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ON THE SUPPORTS OF MEASURE-VALUED CRITICAL BRANCHING BROWNIAN MOTION[1]

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Let $(X_t)_{t\geq 0}$ denote the measure-valued critical branching Brownian motion. When the support of the initial state, X_0, is bounded, temporally global results are given concerning the range, i.e., the size of the supports of $(X_t)_{t\geq 0}$, and the hitting (i.e., charging) probabilities of distant balls are evaluated asymptotically; they depend strongly on the dimension, d, of the underlying Euclidean space $\mathbb{R}^d$. In contrast, in the case $d = 1$ and $X_0 = \lambda$ (Lebesgue measure), it is shown that (spatially) local extinction occurs. Also extensions are indicated for the case of an infinite variance branching mechanism; these results are also dimensionally dependent.

1. Introduction and statement of results. In this article we investigate the global size and, to some extent, the location of the supports of the states of the measure-valued Markov process $(X_t)_{t\geq 0}$, which is an analogue of the classical critical branching Brownian motion (when the offspring distribution has finite variance) on d-dimensional Euclidean space $\mathbb{R}^d$. $(X_t)_{t\geq 0}$ takes its values in the set of Radon measures on the Borel σ-algebra of $\mathbb{R}^d$, and can be obtained as a high-density weak limit (i.e., convergence in distribution) of the latter processes as the mass of the Brownian particles tend to zero in a specific manner [see, e.g., Dawson (1975)]. We refer the reader to the articles of Watanabe (1968), Dawson (1977) and Iscoe (1986a) for the existence and description of $(X_t)_{t\geq 0}$. (The first reference treats the case of finite-measure states, which was extended in the third reference to certain tempered measures; while the second reference is intermediate in that infinite-measure states were considered, but the presentation assumed that the branching mechanism had a finite variance.) See also (1.1) and (1.2) below.

In contrast, the local structure of the support of each state X_t was investigated in Dawson and Hochberg (1979), where it was shown in particular that if $d \geq 2$ and $X_0 = \lambda$, then X_t is almost surely (a.s.) singular with respect to Lebesgue measure, λ, in the sense of absolute continuity. More precisely, they showed that the Hausdorff dimension of a (random) support for X_t is less than or equal to 2; with the singularity in the case $d = 2$ handled separately by a self-similarity argument. Also it was shown in Roelly-Coppoletta (1986) that for $d = 1$, X_t is a.s. absolutely continuous with respect to λ. This result is already implicitly anticipated in equation (6.3) of Dawson (1975) by a stochastic differential equation for the ostensible density.

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Before giving the statements of the main theorems of this article, we introduce some notation, and recall a characterization of $(X_t)_{t\geq 0}$ to put the statement of Theorem 1 into proper perspective.

If $x \in \mathbb{R}^d$ we denote its Euclidean norm by $|x|$ and write simply x^2 for $|x|^2$. We set $B(0; R) := \{x \in \mathbb{R}^d\colon\ |x| < R\}$, $\overline{B}(0; R) := \{x \in \mathbb{R}^d\colon\ |x| \leq R\}$ and $\partial B(0; R) := \{x \in \mathbb{R}^d\colon |x| = R\}$; the complement of a subset of $\mathbb{R}^d$ is indicated by the superscript "c". Write

$$C_0(\mathbb{R}^d) := \left\{\psi\colon \mathbb{R}^d \to \mathbb{R}\,|\,\psi \text{ continuous and } \lim_{|x|\to+\infty} \psi(x) = 0\right\},$$

$$C_c(\mathbb{R}^d) := \{\psi\colon \mathbb{R}^d \to \mathbb{R}\,|\,\psi \text{ continuous and supp } \psi \text{ compact}\},$$

where supp is an abbreviation of support; the subscript "+" indicates "the nonnegative members of ... ." If $G \subset \mathbb{R}^d$ is open and $u\colon \mathbb{R}_+ \times G \to \mathbb{R}$ is appropriately differentiable, we set $\dot{u}(t,x) := (\partial u/\partial t)(t,x)$ and $\Delta u(t,x) := \sum_{i=1}^d(\partial^2/\partial x_i^2)u(t,x)$. Measures on $\mathbb{R}^d$ will be understood to be positive and Radon; λ will always represent Lebesgue measure and for $x \in \mathbb{R}^d$, δ_x denotes the unit atom at x. If ψ is μ-integrable, we will often use the notation $\langle\psi, \mu\rangle$ for $\int\psi\, d\mu$; if ψ depends on a parameter α, then we will write $\langle\psi(\alpha), \mu\rangle \equiv \langle\psi(\alpha,\cdot), \mu\rangle$ for $\int\psi(\alpha, x)\, d\mu(x)$. We denote the set of all Radon measures on $\mathbb{R}^d$ by $M(\mathbb{R}^d)$; it carries the topology of vague convergence and the associated Borel σ-algebra. We fix $p > d$ and set $M_p(\mathbb{R}^d) := \{\mu \in M(\mathbb{R}^d)\colon \int_{\mathbb{R}^d}(1 + |x|)^{-p}\, d\mu(x) < +\infty\}$.

The process $(X_t)_{t\geq 0}$ is an $M_p(\mathbb{R}^d)$-valued Markov process whose transition measures are characterized through their Laplace transforms as

$$E_\mu[\exp(-\langle\psi, X_t\rangle)] = \exp(-\langle u(t), \mu\rangle), \qquad \psi \in C_c(\mathbb{R}^d)_+,\ \mu \in M_p(\mathbb{R}^d), \tag{1.1}$$

where

$$\begin{aligned} \dot{u}(t,x) &= \Delta u(t,x) - u^2(t,x), \\ u(0,x) &= \psi(x). \end{aligned} \tag{1.2}$$

In (1.1), E_μ denotes expectation with respect to the probability P_μ, the law of $(X_t)_{t\geq 0}$ such that $P_\mu(X_0 = \mu) = 1$. We will use the notation $X.$ for a generic sample path of $(X_t)_{t\geq 0}$, which can be assumed to be right continuous with left limits by Theorem 1.1 of Iscoe (1986a). In the case that μ is finite, it is known that there is a continuous version of our process [see, for example, Watanabe (1968)]. This result is probably valid for $\mu \in M_p(\mathbb{R}^d)$ as well, but we shall not have need of it; and moreover it will not be true in the setting of Section 5.

