $\mathscr{S}(\mathscr{D})|_{\mathbf{T}_F} = \mathscr{D}_F$ *for all* $F \in \mathbf{F}(T)$.

PROOF. Assume (C_1). For any $F \in \mathbf{F}(T)$, it is clear that $\mathscr{S}(\mathscr{D})|_{\mathbf{T}_F} \supset \mathscr{D}_F$. $\{\mathbf{E} \in \mathscr{S}(\mathscr{D})\colon \mathbf{E} \cap \mathbf{T}_F \in \mathscr{D}_F\}$ is a σ-ring which contains $\mathscr{D}$, and hence equals $\mathscr{S}(\mathscr{D})$. Conversely, let $F, F' \in \mathbf{F}(T)$, $\mathbf{D} \in \mathscr{D}_F$, $\mathbf{D}' \in \mathscr{D}_{F'}$ be arbitrary. Then $\mathbf{D} = \mathbf{E} \cap \mathbf{T}_F$, $\mathbf{D}' = \mathbf{E}' \cap \mathbf{T}_{F'}$ for some $\mathbf{E}, \mathbf{E}' \in \mathscr{S}(\mathscr{D})$. $\mathbf{D} \cap \mathbf{D}' = (\mathbf{E} \cap \mathbf{T}_F \cap \mathbf{E}') \cap \mathbf{T}_{F'}$ and hence $\mathbf{D} \cap \mathbf{D}' \in \mathscr{D}_{F'}$. Similarly, $\mathbf{D} \cap \mathbf{D}' \in \mathscr{D}_F$. $\square$

Here are some consequences of (C_1).

PROPOSITION 2. *For all* $F, F' \in \mathbf{F}(T)$, $\mathscr{D}_F \cap \mathscr{D}_{F'} = \mathscr{D}_{F \cup F'}$.

PROPOSITION 3. *For all* F, $F' \in \mathbf{F}(T)$, $\mathbf{D} \setminus \mathbf{D}' \in \mathscr{D}_F$ *for all* $\mathbf{D} \in \mathscr{D}_F$ *and* $\mathbf{D}' \in \mathscr{D}_{F'}$.

For any $F \in \mathbf{F}(T)$, let $\mathscr{B}_F$ denote the product σ-algebra $\otimes\{\mathscr{B}_t\colon t \in F\}$ and let $\mathscr{A}_F = \{\psi_F^{-1}(\mathbf{D} \times \Gamma)\colon \mathbf{D} \in \mathscr{D}_F,\ \Gamma \in \mathscr{B}_F\}$. Let $\mathscr{A} = \bigcup\{\mathscr{A}_F\colon F \in \mathbf{F}(T)\}$. From ($C_1$) and Proposition 3 we deduce

PROPOSITION 4. $\mathscr{D}$ *is a pre-ring in* $\mathbf{T}$ *and* $\mathscr{A}$ *is a pre-ring in* Φ.

We shall need the following.

LEMMA 1. *Let λ be a σ-finite measure on a pre-ring $\mathscr{A}$ in a set. Then λ can be extended in a unique manner to a measure $\bar{\lambda}$ on the σ-ring $\mathscr{S}(\mathscr{A})$ generated by $\mathscr{A}$; the measure $\bar{\lambda}$ is σ-finite.*

3. The extension theorem. Suppose for each $F \in \mathbf{F}(T)$, P_F is a measure on $(\Phi_F, \mathscr{S}(\mathscr{A}_F))$ and assume that $\{P_F\colon F \in \mathbf{F}(T)\}$ satisfies the consistency condition: $A \in \mathscr{A}_F \cap \mathscr{A}_{F'} \Rightarrow P_F(A) = P_{F'}(A)$ for all $F, F' \in \mathbf{F}(T)$. Then there is defined unambiguously a map $P\colon \mathscr{A} \to [0, \infty]$ such that $P(A) = P_F(A)$ if $A \in \mathscr{A}_F$.

Let $F \in \mathbf{F}(T)$. A countable collection $\{\Gamma^{(k)}\colon k = 1, 2, \ldots\}$ is called a *proper decomposition of* $(S_F, \mathscr{B}_F)$ *with respect to* P_F if $\Gamma^{(k)} \in \mathscr{B}_F$ for each k, $\Gamma^{(k)} \cap \Gamma^{(k')} = \varnothing$ for $k \neq k'$, $\bigcup\{\Gamma^{(k)}\colon k = 1, 2, \ldots\} = S_F$, and $P_F(\psi_F^{-1}(\cdot \times \Gamma^{(k)}))$ is a σ-finite measure on $(\mathbf{T}_F, \mathscr{D}_F)$ for each k. If $\{\Gamma^{(k)}\colon k = 1, 2, \ldots\}$ is a proper decomposition of $(S_F, \mathscr{B}_F)$ with respect to P_F and $F \subset F'$, then $\{\pi_{F',F}^{-1}(\Gamma^{(k)})\colon k = 1, 2, \ldots\}$ is a proper decomposition of $(S_{F'}, \mathscr{B}_{F'})$ with respect to $P_{F'}$. Hence to ensure that $(S_F, \mathscr{B}_F)$ has a proper decomposition for every $F \in \mathbf{F}(T)$ it is necessary and sufficient to require that $(S_t, \mathscr{B}_t)$ has a proper decomposition with respect to $P_{\{t\}}$ for every $t \in T$, which we shall now assume.