We now present the main results of this article. The first result concerns the range of the process globally in space and time when the initial state, X_0, has compact support. The expression "$X.$ ever charges a (Borel) set B" means: $\exists\ t \geq 0$ such that $X_t(B) > 0$.

THEOREM 1. *Let* $\mu(\mathbb{R}^d) < +\infty$ *with* $\operatorname{supp}\mu \subset B(0; R_0)$ *and let* $R \geq R_0$. *Then*

$$P_\mu\left(X.\ \textit{ever charges}\ [\overline{B}(0; R)]^c\right) = 1 - \exp\left(-R^{-2}\langle u(R^{-1}\cdot), \mu\rangle\right), \tag{1.3}$$

where u is the unique positive (radial) solution of the singular elliptic boundary value problem

$$(1.4)\qquad \begin{aligned} \Delta u(x) &= u^2(x), \qquad x \in B(0;1),\\ u(x) &\to +\infty, \quad \text{as } x \to \partial B(0;1). \end{aligned}$$

In particular, if $\mu = \delta_0$, *then*

$$(1.5)\qquad P_{\delta_0}\big(X.\ \text{ever charges } [\overline{B}(0;R)]^c\big) = 1 - \exp\big(-u(0)R^{-2}\big),$$

with u as in (1.4). *In the case* $d = 1$, *the value of* $u(0)$ *is given by*

$$(1.6)\qquad \begin{aligned} (d=1)\ u(0) = \tfrac{1}{6}\{\Gamma(1/2)\Gamma(1/6)/\Gamma(2/3)\}^2 \in (8.372, 8.388),\\ u(0) \doteq 8.38, \end{aligned}$$

Γ *being the usual gamma function.*

As an immediate corollary we deduce that, under the hypothesis of Theorem 1, the "mass" of the process $(X_t)_{t\geq 0}$ spends its entire "life" within a ball of finite (random) radius, termed the range of $(X_t)_{t\geq 0}$.

COROLLARY. *Under the hypotheses of Theorem* 1, $P_\mu([\bigcup_{t>0} \operatorname{supp} X_t]$ *is bounded*) = 1. *More specifically, we can define a random range* $\mathscr{R} := \inf\{R > 0$: $X.$ *never charges* $[\overline{B}(0;R)]^c\}$; *then* $P_\mu(\mathscr{R} \leq R) = \exp(-R^2\langle u(R^{-1}\,\cdot\,), \mu\rangle)$, *with u as in* (1.4). *In particular* $\mathscr{R}$ *is an absolutely continuous random variable except possibly for an atom at* $\sup\{|x|: x \in \operatorname{supp}\mu\}$, *and* $E_\mu\mathscr{R} < +\infty$; *in the case* $\mu = \delta_0$, $\mathscr{R}$ *is absolutely continuous and* $E_{\delta_0}\mathscr{R} = [u(0)\pi]^{1/2}$.

In the next theorem we given an asymptotic evaluation of a hitting probability of a distant ball. It is qualitatively identical to the result for the classical critical binary branching Brownian motion, as obtained in Sawyer and Fleischman (1979). In the statement of the theorem, the notation $f \sim g$ means $\lim_{|x|\to+\infty}[f(x)/g(x)] = 1$. Note, however, that in "equation" [4.2] of Sawyer and Fleischman (1979), the dependence of the constant on the radius ε is not made explicit when $d \geq 5$. Note also that the exponent $(d-4)$ coincides with that in the covariance kernel $|x-y|^{4-d}$ of Theorem (5.5) of Iscoe (1986a).

THEOREM 2. *For each* $\varepsilon > 0$,

$$P_{\delta_0}\big(X.\ \text{ever charges } B(x;\varepsilon)\big) \sim \begin{cases} 6/x^2, & d = 1,\\ 4/x^2, & d = 2,\\ 2/x^2, & d = 3,\\ 2/\big[x^2 \log|x|\big], & d = 4,\\ c\varepsilon^{d-4}/|x|^{d-2}, & d \geq 5, \end{cases}$$

where c is some constant (depending only on the dimension d).

It was shown in Dawson (1977) that for $X_0 = \lambda$, a unique nontrivial invariant distribution exists and is the weak-$\lim_{t \to +\infty} X_t$ [i.e., $(X_t)_{t \geq 0}$ is ergodic] iff $d \geq 3$; and that if $G \subset \mathbb{R}^d$ is bounded and open, then $\lim_{t \to +\infty} X_t(G) = 0$ in P_λ-probability if $d = 1$ or 2. This latter result was strengthened somewhat in Iscoe (1986a), where it was shown that $\int_0^\infty X_t(G)\,dt < +\infty$ iff $d = 1$. When $d = 1$ it does not follow immediately that $X_t(G) \to 0$ a.s. as $t \to +\infty$; and if so, whether or not $X_t(G)$ is eventually and permanently zero—i.e., local extinction. In the next theorem we verify that the latter scenario is correct. The corresponding result for the classical critical binary branching Brownian motion was obtained in Sawyer and Fleischman (1979); of course, in that model all particles have the same mass, so that $\lim_{t \to +\infty} X_t(G) = 0$ clearly implies extinction in G [here $X_t(G)$ denotes the number of particles in G at time t].

THEOREM 3. *Let $d = 1$ and $X_0 = \lambda$. For each $R > 0$ there exists a finite random time τ_R such that with P_λ-probability* 1, $X_t((-R, R)) = 0$ *for all* $t \geq \tau_R$. *The random variable τ_R is absolutely continuous and*

$$0 < \liminf_{t \to \infty} t^{1/2} P_\lambda(\tau_R > t) \leq \limsup_{t \to \infty} t^{1/2} P_\lambda(\tau_R > t) < \infty.$$

Consequently, $E_\lambda[\tau_R] = +\infty$.

The layout of the remainder of this article is as follows. In Section 2 we present the proofs of Theorems 1–3, with the discussion of the deterministic singular elliptic boundary value problem (1.4) and an analogous "exterior" problem being delayed until Sections 3 and 4, respectively. In Section 3 we give considerable detail so that we can safely refer to it (similarly) in Section 4, to avoid being repetitive. Finally, in Section 5 we indicate some extensions of the results to a class of processes with a branching mechanism which does not possess a finite variance. It is not taken up at the outset in order to simplify the notation in the proofs. An Appendix appears after Section 5 in which some recent results, concerning the temporal asymptotic behavior of solutions of (1.2), are stated. In Sections 2–4 and the Appendix, the numbering of lemmas, propositions, theorems, equations, etc. was done strictly in order of appearance, in the form $(x.y)$, where x denotes the section number ($x = \mathrm{A}$ within the Appendix).

2. Proofs of the theorems. Before presenting the proofs of Theorems 1–3, we introduce some additional notation and recall an important representation.

If $G \subset \mathbb{R}^d$ is open we let $C^k(G)$, $k \in \mathbb{N}$, denote the set of k-times continuously differentiable functions from G to $\mathbb{R}$. Most of the functions we shall see will be radial, i.e., they depend on $x \in \mathbb{R}^d$ only through $|x|$. Strictly speaking then, if $u: G \to \mathbb{R}$ is radial, then $u(x) = \tilde{u}(|x|)$ for some $\tilde{u}: \mathbb{R}_+ \to \mathbb{R}$; however, we shall make no distinction between u and $\tilde{u}$ and abuse the notation by writing simply $u(x) \equiv u(r)$, where the variable r will always represent $|x|$ when viewing u on $G \subset \mathbb{R}^d$. In particular, if $u \in C^2(G)$, then

$$\Delta u(x) = u''(r) + ((d-1)/r)u'(r) \quad \text{for } x \neq 0,$$

where the primes denote differentiation with respect to r; if $0 \in G$, then

$u'(0^+) = 0$, $u''(0^+)$ exists, and $\Delta u(0) = d \cdot u''(0^+)$. Conversely, it is easy to see that given an interval $I \subset \mathbb{R}_+$ and a function $f\colon I \to \mathbb{R}$, $u(x) := f(|x|)$ defines a C^2 function iff $f'(0^+) = 0$ and $f''(0^+)$ exists in case $0 \in I$; in which case $\Delta u(x) = f''(|x|) + ((d-1)/|x|)f'(|x|)$ for $x \neq 0$, and if $0 \in I$, then $\Delta u(0) = d \cdot f''(0^+)$. We would then apply our convention and write simply $u(r)$ for $f(|x|)$. We shall take advantage of this symmetry in order to avoid unnecessary use of elliptic regularity theory, etc., for the p.d.e.'s which arise in the proofs.

LEMMA (2.0). *Let $t \geq 0$, $G \subset \mathbb{R}^d$ open. If $\int_t^\infty X_s(G)\,ds = 0$, then for all $s \geq t$, $X_s(G) = 0$ (a.s.).*

PROOF. Choose $(\psi_n)_{n\in\mathbb{N}} \subset C_c(\mathbb{R}^d)_+$ such that $\psi_n \uparrow 1_G$ as $n \to +\infty$. Then $0 = \int_t^\infty X_s(G)\,ds \geq \int_t^\infty \langle \psi_n, X_s\rangle\,ds$. Therefore $\langle \psi_n, X_s\rangle = 0$ for a.e. $s \geq t$; but $s \mapsto \langle \psi_n, X_s\rangle$ is a.s. right continuous, and therefore $\langle \psi_n, X_s\rangle = 0$ for all $s \geq t$. By the monotone convergence theorem, $X_s(G) = \lim_{n\to+\infty}\langle \psi_n, X_s\rangle = 0$ for all $s \geq t$ (a.s.). □

The main tool used in the proofs of Theorems 1–3 is the following representation derived in Theorem (3.1) of Iscoe (1986a):

$$E_\mu\left[\exp\left(-\int_0^t \langle \varphi, X_s\rangle\,ds\right)\right] = \exp(-\langle u(t), \mu\rangle), \quad \mu \in M_p(\mathbb{R}^d),\ \varphi \in C_c(\mathbb{R}^d)_+, \tag{2.1}$$

where $u(t) \equiv u(t,x)$ is the solution of the evolution equation

$$\begin{aligned} \dot{u}(s) &= \Delta u(s) - u^2(s) + \varphi, \qquad 0 \leq s \leq t, \\ u(0) &= 0. \end{aligned} \tag{2.2}$$

More generally, we shall need a temporally inhomogeneous version of (2.1) and (2.2), which was derived in Theorem (3.2) of Iscoe (1986a), for Theorem 3,

$$E_\mu\left[\exp\left(-\int_0^t \langle \varphi(s), X_s\rangle\,ds\right)\right] = \exp(-\langle u(t), \mu\rangle), \tag{2.3}$$

where u is the mild solution of the evolution equation

$$\begin{aligned} \dot{u}(s) &= \Delta u(s) - u^2(s) + \varphi(t-s), \qquad 0 < s \leq t, \\ u(0) &= 0. \end{aligned} \tag{2.4}$$

In (2.3) and (2.4), $\mu \in M_p(\mathbb{R}^d)$ and $\varphi\colon [0,t] \to C_0(\mathbb{R}^d)_+$ is right continuous with left limits [$C_0(\mathbb{R}^d)$ carries the sup-norm topology] and such that for some $K > 0$ and all $(s,x) \in [0,t] \times \mathbb{R}^d$: $\varphi(s,x) \leq K(1+|x|)^{-p}$; in particular, the latter condition holds if each $\varphi(s,\cdot) \in C_c(\mathbb{R}^d)_+$.

It was also shown, in Theorem (3.3) of Iscoe (1986a), that as $t \to +\infty$, $u(t)$ [of (2.2)] increases and converges uniformly to $u \in C_0(\mathbb{R}^d)_+$, where

$$\Delta u - u^2 + \varphi = 0. \tag{2.5}$$

PROOF OF THEOREM 1 AND ITS COROLLARY. With R and μ as in the statement of the theorem, $\theta > 0$, and $\varphi_{n,m}$ as in (3.3), we can calculate, using Lemma (2.0), the monotone convergence theorem, and the representation (2.1), (2.2),

$$\begin{aligned}
&P_\mu\big(X.\text{ never charges } [\overline{B}(0;R)]^c\big)\\
&= P_\mu\Big(\int_0^\infty X_t([\overline{B}(0;R)]^c)\,dt = 0\Big)\\
&= \lim_{\theta\to+\infty} E_\mu\Big[\exp\Big(-\theta^2\int_0^\infty X_t([\overline{B}(0;R)]^c)\,dt\Big)\Big]\\
&= \lim_{\theta\to+\infty}\lim_{n\to+\infty}\lim_{m\to+\infty}\lim_{T\to+\infty} E_\mu\Big[\exp\Big(-\int_0^T\langle\theta^2\varphi_{n,m}, X_t\rangle\,dt\Big)\Big]\\
&= \lim_{\theta\to+\infty}\lim_{n\to+\infty}\lim_{m\to+\infty}\lim_{T\to+\infty} \exp(-\langle u_{n,m}(T,\cdot\,;R,\theta),\mu\rangle)\\
&= \exp(-\langle u(\cdot\,;R),\mu\rangle).
\end{aligned}$$