Let $F_0 \in \mathbf{F}(T)$ be arbitrarily fixed, and let $\{\Gamma^{(k)}: k = 1, 2, \dots\}$ be a proper decomposition of $(S_{F_0}, \mathscr{B}_{F_0})$ with respect to P_{F_0}. For any $F \in \mathbf{F}_{F_0}(T)$, $\{\pi_{F,F_0}^{-1}(\Gamma^{(k)}): k = 1, 2, \dots\}$ is a proper decomposition of $(S_F, \mathscr{B}_F)$ with respect to P_F and $\mu_F^{(k)} = P_F(\psi_F^{-1}(\cdot \times \pi_{F,F_0}^{-1}(\Gamma^{(k)})))$ is a σ-finite measure for each k. For any k, $\{\mu_F^{(k)}: F \in \mathbf{F}_{F_0}(T)\}$ satisfies the consistency condition: $\mathbf{D} \in \mathscr{D}_F \cap \mathscr{D}_{F'} \Rightarrow \mu_F^{(k)}(\mathbf{D}) = \mu_{F'}^{(k)}(\mathbf{D})$ for all $F, F' \in \mathbf{F}_{F_0}(T)$. Hence there is defined unambiguously a map $\mu^{(k)}: \bigcup\{\mathscr{D}_F: F \in \mathbf{F}_{F_0}(T)\} \to [0, \infty]$ such that $\mu^{(k)}(\mathbf{D}) = \mu_F^{(k)}(\mathbf{D})$ if $\mathbf{D} \in \mathscr{D}_F$. Since by (C_1) $\bigcup\{\mathscr{D}_F: F \in \mathbf{F}_{F_0}(T)\} = \mathscr{D}_{F_0}$, we can regard $\mu^{(k)} = \mu_{F_0}^{(k)}$. We shall let $\mu_{F_0} = \sum_{k=1}^{\infty} \mu_{F_0}^{(k)} = P_{F_0}(\psi_{F_0}^{-1}(\cdot \times S_{F_0}))$ as a measure on $(\mathbf{T}_{F_0}, \mathscr{D}_{F_0})$.

From (C_1) it is easy to deduce

PROPOSITION 5. *For any* $k = 1, 2, \dots$, $F \in \mathbf{F}_{F_0}(T)$ *and* $\mathbf{D} \in \mathscr{D}_F$, *we have* (i) $\mathbf{D} \in \mathscr{D}_{F_0}$, *and* (ii) $(\mathbf{D}, \mathscr{D}_F|_{\mathbf{D}}, \mu_F^{(k)})$ *and* $(\mathbf{D}, \mathscr{D}_{F_0}|_{\mathbf{D}}, \mu_{F_0}^{(k)})$ *are identical measure spaces.*

For each $k = 1, 2, \dots$ and each $F \in \mathbf{F}_{F_0}(T)$, let $\nu_F^{(k)}(\Gamma|D)$ $(\Gamma \in \mathscr{B}_F,\ D \in \mathbf{T}_F)$ take values in $[0, 1]$ such that

(i) for each $D \in \mathbf{T}_F$, $\nu_F^{(k)}(\cdot|D)$ is a probability measure on $(S_F, \mathscr{B}_F)$;

(ii) for each $\Gamma \in \mathscr{B}_F$, $\nu_F^{(k)}(\Gamma|\cdot)$ is a measurable function on $(\Gamma_F, \mathscr{D}_F)$; and

(iii) for each $\Gamma \in \mathscr{B}_F$ and each $\mathbf{D} \in \mathscr{D}_F$, $P_F(\psi_F^{-1}(\mathbf{D} \times (\Gamma \cap \pi_{F,F_0}^{-1}(\Gamma^{(k)})))) = \int_{\mathbf{D}} \nu_F^{(k)}(\Gamma|D)\mu_F^{(k)}(dD)$.

The existence of $\nu_F^{(k)}$ can be proved in a similar way as we prove the existence of a regular conditional probability distribution.

LEMMA 2. *For each* $k = 1, 2, \dots$ *and each* $F, F' \in \mathbf{F}_{F_0}(T)$, *there exists* $\mathbf{E}_{F,F'}^{(k)} \in \mathscr{D}_{F \cup F'}$ *such that* $\mu_{F_0}^{(k)}(\mathbf{E}_{F,F'}^{(k)}) = 0$ *and for all* $D \in \mathbf{T}_{F \cup F'} \setminus \mathbf{E}_{F,F'}^{(k)}$, $\nu_F^{(k)}(\pi_{F,F\cap F'}^{-1}(\Gamma)|D) = \nu_{F'}^{(k)}(\pi_{F',F\cap F'}^{-1}(\Gamma)|D)$ *for all* $\Gamma \in \mathscr{B}_{F \cap F'}$.

PROOF. For any $\Gamma \in \mathscr{B}_{F\cap F'}$ and any $\mathbf{D} \in \mathscr{D}_{F \cup F'} = \mathscr{D}_F \cap \mathscr{D}_{F'}$, we have

$$\begin{aligned}
&\int_{\mathbf{D}} \nu_F^{(k)}\left(\pi_{F,F\cap F'}^{-1}(\Gamma)|D\right)\mu_{F_0}^{(k)}(dD) \\
&= \int_{\mathbf{D}} \nu_F^{(k)}\left(\pi_{F,F\cap F'}^{-1}(\Gamma)|D\right)\mu_F^{(k)}(dD) \\
&= P_F\left(\psi_F^{-1}\left(\mathbf{D} \times \left(\pi_{F,F\cap F'}^{-1}(\Gamma) \cap \pi_{F,F_0}^{-1}(\Gamma^{(k)})\right)\right)\right) \\
&= P_{F'}\left(\psi_{F'}^{-1}\left(\mathbf{D} \times \left(\pi_{F',F\cap F'}^{-1}(\Gamma) \cap \pi_{F',F_0}^{-1}(\Gamma^{(k)})\right)\right)\right) \\
&= \int_{\mathbf{D}} \nu_{F'}^{(k)}\left(\pi_{F',F\cap F'}^{-1}(\Gamma)|D\right)\nu_{F'}^{(k)}(dD) \\
&= \int_{\mathbf{D}} \mu_{F'}^{(k)}\left(\pi_{F',F\cap F'}^{-1}(\Gamma)|D\right)\mu_{F_0}^{(k)}(dD).
\end{aligned}$$