The last equality follows from (2.5), Lemma (3.4) and Proposition (3.15) invoked in that order. The function $v \equiv u_{n,m}(t,\cdot\,;R,\theta)$ is the solution of

$$\begin{aligned}
\dot v(t) &= \Delta v(t) - v^2(t) + \theta^2\varphi_{n,m},\\
v(0) &= 0.
\end{aligned}$$

Also by Proposition (3.15) $u(x;R) = R^{-2}u(R^{-1}x;1)$. Thus (1.3) and (1.4) are established, and (1.5) follows upon setting $x = 0$. In the special case $d = 1$, (1.6) follows from (3.8) of Proposition (3.5). The numerical estimation was obtained using Table (6.1) and the values given at (6.1.8) and (6.1.13) in Abramowitz and Stegun (1964).

Finally, to establish the corollary, we observe that if $\operatorname{supp}\mu \subset B(0;R_0)$, then with $B \equiv B(0,R_0)$ and $B(R) \equiv B(0,R_0/R)$,

$$\begin{aligned}
\limsup_{R\to+\infty}\langle u(\cdot\,;R),\mu\rangle &= \limsup_{R\to+\infty}\int_B R^{-2}u(R^{-1}x;1)\mu(dx)\\
&\le \limsup_{R\to+\infty} R^{-2}\Big[\max_{y\in B(R)} u(y;1)\Big]\mu(\mathbb{R}^d) = 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
P_\mu\Big(\Big[\bigcup_{t\ge 0}\operatorname{supp} X_t\Big]\text{ is bounded}\Big) &= P_\mu\Big(\bigcup_{n\ge R_0}\{X.\text{ never charges } [\overline{B}(0;n)]^c\}\Big)\\
&= \lim_{n\to+\infty}\exp(-\langle u(\cdot\,;n),\mu\rangle) = 1.
\end{aligned}$$

That $E_\mu\mathscr{R} < +\infty$ is clear from (1.3) since for large R, $\langle u(R^{-1}\cdot),\mu\rangle$ is bounded [actually $\lim_{R\to+\infty}\langle u(R^{-1}\cdot),\mu\rangle = 0$]. That $\mathscr{R}$ is absolutely continuous, except for a possible atom at $\sup\{|x|: x\in\operatorname{supp}\mu\}$, is clear from (1.3). In the case $\mu = \delta_0$, an atom at $R = 0$ is clearly ruled out by (1.5).

To compute $E_{\delta_0}[\mathscr{R}] = \int_0^\infty [1 - \exp(-u(0)R^{-2})]\,dR$, we make the change of variables $R = \sqrt{u(0)/r}$:

$$E_{\delta_0}[\mathscr{R}] = \sqrt{u(0)} \int_0^\infty r^{-2}[1 - \exp(-r^2)]\,dr.$$

Consider

$$f(t) := \int_0^\infty r^{-2}[1 - \exp(-tr^2)]\,dr, \qquad f(0) = f(0^+) = 0.$$

Then

$$(df/dt)(t) = \int_0^\infty \exp(-tr^2)\,dr = \sqrt{\pi}/(2\sqrt{t}).$$

Therefore $f(t) = [\pi t]^{1/2}$ and $E_{\delta_0}[\mathscr{R}] = \sqrt{u(0)}\,f(1) = [u(0)\pi]^{1/2}$. $\square$

PROOF OF THEOREM 2. It follows easily from the spatial homogeneity of the Laplacian, Δ, and the branching mechanism [through the term u^2 in (2.2)] that we can interchange the roles of 0 and x, i.e.,

$$\begin{aligned} P_{\delta_0}(X.\text{ ever charges } B(x;\varepsilon)) &= 1 - P_{\delta_0}\left(\int_0^\infty X_t(B(x;\varepsilon))\,dt = 0\right) \\ &= 1 - P_{\delta_x}\left(\int_0^\infty X_t(B(0;\varepsilon))\,dt = 0\right). \end{aligned}$$

As in the proof of Theorem 1, we approximate $1_{B(0;\varepsilon)} = \lim_{n\to+\infty}\varphi_n$ with φ_n as in the statement of Proposition (4.2). Then by (2.1), (2.2), (2.5) and Proposition (4.2)

$$\begin{aligned} P_{\delta_x}\left(\int_0^\infty X_t(B(0;\varepsilon))\,dt = 0\right) &= \lim_{\theta\to+\infty} E_{\delta_x}\left[\exp\left(-\theta^2\int_0^\infty X_t(B(0;\varepsilon))\,dt\right)\right] \\ &= \lim_{\theta\to+\infty}\ \lim_{n\to+\infty}\ \lim_{T\to+\infty} E_{\delta_x}\left[\exp\left(-\int_0^T \langle\theta^2\varphi_n, X_t\rangle\,dt\right)\right] \\ &= \lim_{\theta\to+\infty}\ \lim_{n\to+\infty}\ \lim_{T\to+\infty} \exp[-u_n(T,x;\varepsilon,\theta)] \\ &= \exp[-u(x;\varepsilon)], \end{aligned}$$

where $u_n = u_n(t,x;\varepsilon,\theta)$ satisfies

$$\begin{aligned} \dot{u}_n(t) &= \Delta u_n(t) - u_n^2(t) + \theta^2\varphi_n, \\ u_n(0) &= 0, \end{aligned}$$

and $u \equiv u(x;\varepsilon)$ is the solution of the problem (4.1).

As stated at (4.7), $u(x) \sim v(|x|)$ which suffices for the assertion of this theorem when $d \le 4$. It remains to make explicit the dependence of $c \equiv c(\varepsilon)$, of (4.7), on ε. To this end, we note the identity $u(x;\varepsilon) = \varepsilon^{-2}u(\varepsilon^{-1}x;1)$ established in Proposition (4.2). Multiplying this identity by $|x|^{d-2}$ and letting $|x| \to +\infty$ yields the result: $c(\varepsilon) = \varepsilon^{d-4}c(1)$.