Hence for arbitrary $\Gamma \in \mathscr{B}_{F \cap F'}$, $\nu_F^{(k)}(\pi_{F, F \cap F'}^{-1}(\Gamma)|D) = \nu_{F'}^{(k)}(\pi_{F', F \cap F'}^{-1}(\Gamma)|D)$ for almost all D in $(\mathbf{T}_{F \cup F'}, \mathscr{D}_{F \cup F'}, \mu_{F_0}^{(k)})$. Since $\mathscr{B}_{F \cap F'}$ is countably generated and each of $\nu_F^{(k)}(\pi_{F, F \cap F'}^{-1}(\cdot)|D)$ and $\nu_{F'}^{(k)}(\pi_{F', F \cap F'}^{-1}(\cdot)|D)$ is a finite measure, the result follows. □

PROPOSITION 6. *P is countably additive on $\mathscr{A}$.*

PROOF. Let $I = \{1, 2, \dots\}$ and suppose $\psi_{F_0}^{-1}(\mathbf{D}_0 \times \Gamma_0) = \bigcup\{\psi_{F_i}^{-1}(\mathbf{D}_i \times \Gamma_i): i \in I\}$ is a disjoint union. It is clear that $\mathbf{D}_0 = \bigcup\{\mathbf{D}_i: i \in I\}$ and hence $\mathbf{D}_i = \mathbf{D}_i \cap \mathbf{D}_0 \in \mathscr{D}_{F_i} \cap \mathscr{D}_{F_0} = \mathscr{D}_{F_i \cup F_0}$ and $\psi_{F_i}^{-1}(\mathbf{D}_i \times \Gamma_i) = \psi_{F_i \cup F_0}^{-1}(\mathbf{D}_i \times \pi_{F_i \cup F_0, F_i}^{-1}(\Gamma_i))$. Fix a proper decomposition $\{\Gamma^{(k)}: k = 1, 2, \dots\}$ of $(S_{F_0}, \mathscr{B}_{F_0})$ so that we can apply the results obtained in the earlier part of this section. Let $C = (\bigcup\{F_i: i \in I\}) \cup F_0$. Then $\mathbf{F}_{F_0}(C)$ is countable. For any k, any $F, F' \in \mathbf{F}_{F_0}(C)$, by Lemma 2 there exists $\mathbf{E}_{F, F'}^{(k)} \in \mathscr{D}_{F \cup F'}$ such that $\mu^{(k)}(\mathbf{E}_{F, F'}^{(k)}) = 0$ and for all $D \in \mathbf{T}_{F \cup F'} \setminus \mathbf{E}_{F, F'}^{(k)}$, $\nu_F^{(k)}(\pi_{F, F \cap F'}^{-1}(\Gamma)|D) = \nu_{F'}^{(k)}(\pi_{F', F \cap F'}^{-1}(\Gamma)|D)$ for all $\Gamma \in \mathscr{B}_{F \cap F'}$. We let $\mathbf{E}^{(k)} = \bigcup\{\mathbf{D}_0 \cap \mathbf{E}_{F, F'}^{(k)}: F, F' \in \mathbf{F}_{F_0}(C)\}$. Then $\mathbf{E}^{(k)} \in \mathscr{D}_{F_0}$ and $\mu_{F_0}^{(k)}(\mathbf{E}^{(k)}) = 0$. Let $D \in \mathbf{D}_0 \setminus \mathbf{E}^{(k)}$ be arbitrary. For any $F, F' \in \mathbf{F}_{F_0}(D \cap C)$, $D \in \mathbf{T}_{F \cup F'}$ and hence $D \in \mathbf{T}_{F \cup F'} \setminus \mathbf{E}_{F, F'}^{(k)}$. Moreover, for $F, F' \in \mathbf{F}_{F_0}(D \cap C)$, $\Gamma \in \mathscr{B}_F$, and $\Gamma' \in \mathscr{B}_{F'}$, $\pi_{D \cap C, F}^{-1}(\Gamma) = \pi_{D \cap C, F'}^{-1}(\Gamma') \Rightarrow \Gamma = \pi_{F, F \cap F'}^{-1}(\tilde{\Gamma})$ and $\Gamma' = \pi_{F', F \cap F'}^{-1}(\tilde{\Gamma})$ for some

$$\tilde{\Gamma} \in \mathscr{B}_{F \cap F'} \Rightarrow \nu_F^{(k)}(\Gamma|D) = \nu_F^{(k)}\big(\pi_{F, F \cap F'}^{-1}(\tilde{\Gamma})|D\big) = \nu_{F'}^{(k)}\big(\pi_{F', F \cap F'}^{-1}(\tilde{\Gamma})|D\big) = \nu_{F'}^{(k)}(\Gamma'|D).$$

Hence we can define unambiguously for any $D \in \mathbf{D}_0 \setminus \mathbf{E}^{(k)}$ a map $Q_{D \cap C}^{(k)}$: $\bigcup\{\pi_{D \cap C, F}^{-1}[\mathscr{B}_F]: F \in \mathbf{F}_{F_0}(D \cap C)\} \to [0, 1]$ such that $Q_{D \cap C}^{(k)}(\pi_{D \cap C, F}^{-1}(\Gamma)) = \nu_F^{(k)}(\Gamma|D)$. For each $F \in \mathbf{F}_{F_0}(D \cap C)$, $Q_{D \cap C}^{(k)}$ restricted to $\pi_{D \cap C, F}^{-1}[\mathscr{B}_F]$ is a probability measure. Hence by Kolmogorov's extension theorem $Q_{D \cap C}^{(k)}$ is countably additive. For any $D \in \mathbf{D}_0 \setminus \mathbf{E}^{(k)}$, let $I_D = \{i \in I: D \in \mathbf{D}_i\}$. Then $\{\pi_{D \cap C, F_i \cup F_0}^{-1}(\pi_{F_i \cup F_0, F_i}^{-1}(\Gamma_i)): i \in I_D\}$ is a pairwise disjoint family and

$$\bigcup\Big\{\pi_{D \cap C, F_i \cup F_0}^{-1}\Big(\pi_{F_i \cup F_0, F_i}^{-1}(\Gamma_i)\Big): i \in I_D\Big\} = \pi_{D \cap C, F_0}^{-1}(\Gamma_0).$$