To conclude, it only remains to observe that as $a \to 0^+$, $1 - e^{-a} \sim a$. $\square$

Proof of Theorem 3. Fix $R > 0$, and choose an even $\varphi \in C_c(\mathbb{R}^1)_+$ with $0 < \varphi(x) \le 1$ for $x \in (-R, R)$ and $\varphi(x) = 0$ for $x \notin (-R, R)$. Then by Theorem (3.2) of Iscoe (1986a), cited as (2.3) and (2.4) in this article,

$$
\begin{aligned}
&P_\lambda\left(\int_t^\infty X_s((-R, R))\,ds = 0\right)\\
&\quad = P_\lambda\left(\int_t^\infty \langle \varphi, X_s\rangle\,ds = 0\right)\\
&\quad = \lim_{\theta\to+\infty}\ \lim_{T\to+\infty} E_\lambda\left[\exp\left(-\int_0^T \langle \theta^2\varphi\cdot 1_{[t,+\infty)}, X_s\rangle\,ds\right)\right]\\
&\quad = \lim_{\theta\to+\infty}\ \lim_{T\to+\infty} \exp(-\langle u(T), \lambda\rangle),
\end{aligned}
\tag{2.6}
$$

where u is the mild solution of the evolution equation

$$
\begin{aligned}
&\dot u(s) = \Delta u(s) - u^2(s) + \theta^2\varphi\cdot 1_{[t,+\infty)}(T-s), \qquad 0 < s < T,\\
&u(0) = 0,\\
&\Leftrightarrow\\
&\dot u(s) = \Delta u(s) - u^2(s) + \theta^2\varphi\cdot 1_{[0,T-t]}(s), \qquad 0 < s < T,\\
&u(0) = 0.
\end{aligned}
\tag{2.7}
$$

Actually, it follows [e.g., directly from the proof of Theorem (3.2) of Iscoe (1986a)] that $u(T) = v_T(t)$, where

$$
\begin{aligned}
&\dot v_T(s) = \Delta v_T(s) - v_T^2(s),\\
&v_T(0) = u(T-t),
\end{aligned}
\tag{2.8}
$$

and where $u(T-t)$ is determined, of course, from (2.7) which simplifies to

$$
\begin{aligned}
&\dot u(s) = \Delta u(s) - u^2(s) + \theta^2\varphi, \qquad 0 \le s \le T - t,\\
&u(0) = 0.
\end{aligned}
$$

By (2.5) as $T \uparrow +\infty$, $u(T-t, x) \uparrow u(x)$, where $u \in C_0(\mathbb{R}^1)$ is the positive solution of the equation $\Delta u - u^2 + \theta^2\varphi = 0$ such that $0 < u \le \theta$ [the uniqueness is established in Lemma (3.2)]. Then by the continuity of the monotone semigroup associated with (2.8), v_T increases as $T \to +\infty$ to the unique solution v of the evolution equation

$$
\begin{aligned}
&\dot v(s) = \Delta v(s) - v^2(s), \qquad 0 \le s < +\infty,\\
&v(0) = u.
\end{aligned}
\tag{2.9}
$$

The function v depends, of course, on θ since u does; we make this explicit in our notation by writing $v \equiv v(s, x; \theta)$, when necessary. Now $0 < u \le \theta$ implies, by the monotonicity of the semigroup associated with (2.9), that v is dominated by the solution of (2.9) with u replaced by θ; namely, $\theta/[1 + \theta t]$. In particular, $v(t, \cdot; \theta) \le 1/t$. Thus as $\theta \uparrow +\infty$, $v(t, \cdot; \theta) \uparrow v(t, \cdot)$, say, and

$$
v(t, x) \le 1/t, \quad \text{for all } x \in \mathbb{R}^1. \tag{2.10}
$$

Suppose for the moment that we have established the estimate

$$(2.11)\qquad v(t,x) \le 6/[|x| - R]^2, \quad \text{for all } |x| > R.$$

Then returning to (2.6) we can continue the calculation with two applications of the monotone convergence theorem:

$$(2.12)\qquad P_\lambda\Big(\int_t^\infty X_s((-R,R))\,ds = 0\Big) = \exp(-\langle v(t,\cdot),\lambda\rangle).$$

By (2.10) and (2.11) we can apply the dominated convergence theorem to the right-hand side of (2.12) to obtain

$$P_\lambda\Big(\exists\, t > 0\colon \int_t^\infty X_s((-R,R))\,ds = 0\Big) = \lim_{t\to+\infty} P_\lambda\Big(\int_t^\infty X_s((-R,R))\,ds = 0\Big)$$
$$= \exp\Big(-\lim_{t\to+\infty}\langle v(t,\cdot),\lambda\rangle\Big) = e^0 = 1.$$

We now verify the estimate (2.11). It suffices for it to hold for each $v(\cdot,\cdot;\theta)$ in place of v; we verify this, but omit writing θ explicitly for brevity. Set $w(x) = 6/[|x| - R]^2$. Since v and w are both even functions it suffices to restrict attention to $x \in [R, +\infty)$. Suppose for the moment that (2.11) is valid for $t = 0$, i.e., that $u(x) \le w(x)$ for $x \ge R$. Then on $\partial(\mathbb{R}_+\times[R,+\infty))$, $0 < w - v \le +\infty$; and also $\lim_{\sigma\to+\infty}\inf_{t\ge 0,\, x\ge\sigma}(w(x) - v(t,x)) = 0$.

Therefore if $w - v$ were ever negative on $\mathbb{R}_+\times[R,+\infty)$, it would assume a negative minimum at a point in the interior of $\mathbb{R}_+\times[R,+\infty)$. Now

$$(2.13)\qquad (w-v)^{\cdot} - \Delta(w-v) = -(w^2 - v^2),$$

so that at such a point of minimization, the left-hand side of (2.13) would be nonpositive, while the right-hand side would be positive, which is absurd. Therefore $w - v \ge 0$, and the estimate is established for $t > 0$.

To obtain (2.11) at $t = 0$, we repeat the argument at (2.13), working on $[R, +\infty)$ and omitting the derivative with respect to "t".