Thus $\sum_{i \in I_D} Q_{D \cap C}^{(k)}(\pi_{D \cap C, F_i \cup F_0}^{-1}(\pi_{F_i \cup F_0, F_i}^{-1}(\Gamma_i))) = Q_{D \cap C}^{(k)}(\pi_{D \cap C, F_0}^{-1}(\Gamma_0))$ which implies that $\sum_{i \in I_D} \nu_{F_i \cup F_0}^{(k)}(\pi_{F_i \cup F_0, F_i}^{-1}(\Gamma_i)|D) = \nu_{F_0}^{(k)}(\Gamma_0|D)$, or $\sum_{i \in I} \chi_{\mathbf{D}_i}(D) \nu_{F_i \cup F_0}^{(k)}(\pi_{F_i \cup F_0, F_i}^{-1}(\Gamma_i)|D) = \nu_{F_0}^{(k)}(\Gamma_0|D)$. By Proposition 5 and the monotone convergence theorem we conclude that

$$\begin{aligned}
&\sum_{i \in I} P\big(\psi_{F_i}^{-1}(\mathbf{D}_i \times \Gamma_i)\big)\\
&\quad = \sum_{i \in I} P_{F_i}\big(\psi_{F_i}^{-1}(\mathbf{D}_i \times \Gamma_i)\big)\\
&\quad = \sum_{i \in I} P_{F_i \cup F_0}\Big(\psi_{F_i \cup F_0}^{-1}\Big(\mathbf{D}_i \times \pi_{F_i \cup F_0, F_i}^{-1}(\Gamma_i)\Big)\Big)\\
&\quad = \sum_{i \in I} \sum_{k=1}^{\infty} P_{F_i \cup F_0}\Big(\psi_{F_i \cup F_0}^{-1}\Big(\mathbf{D}_i \times \Big(\pi_{F_i \cup F_0, F_i}^{-1}(\Gamma_i) \cap \pi_{F_i \cup F_0, F_0}^{-1}(\Gamma^{(k)})\Big)\Big)\Big)
\end{aligned}$$

$$= \sum_{i \in I} \sum_{k=1}^{\infty} \int_{\mathbf{D}_i} \nu^{(k)}_{F_i \cup F_0}\Big(\pi^{-1}_{F_i \cup F_0,\, F_i}(\Gamma_i) | D\Big) \mu^{(k)}_{F_i \cup F_0}(dD)$$

$$= \sum_{i \in I} \sum_{k=1}^{\infty} \int_{\mathbf{D}_i} \nu^{(k)}_{F_i \cup F_0}\Big(\pi^{-1}_{F_i \cup F_0,\, F_i}(\Gamma_i) | D\Big) \mu^{(k)}_{F_0}(dD)$$

$$= \sum_{i \in I} \sum_{k=1}^{\infty} \int_{\mathbf{D}_0} \chi_{\mathbf{D}_i}(D) \nu^{(k)}_{F_i \cup F_0}\Big(\pi^{-1}_{F_i \cup F_0,\, F_i}(\Gamma_i) | D\Big) \mu^{(k)}_{F_0}(dD)$$

$$= \sum_{k=1}^{\infty} \int_{\mathbf{D}_0} \sum_{i \in I} \chi_{\mathbf{D}_i}(D) \nu^{(k)}_{F_i \cup F_0}\Big(\pi^{-1}_{F_i \cup F_0,\, F_i}(\Gamma_i) | D\Big) \mu^{(k)}_{F_0}(dD)$$

$$= \sum_{k=1}^{\infty} \int_{\mathbf{D}_0} \nu^{(k)}_{F_0}(\Gamma_0 | D) \mu^{(k)}_{F_0}(dD)$$

$$= \sum_{k=1}^{\infty} P_{F_0}\Big(\psi^{-1}_{F_0}\big(\mathbf{D}_0 \times (\Gamma_0 \cap \Gamma^{(k)})\big)\Big)$$

$$= P_{F_0}\big(\psi^{-1}_{F_0}(\mathbf{D}_0 \times \Gamma_0)\big) = P\big(\psi^{-1}_{F_0}(\mathbf{D}_0 \times \Gamma_0)\big). \qquad \square$$

We shall prove the following.

THEOREM 1. *Let $\{\mathscr{D}_F\colon F \in \mathbf{F}(T)\}$ satisfy (C_1). Suppose for each $F \in \mathbf{F}(T)$ there is defined on $(\Phi_F, \mathscr{S}(\mathscr{A}_F))$ a measure P_F with respect to which there is a proper decomposition of $(S_F, \mathscr{B}_F)$. Assume that $\{P_F\colon F \in \mathbf{F}(T)\}$ satisfies the consistency condition: $A \in \mathscr{A}_F \cap \mathscr{A}_{F'} \Rightarrow P_F(A) = P_{F'}(A)$ for all $F, F' \in \mathbf{F}(T)$. Then there is a unique σ-finite measure $\bar{P}$ on $(\Phi, \mathscr{S}(\mathscr{A}))$ such that $\bar{P}|_{\mathscr{S}(\mathscr{A}_F)} = P_F$ for all $F \in \mathbf{F}(T)$.*

PROOF. P is countably additive on the pre-ring $\mathscr{A}$ by Proposition 6. It is also σ-finite on $\mathscr{A}$. By Lemma 1 there is a unique σ-finite measure $\bar{P}$ on $(\Phi, \mathscr{S}(\mathscr{A}))$ such that $\bar{P}|_{\mathscr{A}} = P$. For any $F \in \mathbf{F}(T)$ it is clear that $\mathscr{S}(\mathscr{A}) \supset \mathscr{S}(\mathscr{A}_F)$. Hence $\bar{P}|_{\mathscr{S}(\mathscr{A}_F)}$ is a measure on $(\Phi_F, \mathscr{S}(\mathscr{A}_F))$ which extends $P|_{\mathscr{A}_F}$. As $P|_{\mathscr{A}_F}$ is σ-finite the extension is unique and hence $\bar{P}|_{\mathscr{S}(\mathscr{A}_F)} = P_F$. $\square$

We shall furthermore assume that $\{\mathscr{D}_F\colon F \in \mathbf{F}(T)\}$ satisfies

(C_2) $\mathbf{T} = \bigcup_{i=1}^{\infty} \mathbf{D}_i$ where $\mathbf{D}_i \in \mathscr{D}_{F_i}$ for some $F_i \in \mathbf{F}(T)$.

By Proposition 3 we can regard the $\mathbf{D}_i$'s as pairwise disjoint. Moreover, we have the following characterization of (C_2).

PROPOSITION 7. *(C_2) is fulfilled if and only if $\mathbf{T} \in \mathscr{S}(\mathscr{D})$.*

PROOF. Proposition 7 follows from the fact that the class of all subsets of $\mathbf{T}$ that are covered by countably many members of $\mathscr{D}$ is a σ-ring. $\square$

THEOREM 2. *Let the assumptions in Theorem* 1 *be satisfied. If* (C_2) *is also fulfilled for some pairwise disjoint* $\{\mathbf{D}_i\}$, *then* $\Phi \in \mathscr{S}(\mathscr{A})$ *and* $\overline{P}(\Phi) = \sum_{i=1}^{\infty} \mu_{F_i}(\mathbf{D}_i)$.

PROOF. $\Phi = \bigcup_{i=1}^{\infty} \psi_{F_i}^{-1}(\mathbf{D}_i \times S_{F_i}) \in \mathscr{S}(\mathscr{A})$ is a disjoint union. Hence

$$\begin{aligned}\overline{P}(\Phi) &= \sum_{i=1}^{\infty} \overline{P}\big(\psi_{F_i}^{-1}(\mathbf{D}_i \times S_{F_i})\big)\\ &= \sum_{i=1}^{\infty} P_{F_i}\big(\psi_{F_i}^{-1}(\mathbf{D}_i \times S_{F_i})\big)\\ &= \sum_{i=1}^{\infty} \mu_{F_i}(\mathbf{D}_i).\end{aligned}$$ □

REMARK. (i) If $\mathbf{T} = \{T\}$, $\mathscr{D}_F = \{\varnothing, \{T\}\}$ for each $F \in \mathbf{F}(T)$, then (C_1) and (C_2) are satisfied. If each P_F is a probability measure, then each μ_F is a probability measure and $\overline{P}(\Phi) = 1$. In this situation Theorems 1 and 2 reduce to the usual Kolmogorov extension theorem.

(ii) If T is a subset of the real line and $\mathbf{T}$ is a set of intervals, (C_2) is fulfilled because $\mathbf{T} = \bigcup\{\mathbf{T}_{\{r\}}: r \in T, r \text{ is rational}\}$. If T is any countable set, (C_2) is also fulfilled.

4. Construction of a stochastic process with random time domains. Let $T = R$ (the real numbers) and let $\mathbf{T}$ be a set of subsets of T. For each $F \in \mathbf{F}(T)$ suppose $\mathscr{D}_F$ is a σ-algebra in $\mathbf{T}_F$ such that $\{\mathscr{D}_F: F \in \mathbf{F}(T)\}$ satisfies condition (C_1). We define a *stochastic process with random time domains in* $\mathbf{T}$ to be a measurable map $X: (\Omega, \mathscr{G}, P) \to (\Phi, \mathscr{S}(\mathscr{A}))$ for some σ-finite measure space $(\Omega, \mathscr{G}, P)$. For any $F \in \mathbf{F}(T)$, a map $\eta_F: \mathscr{D}_F \times \mathscr{B}_F \to [0, \infty]$ is called a *bimeasure* if for each $\mathbf{D} \in \mathscr{D}_F$, $\eta_F(\mathbf{D}, \cdot)$ is countably additive on $\mathscr{B}_F$ and for each $\Gamma \in \mathscr{B}_F$, $\eta_F(\cdot, \Gamma)$ is countably additive on $\mathscr{D}_F$. Let η_F be a bimeasure. A pairwise disjoint family $\{\Gamma^{(k)}: k = 1, 2, \dots\}$ of members of $\mathscr{B}_F$ is called a *proper decomposition of* $(S_F, \mathscr{B}_F)$ *for* η_F if for each k, $\eta_F(\cdot, \Gamma^{(k)})$ is σ-finite on $\mathscr{D}_F$. A family $\{\eta_F: F \in \mathbf{F}(T)\}$ of bimeasures is said to be consistent if for any $F \subset F'$, $\eta_{F'}(\mathbf{D}', \pi_{F',F}^{-1}(\Gamma)) = \eta_F(\mathbf{D}', \Gamma)$ for all $\mathbf{D}' \in \mathscr{D}_{F'}$ and $\Gamma \in \mathscr{B}_F$. If $\{\eta_F: F \in \mathbf{F}(T)\}$ is consistent and if $\{\Gamma^{(k)}: k = 1, 2, \dots\}$ is a proper decomposition of $(S_F, \mathscr{B}_F)$ for η_F, then $\{\pi_{F',F}^{-1}(\Gamma^{(k)}): k = 1, 2, \dots\}$ is a proper decomposition of $(S_{F'}, \mathscr{B}_{F'})$ for $\eta_{F'}$, for any $F' \supset F$.

LEMMA 3. *Let* η_F *be a bimeasure for which there is a proper decomposition of* $(S_F, \mathscr{B}_F)$. *Then there is a* σ*-finite measure* λ_F *on* $(\mathbf{T}_F \times S_F, \mathscr{D}_F \otimes \mathscr{B}_F)$ *such that* $\lambda_F(\mathbf{D} \times \Gamma) = \eta_F(\mathbf{D}, \Gamma)$ *for all* $\mathbf{D} \in \mathscr{D}_F$ *and* $\Gamma \in \mathscr{B}_F$, λ_F *is uniquely determined by* η_F.