To derive the asymptotic behavior of $P_\lambda(\tau_R > 0)$ as $t \to +\infty$, we must take a closer look at the limiting function $v(t,x) \equiv \lim_{\theta\to+\infty} v(t,x;\theta)$, since

$$(2.14)\quad P_\lambda(\tau_R > t) = 1 - \exp(-\langle v(t,\cdot),\lambda\rangle) \sim \langle v(t,\cdot),\lambda\rangle, \quad \text{as } t \to +\infty,$$

which follows from (2.10)–(2.12). The asymptotic result will follow immediately from Lemma (A.10) (with $d = 1$ and $\beta = 1$) of the Appendix, once it is established that v satisfies the p.d.e.

$$(2.15)\qquad \dot v(t,x) = \Delta v(t,x) - v^2(t,x), \qquad t > 0,\ x \in \mathbb{R}^d.$$

[The determination of whether or not $E_\lambda[\tau_R]$ is infinite is effected through its expression as $E_\lambda[\tau_R] = \int_0^\infty P_\lambda(\tau_R > t)\,dt$.]

To this end, we cast (2.9) into its equivalent "mild" form [see the Appendix to Iscoe (1986a)] on the interval $[t_1, +\infty)$ for any fixed $t_1 > 0$,

$$(2.16)\quad v(t;\theta) = S_{t-t_1}(v(t_1;\theta)) - \int_{t_1}^t S_{t-s+t_1}(v^2(s;\theta))\,ds, \qquad t \ge t_1,$$

where $(S_r(\psi))(x) \equiv \int_{\mathbb{R}^d}\psi(y)[\exp(-(x-y)^2/4r)]/(4\pi r)^{d/2}\,dr$, for $r > 0$, $x \in \mathbb{R}^d$

(here $d = 1$), and $\psi \in C_0(\mathbb{R}^d)$, say. Using (2.10) and the bounded convergence theorem, we obtain from (2.16) as $\theta \to +\infty$,

$$v(t) = S_{t-t_1}(v(t_1)) - \int_{t_1}^{t} S_{t-s+t_1}(v^2(s))\, ds, \qquad t \geq t_1. \tag{2.17}$$

Note also that the function $r \mapsto v(r, \cdot)$ from $(t_1, +\infty) \to L^2(\mathbb{R}^d, \lambda)$ [L^2 corresponds to the term v^2 in (2.17)] is continuous. This follows easily using (2.10), (2.11) and the dominated convergence theorem a few times. We can them employ the bootstrapping technique used to prove Proposition (3.28) of Iscoe (1986b) to conclude that v satisfies (2.15) for $t > t_1$, which was arbitrary. Three comments are in order, to compare the two situations. First, our initial time is not 0; this only involves notational complications. Second, in Iscoe (1986b), the solution to the p.d.e. at our (2.15) had a singularity at $x = 0$, which is not the case here. This leads to the occasional simplification. Third, we must note that due to (2.10), $(\partial/\partial t - \Delta)S_{t-t_1}(v(t_1)) = 0$ for $t > t_1$.

To conclude the proof, we show that τ_R is an absolutely continuous random variable. Indeed, from (2.14), we see that its distribution function is given by

$$F(t) \equiv P_\lambda(\tau_R \leq t) = \exp(-\langle v(t), \lambda \rangle), \qquad t > 0.$$

Clearly, $P_\lambda(\tau_R = 0) = 0$ follows from Lemma (2.0). We shall show that F is continuously differentiable on $(0, +\infty)$. Fix $t_1 > 0$. From (2.17) we obtain that

$$\langle v(t), \lambda \rangle = \langle v(t_1), \lambda \rangle - \int_{t_1}^{t} \langle v^2(s), \lambda \rangle\, ds,$$

for $t > t_1$, since $\int_{\mathbb{R}^d} (4\pi r)^{-d/2} \exp(-x^2/4r)\, dx = 1$ for all $r > 0$. It was previously noted that v: $(0, +\infty) \to L^2(\mathbb{R}^d, \lambda)$ is continuous; so we are done. □

3. A singular elliptic boundary value problem. In this section we examine the problem of existence and uniqueness for the problem

$$\begin{aligned} \Delta u(x) &= u^2(x), \qquad x \in B(0; R), \\ u(x) &\to +\infty, \qquad x \to \partial B(0; R). \end{aligned} \tag{3.1}$$

We restrict our attention to positive (radial) solutions and discuss the one-dimensional case in great detail. Note that the transformation $u \mapsto 1/u$ converts (3.1) into a quasilinear Dirichlet problem with the singularity shifted into the coefficients. However, in making this transformation, the utility of the maximum principle is lost. Analogous problems, where its utility is present, were considered by Crandall, Rabinowitz and Tartar (1977). It would be interesting to know if some transformation of (3.1) would convert it exactly into the setting considered there.

LEMMA (3.2). *Let $u, v \in C_0(\mathbb{R}^d)_+$ such that $\Delta u - u^2 + \varphi = 0$ and $\Delta v - v^2 + \psi = 0$, where $\varphi, \psi \in C_c(\mathbb{R}^d)_+$ with $\varphi \leq \psi$. Then $u \leq v$. In particular, the equations have unique solutions.*

PROOF. Suppose u were greater than v at some point $x \in \mathbb{R}^d$. Then at a point where $u - v$ assumes a positive maximum (note that $|u|^2 \leq \sup \psi$; for otherwise at a point where $|u|$ assumes its maximum: $0 > \Delta u - u^2 + \varphi = 0$)

$$0 \geq \Delta(u - v) = (u^2 - v^2) + (\psi - \varphi) > 0,$$

which is absurd. The unicity follows by taking $\varphi = \psi$ so that $u \leq v$ and by symmetry $v \leq u$; existence was announced at (2.5). □

Define for each $m, n \in \mathbb{N}$, $\varphi_{n,m}: \mathbb{R}_+ \to [0,1]$ by

$$\varphi_{n,m}(r) := \begin{cases} 0, & r \leq R \text{ or } r \geq m+1, \\ 1, & R + 1/n \leq r \leq m, \end{cases} \tag{3.3}$$

and extended uniquely to $\mathbb{R}_+$ as a continuous piecewise linear function. Suppose $u \equiv u_{n,m} \in C_0(\mathbb{R}^d)$ is the positive, radial solution of the equation

$$\Delta u(x) - u^2(x) + \theta^2 \varphi_{n,m} = 0$$

such that $0 < u < \theta$ ($\theta > 0$); see (2.5).