PROOF. Let $\{\Gamma^{(k)}: k = 1, 2, \dots\}$ be a proper decomposition of $(S_F, \mathscr{B}_F)$ for η_F. For each k there is a pairwise disjoint family $\{\mathbf{D}_i: i = 1, 2, \dots\} \subset \mathscr{D}_F$ such that $\bigcup\{\mathbf{D}_i: i = 1, 2, \dots\} = \mathbf{T}_F$ and $\eta_F(\mathbf{D}_i, \Gamma^{(k)}) < \infty$ for all i. Let $\eta_F^{(i,k)}: \mathscr{D}_F \times$

$\mathscr{B}_F \to [0, \infty]$ be defined by $\eta_F^{(i,k)}(\mathbf{D}, \Gamma) = \eta_F(\mathbf{D} \cap \mathbf{D}_i, \Gamma \cap \Gamma^{(k)})$. Then $\eta_F^{(i,k)}$ is a bimeasure with $\eta_F^{(i,k)}(\mathbf{T}_F, S_F) < \infty$. Since $(S_F, \mathscr{B}_F)$ is a standard space, by a theorem due to Morando [4], page 222, there is a unique finite measure $\lambda_F^{(i,k)}$ on $(\mathbf{T}_F \times S_F, \mathscr{D}_F \otimes \mathscr{B}_F)$ such that $\lambda_F^{(i,k)}(\mathbf{D} \times \Gamma) = \eta_F^{(i,k)}(\mathbf{D}, \Gamma)$. We let $\lambda_F = \sum_k \sum_i \lambda_F^{(i,k)}$. Then λ_F is σ-finite measure on $(\mathbf{T}_F \times S_F, \mathscr{D}_F \otimes \mathscr{B}_F)$. For any $\mathbf{D} \in \mathscr{D}_F$ and $\Gamma \in \mathscr{B}_F$,

$$\begin{aligned} \lambda_F(\mathbf{D} \times \Gamma) &= \sum_k \sum_i \lambda_F^{(i,k)}(\mathbf{D} \times \Gamma) \\ &= \sum_k \sum_i \eta_F^{(i,k)}(\mathbf{D}, \Gamma) \\ &= \sum_k \sum_i \eta_F\big(\mathbf{D} \cap \mathbf{D}_i, \Gamma \cap \Gamma^{(k)}\big) \\ &= \eta_F(\mathbf{D}, \Gamma). \end{aligned}$$

The uniqueness is obvious. □

We can prove the following alternate version of Theorem 1.

THEOREM 1′. *Let $\{\mathscr{D}_F\colon F \in \mathbf{F}(T)\}$ satisfy condition* (C_1). *Suppose for each $F \in \mathbf{F}(T)$ there is a bimeasure $\eta_F\colon \mathscr{D}_F \times \mathscr{B}_F \to [0, \infty]$ for which there is a proper decomposition of $(S_F, \mathscr{B}_F)$. Assume that $\{\eta_F\colon F \in \mathbf{F}(T)\}$ satisfies the consistency condition: For all F, $F' \in \mathbf{F}(T)$ such that $F \subset F'$, $\eta_{F'}(\mathbf{D}', \pi_{F',F}^{-1}(\Gamma)) = \eta_F(\mathbf{D}', \Gamma)$ for all $\mathbf{D}' \in \mathscr{D}_{F'}$ and $\Gamma \in \mathscr{B}_F$. Then there is a stochastic process X with random time domains in $\mathbf{T}$ so that for any $F \in \mathbf{F}(T)$, $P(X^{-1}(\psi_F^{-1}(\mathbf{D} \times \Gamma))) = \eta_F(\mathbf{D}, \Gamma)$ for all $\mathbf{D} \in \mathscr{D}_F$ and $\Gamma \in \mathscr{B}_F$.*

PROOF. For any $F \in \mathbf{F}(T)$, let P_F be the measure on $(\Phi_F, \mathscr{S}(\mathscr{A}_F))$ such that $P_F(A) = \lambda_F(\psi_F(A))$ for $A \in \mathscr{S}(\mathscr{A}_F)$. Then any proper decomposition of $(S_F, \mathscr{B}_F)$ for η_F is a proper decomposition of $(S_F, \mathscr{B}_F)$ with respect to P_F. Let us prove the consistency of $\{P_F\colon F \in \mathbf{F}(T)\}$. Suppose $F \cap F' \neq \varnothing$. Then $A \in \mathscr{A}_F \cap \mathscr{A}_{F'} \Rightarrow A = \psi_F^{-1}(\mathbf{D} \times \Gamma) = \psi_{F'}^{-1}(\mathbf{D}' \times \Gamma')$, where $\mathbf{D} = \mathbf{D}' \in \mathscr{D}_F \cap \mathscr{D}_{F'} = \mathscr{D}_{F \cup F'}$ and $\Gamma = \pi_{F, F \cap F'}^{-1}(\tilde{\Gamma})$ and $\Gamma' = \pi_{F', F \cap F'}^{-1}(\tilde{\Gamma})$ for some

$$\begin{aligned} \tilde{\Gamma} \in \mathscr{B}_{F \cap F'} \Rightarrow P_F(A) &= P_F\big(\psi_F^{-1}(\mathbf{D} \times \Gamma)\big) = \lambda_F(\mathbf{D} \times \Gamma) = \eta_F(\mathbf{D}, \Gamma) \\ &= \eta_F\big(\mathbf{D}, \pi_{F, F \cap F'}^{-1}(\tilde{\Gamma})\big) = \eta_{F \cap F'}(\mathbf{D}, \tilde{\Gamma}) = \eta_{F'}\big(\mathbf{D}', \pi_{F', F \cap F'}^{-1}(\tilde{\Gamma})\big) \\ &= \eta_{F'}(\mathbf{D}', \Gamma') = \lambda_{F'}(\mathbf{D}' \times \Gamma') = P_{F'}\big(\psi_{F'}^{-1}(\mathbf{D}' \times \Gamma')\big) = P_{F'}(A). \end{aligned}$$