LEMMA (3.4). *$u := \lim_{n \to +\infty} \lim_{m \to +\infty} u_{n,m}$ exists, is C^1, and is a positive radial solution of the equation $\Delta u - u^2 + \theta^2 1_{B^c(0;R)} = 0$, for $|x| \neq R$, such that $0 < u \leq \theta$.*

PROOF. As $m \uparrow +\infty$, $\varphi_{n,m} \uparrow \varphi_n$, say; and by Lemma (3.2), $u_{n,m}$ also increases to some function, say u_n, such that $0 < u_n \leq \theta$. It follows from the monotone convergence theorem applied to the integral equation (which is equivalent to the p.d.e.)

$$u_{n,m}(r) = u_{n,m}(1) + \int_1^r s^{1-d} \int_0^s \left[u_{n,m}^2(t) - \theta^2 \varphi_{n,m}(t)\right] t^{d-1}\, dt\, ds,$$

that

$$u_n(t) = u_n(1) + \int_1^r s^{1-d} \int_0^s \left[u_n^2(t) - \theta^2 \varphi_n(t)\right] t^{d-1}\, dt\, ds;$$

we follow the usual convention if $0 \leq r < 1$. Repeating the argument as $n \to +\infty$ yields that u satisfies

$$u(r) = u(1) + \int_1^r s^{1-d} \int_0^s \left[u^2(t) - \theta^2 1_{B^c(0;R)}(t)\right] t^{d-1}\, dt\, ds.$$

It is clear from this representation that $u \in C^1((0,+\infty)) \cap C^2((0,+\infty) \setminus \{R\})$; and that $u'(0^+) = 0$ and that $u''(0^+)$ exists. Differentiating the last integral equation yields the lemma; that $u \leq \theta$ follows as in Lemma (3.2). □

PROPOSITION (3.5). *Let the dimension $d = 1$ and $u \equiv u(\cdot;\theta) \in C^1(\mathbb{R}^1) \cap C^2(\mathbb{R}^1 \setminus \{\pm R\})$ be a positive, even (symmetric) solution of the equation*

$$u'' - u^2 + \theta^2 1_{[-R,R]^c} = 0, \tag{3.6}$$

such that $0 < u \leq \theta$, $\theta > 0$. Then $u := \lim_{\theta \to +\infty} u(\cdot;\theta)$ exists on $(-R,R)$, on which it is the unique positive, even solution of the singular boundary value *problem*

$$u''(r) = u^2(r), \qquad r \in (-R, R), \tag{3.7}$$
$$u(r) \to +\infty, \qquad \text{as } r \to \pm R.$$

Equivalently, u is the unique maximal solution of the initial value problem

$$u''(r) = u^2(r),$$
$$u(0) = [6R^2]^{-1}\{\Gamma(1/2)\Gamma(1/6)/\Gamma(2/3)\}^2, \tag{3.8}$$
$$u'(0) = 0,$$

Γ *denoting the usual gamma function.*

[*Remark*: The assumption of symmetry at (3.7) is actually unnecessary, as will be shown in Proposition (3.15).]

PROOF. For the moment we suppress the implicit dependence of $u(\cdot;\theta)$ on θ, and write simply u; we restrict attention to $r \in \mathbb{R}_+$. Multiplying (3.6) by u' and integrating yields

$$[u']^2/2 = u^3/3 + c_1 \quad \text{on } [0, R], \tag{3.9}$$

$$[u']^2/2 = u^3/3 - \theta^2 u + c_2 \quad \text{on } [R, +\infty). \tag{3.10}$$

As $u'(0) = 0$, we deduce that $c_1 = -[u(0)]^3/3$. Directly from (3.6) and the assumptions that $0 < u < \theta$ and $u'(0) = 0$, we see that u is convex and increasing on $[0, R]$; and u is concave and increasing on $[R, +\infty)$. Indeed, if u were eventually decreasing for $r > r_0$, then from $u'' = u^2 - \theta^2 \le u^2(r_0) - \theta^2 < 0$ we see that u is eventually negative [note that $u(r_0) = \theta$ is impossible since at that maximizing point $u'(r_0) = 0$ and the unique solution to the initial value problem $u'' = u^2 - \theta^2$, $u(r_0) = \theta$, $u'(r_0) = 0$ is simply $u \equiv \theta$; however, this implies that $0 < u'(R^-) = u'(R^+) = 0$, which is absurd]. Thus $u(r)$ increases to a limit; $u'(r)$ decreases necessarily to 0 [since $u(r) < \theta$] as $r \to +\infty$. If $u(+\infty) < \theta$, then we can repeat an earlier argument to obtain that u would eventually be negative. Thus $\lim_{r\to+\infty} u(r) = \theta$ and $\lim_{r\to+\infty} u'(r) = 0$. From (3.10) we deduce that $c_2 = 2\theta^3/3$.

From (3.9) and (3.10) we see that u being C^1 at $r = R$ implies that

$$3\theta^2 u(R) - 2\theta^3(0) = u^3(0) > 0.$$

Therefore $u(R) > 2\theta/3 \to +\infty$ as $\theta \to +\infty$.

We return to the separable equation (3.9) and "integrate" it (implicitly) on $[r, R]$:

$$[u']^2/2 = u^3/3 - u^3(0)/3 \leftrightarrow \left[(2/3)\left(u^3(r) - u^3(0)\right)\right]^{-1/2} u'(r) = 1$$
$$\leftrightarrow \int_{u(r)}^{u(R)} \left[(2/3)\left(u^3 - u^3(0)\right)\right]^{-1/2} du = R - r.$$

Since u is strictly increasing on $[0, R]$ it was permissible to make the change of variables $r \mapsto u(r)$ in the integral, where now u is just a dummy variable. Making the further change $u = u(0)v$ yields the following equation, in which we reinstate the dependence of the function u on θ,

$$(3.11)\quad \int_{l_2(r;\,\theta)}^{l_1(\theta)} \left[(2/3)(v^3 - 1)\right]^{-1/2} dv = \sqrt{u(0;\theta)}\,(R - r), \qquad 0 \le r \le R,$$

where $l_1(r;\theta) = u(r;\theta)/u(0,\theta)$ and $l_2(\theta) = u(R;\theta)/u(0;\theta)$.