Suppose $F \cap F' = \varnothing$. Then $A \in \mathscr{A}_F \cap \mathscr{A}_{F'} \Rightarrow P_F(A) = \lambda_F(\mathbf{D} \times S_F) = \eta_F(\mathbf{D}, S_F) = \eta_{F \cup F'}(\mathbf{D}, \pi_{F \cup F', F}^{-1}(S_F)) = \eta_{F \cup F'}(\mathbf{D}', \pi_{F \cup F', F'}^{-1}(S_{F'})) = \eta_{F'}(\mathbf{D}', S_{F'}) = \lambda_{F'}(\mathbf{D}' \times S_{F'}) = P_{F'}(A)$. Hence by Theorem 1 there is a unique σ-finite measure P on $(\Phi, \mathscr{S}(\mathscr{A}))$ such that $P|_{\mathscr{S}(\mathscr{A}_F)} = P_F$ for all $F \in \mathbf{F}(T)$. If $X\colon (\Phi, \mathscr{S}(\mathscr{A}), P) \to (\Phi, \mathscr{S}(\mathscr{A}))$ is the identity map, then $P(X^{-1}(\psi_F^{-1}(\mathbf{D} \times \Gamma))) = \eta_F(\mathbf{D}, \Gamma)$ for all $\mathbf{D} \in \mathscr{D}_F$ and $\Gamma \in \mathscr{B}_F$. □

REMARK. Under consistency it is sufficient to assume that each $(S_t, \mathscr{B}_t)$ has a proper decomposition for $\eta_{\{t\}}$, for each $t \in T$.

5. Relation with a result of Dynkin. Let $T = R$ and $\mathbf{T} = \{(\alpha, \beta): -\infty \leq \alpha < \beta \leq +\infty\}$ be the set of all open intervals. For each $F \in \mathbf{F}(T)$ let $a_F = \inf F$, $b_F = \sup F$, $\iota_F: \mathbf{T}_F \to [-\infty, a_F) \times (b_F, +\infty]$ be the bijection which associates each $(\alpha, \beta) \in \mathbf{T}_F$ with the point (α, β), and $\mathscr{D}_F$ be the σ-algebra $\iota_F^{-1}[\mathscr{B}[-\infty, a_F) \otimes \mathscr{B}(b_F, +\infty]]$. In this situation Dynkin [1] gave the definition of a stochastic process on a random time interval and conditions for its existence. His definition coincides with ours and his result can be derived from Theorem 1′.

Formulated in the framework of this paper, Dynkin's result reads as follows.

THEOREM (Dynkin [1]). *For each $F \in \mathbf{F}(T)$ let m_F be a σ-finite measure on $(S_F, \mathscr{B}_F)$. Then there exists a stochastic process X on a random interval such that $m_F(\Gamma) = P(X^{-1}(\psi_F^{-1}(\mathbf{T}_F \times \Gamma)))$ for all $F \in \mathbf{F}(T)$ and $\Gamma \in \mathscr{B}_F$ if and only if $\{m_F: F \in \mathbf{F}(T)\}$ satisfies the following conditions for all $F, F' \in \mathbf{F}(T)$, $\Gamma \in \mathscr{B}_F$ and $\Gamma' \in \mathscr{B}_{F'}$:*

(i) $m_{FtF'}(\Gamma \times S_t \times \Gamma') = m_{FF'}(\Gamma \times \Gamma')$ *if* $F < t < F'$;
(ii) $m_{sF}(S_s \times \Gamma) \uparrow m_F(\Gamma)$ *as* $s \uparrow F$, *if* $m_F(\Gamma) < \infty$;
(iii) $m_{Fu}(\Gamma \times S_u) \uparrow m_F(\Gamma)$ *as* $u \downarrow F$, *if* $m_F(\Gamma) < \infty$;
(iv) $m_{sF}(S_s \times \Gamma) + m_{Fu}(\Gamma \times S_u) \leq m_F(\Gamma) + m_{sFu}(S_s \times \Gamma \times S_u)$ *if* $s < F < u$.

Here $m_{FtF'}$ denotes $m_{F \cup \{t\} \cup F'}$ and $F < t < F'$ means $s < t < s'$ for all $s \in F$ and $s' \in F'$. Similar meanings are attached to other similar expressions.

PROOF. The necessity was proved in Dynkin [1] but the sufficiency was not proved there. The author was unable to trace Dynkin's papers for a sufficiency proof. Let us prove it by using Theorem 1′. From Proposition 1 we see that condition (C_1) is fulfilled for $\{\mathscr{D}_F: F \in \mathbf{F}(T)\}$. For any $F \in \mathbf{F}(T)$ let $\{\Gamma^{(k)}: k = 1, 2, \dots\}$ be a family of pairwise disjoint members of $\mathscr{B}_F$ such that $\bigcup\{\Gamma^{(k)}: k = 1, 2, \dots\} = S_F$ and $m_F(\Gamma^{(k)}) < \infty$ for each k. For an arbitrary k and any $\Gamma \in \mathscr{B}_F$, the function

$$G(x, y) = \begin{cases} m_{a_F - 1/x F b_F + 1/y}\left(S_{a_F - 1/x} \times (\Gamma \cap \Gamma^{(k)}) \times S_{b_F + 1/y}\right), & \text{if } x, y > 0, \\ 0, & \text{if } x \leq 0 \text{ or } y \leq 0, \end{cases}$$

is a two-dimensional distribution function by (i)–(iv), except for the normalization condition: $\lim_{x, y \to \infty} G(x, y) = m_F(\Gamma \cap \Gamma^{(k)})$ which may not be 1. It defines a finite measure $\zeta_F^{(k)}(\cdot, \Gamma)$ on $([-\infty, a_F) \times (b_F, +\infty], \mathscr{B}[-\infty, a_F) \otimes \mathscr{B}(b_F, +\infty])$ such that $\zeta_F^{(k)}([-\infty, s) \times (u, +\infty], \Gamma) = m_{sFu}(\pi_{sFu, F}^{-1}(\Gamma \cap \Gamma^{(k)}))$ if $s \leq F \leq u$. For each $B \in \mathscr{B}[-\infty, a_F) \otimes \mathscr{B}(b_F, +\infty]$, $\zeta_F^{(k)}(B, \cdot)$ is a finite measure on $(S_F, \mathscr{B}_F)$ because the class $\{B \in \mathscr{B}[-\infty, a_F) \otimes \mathscr{B}(b_F, +\infty]: \zeta_F^{(k)}(B, \cdot)$ is countably additive on $\mathscr{B}_F\}$ is a λ-system and contains the π-system $\{[-\infty, s) \times (u, +\infty]: s \leq F \leq u\}$. We let $\eta_F: \mathscr{D}_F \times \mathscr{B}_F \to [0, \infty]$ such that $\eta_F(\mathbf{D}, \Gamma) = \sum_{k=1}^{\infty} \zeta_F^{(k)}(\iota_F(\mathbf{D}), \Gamma)$. η_F so defined is independent of the choice of $\{\Gamma^{(k)}: k = 1, 2, \dots\}$ and is a bimeasure for which there is a proper decomposition of $(S_F, \mathscr{B}_F)$. As a matter of fact, $\eta_F(\cdot, \Gamma^{(k)})$ is a finite measure in this situation. If

$F' \supset F$, then by (ii), (iii) and induction $m_{F'}(\pi_{F',F}^{-1}(\Gamma^{(k)})) < \infty$ for each k and we can regard $\{\zeta_{F'}^{(k)}: k = 1, 2, \dots\}$ and $\eta_{F'}$ as constructed from the decomposition $\{\pi_{F',F}^{-1}(\Gamma^{(k)}): k = 1, 2, \dots\}$ of $(S_{F'}, \mathscr{B}_{F'})$. Thus

$$\begin{aligned}
&\zeta_{F'}^{(k)}\big([-\infty, s) \times (u, +\infty], \pi_{F',F}^{-1}(\Gamma)\big) \\
&= m_{sF'u}\Big(\pi_{sF'u,F'}^{-1}\big(\pi_{F',F}^{-1}(\Gamma) \cap \pi_{F',F}^{-1}(\Gamma^{(k)})\big)\Big) \\
&= m_{sF'u}\Big(\pi_{sF'u,F'}^{-1}\big(\pi_{F',F}^{-1}(\Gamma \cap \Gamma^{(k)})\big)\Big) \\
&= m_{sF'u}\big(\pi_{sF'u,F}^{-1}(\Gamma \cap \Gamma^{(k)})\big) \\
&= m_{sFu}\big(\pi_{sFu,F}^{-1}(\Gamma \cap \Gamma^{(k)})\big) \\
&= \zeta_F^{(k)}([-\infty, s) \times (u, +\infty], \Gamma)
\end{aligned}$$

for all $s \le F' \le u$ and $\Gamma \in \mathscr{B}_F$ which implies that $\zeta_{F'}^{(k)}(B, \pi_{F',F}^{-1}(\Gamma)) = \zeta_F^{(k)}(B, \Gamma)$ for all $B \in \mathscr{B}[-\infty, a_F) \otimes \mathscr{B}(b_F, +\infty]$ and $\Gamma \in \mathscr{B}_F$. Hence

$$\begin{aligned}
\eta_{F'}\big(\mathbf{D}', \pi_{F',F}^{-1}(\Gamma)\big) &= \sum_{k=1}^{\infty} \zeta_{F'}^{(k)}\big(\iota_{F'}(\mathbf{D}'), \pi_{F',F}^{-1}(\Gamma)\big) \\
&= \sum_{k=1}^{\infty} \zeta_F^{(k)}\big(\iota_F(\mathbf{D}'), \Gamma\big) = \eta_F(\mathbf{D}', \Gamma),
\end{aligned}$$

for all $\mathbf{D}' \in \mathscr{D}_{F'}$ and $\Gamma \in \mathscr{B}_F$ and the consistency is concluded. By Theorem 1′ there exists a stochastic process $X: (\Omega, \mathscr{G}, P) \to (\Phi, \mathscr{S}(\mathscr{A}))$ such that for all $F \in \mathbf{F}(T)$, $P(X^{-1}(\psi_F^{-1}(\mathbf{D} \times \Gamma))) = \eta_F(\mathbf{D}, \Gamma)$ for all $\mathbf{D} \in \mathscr{D}_F$ and $\Gamma \in \mathscr{B}_F$; in particular,

$$\begin{aligned}
P\big(X^{-1}\big(\psi_F^{-1}(\mathbf{T}_F \times \Gamma)\big)\big) = \eta_F(\mathbf{T}_F, \Gamma) &= \sum_{k=1}^{\infty} \zeta_F^{(k)}\big(\iota_F(\mathbf{T}_F), \Gamma\big) \\
&= \sum_{k=1}^{\infty} m_F(\Gamma \cap \Gamma^{(k)}) = m_F(\Gamma). \qquad \square
\end{aligned}$$

Theorem 1′ is a generalization of Dynkin's result in the following two aspects: (i) $\mathbf{T}$, $\{\mathscr{D}_F: F \in \mathbf{F}(T)\}$ (and even T) can be arbitrary; and (ii) the concept of proper decomposition extends that of σ-finiteness of m_F, allowing for a broader consideration of the class of measures on the time domains.

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DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE
KENT RIDGE 0511
REPUBLIC OF SINGAPORE