The following monotonicity property is inherited from that of the $\{u_{n,m}(\cdot;\theta)\}$ of Lemma (3.4), as results from Lemma (3.2): As $\theta \uparrow +\infty$, $u(\cdot;\theta) \uparrow u$, say, where u is a possibly infinite but positive function. Setting $r = 0$ in (3.11) yields that $\sqrt{u(0;\theta)}\,R \le \int_1^\infty [2/3(v^3 - 1)]^{-1/2}\,dv < +\infty$; so that $u(0) < +\infty$. Thus again from (3.11) it follows that $u(r) < +\infty$ for $0 \le r < R$; otherwise the left-hand side of (3.11) would tend to 0 as $\theta \to +\infty$ since $u(R;\theta) \to +\infty$, while the right-hand side would not do so. Setting $r = 0$ again in (3.11) and letting $\theta \to +\infty$ yields the value

$$(3.12)\quad \begin{aligned} u(0) &= (3/2R^2)\left(\int_1^\infty [v^3 - 1]^{-1/2}\,dv\right)^2 \\ &= (1/6R^2)\left(\int_0^1 (1 - z)^{1/2 - 1} z^{1/6 - 1}\,dz\right)^2 \\ &= (1/6R^2)\{\Gamma(1/2)\Gamma(1/6)/\Gamma(2/3)\}^2, \end{aligned}$$

by making the change of variables $z = v^{-3}$; Γ is the usual gamma function and the "beta" identity between the two braces is well known [see, e.g., (6.2.1) and (6.2.2) of Abramowitz and Stegun (1964)].

Letting $\theta \to +\infty$ in (3.11) yields the equation

$$(3.13)\quad \int_{[u(r)/u(0)]}^{\infty} \left[(2/3)(v^3 - 1)\right]^{-1/2} dv = \sqrt{u(0)}\,(R - r), \qquad 0 \le r \le R.$$

Clearly, as $r \to R$, $u(r) \to +\infty$ since the right-hand side of (3.13) tends to 0. Since $u''(r;\theta) = u^2(r;\theta)$ for $r \in [0, R)$ and $u'(0;\theta) = 0$, it follows that $u(\cdot;\theta)$ satisfies the integral equation $u(r;\theta) = u(0;\theta) + \int_0^r\int_0^s u^2(t;\theta)\,dt\,ds$. Letting $\theta \to +\infty$ and applying the monotone convergence theorem yields that u satisfies the integral equation $u(r) = u(0) + \int_0^r\int_0^s u^2(t)\,dt\,ds$ with $u(0)$ given by (3.12). It follows that u is the solution to the initial value problem (3.8).

Conversely, if u is a positive, even solution to the singular boundary value problem (3.7), then we can "integrate" the equation to arrive at (3.13), from which it follows that (3.12) is valid for $u(0)$. Thus u coincides with the solution of the initial value problem (3.8). □

LEMMA (3.14). *Denote the solution in Proposition* (3.5) *by* v, *and let* $u \equiv u(x;\theta)$ *be the solution in Lemma* (3.4). *There exists a constant* C, *which is independent of* θ, *such that* $u(x;\theta) \le Cv(x^2/R)$, *for all* $x \in B(0;R)$.

PROOF. By (3.9), which appears in the proof of Proposition (3.5), we see that for $0 \le r < R$,

$$v'(r) = \left[(2/3)\left(v^3(r) - v^3(0)\right)\right]^{1/2} \le \sqrt{2/3}\,v^{3/2}(r) \le Kv^2(r),$$

for sufficiently large K, since $v(0) > 0$ and v is increasing on $[0, R]$ [for example, $K = [3v(0)/2]^{-1/2}$]. Now, if $x^2 < R^2$, then

$$\begin{aligned}\Delta[v(x^2/R)] &= v''(x^2/R)[\nabla(x^2/R)]^2 + v'(x^2/R)[\Delta(x^2/R)]\\ &= [4x^2/R^2]v^2(x^2/R) + [2/R]v'(x^2/R)\\ &\le [4 + 2K/R]v^2(x^2/R).\end{aligned}$$

Set $C = 4 + 2K/R$; then $u(x;\theta) < Cv(x^2/R)$ $(= +\infty$ if $|x| = R)$. If $u(x,\theta) > Cv(x^2/R)$ for some $x \in B(0; R)$, then $u(x;\theta) - Cv(x^2/R)$ would have to assume a positive maximum at some point $x \in B(0; R)$ at which

$$\begin{aligned}0 &\ge \Delta[u(x;\theta) - Cv(x^2/R)]\\ &\ge u^2(x;\theta) - C[4 + 2K/R]v^2(x^2/R)\\ &= u^2(x;\theta) - C^2v^2(x^2/R) > 0.\end{aligned}$$

Thus $u(x;\theta) \le Cv(x^2/R)$ for all $x \in B(0; R)$ and the lemma is established. □

PROPOSITION (3.15). *Let $u(x; R, \theta)$ denote the solution in Lemma (3.4). Then $u(x; R) := \lim_{\theta\to+\infty} u(x; R,\theta)$ exists for $x \in B(0; R)$ and is the unique positive solution of the singular boundary value problem*

$$(3.16)_R \qquad \begin{aligned}\Delta u(x) &= u^2(x), && \text{for } x \in B(0; R),\\ u(x) &\to +\infty, && \text{as } x \to \partial B(0; R).\end{aligned}$$

Furthermore, $u(x, R) = R^{-2}u(R^{-1}x; 1)$ and is radial.

PROOF. As θ increases, so does $u(\cdot; R,\theta)$ (R fixed); this is inherited from the corresponding behavior of the $u_{n,m}$ in Lemma (3.4), due to Lemma (3.2). By Lemma (3.14) for $d \ge 2$, and Proposition (3.5) for $d = 1$, $u(x; R) := \lim_{\theta\to+\infty} u(x; R,\theta)$ exists and is finite for $x \in B(0; R)$. It can be shown as in the proof of Proposition (3.5), that $\lim_{|x|\to R} u(x; R) = +\infty$: One argues again that $\lim_{r\to+\infty} u(r;\theta) = \theta$ and integrates the radial version of the equation for u, multiplied by $u'(r;\theta)$, on $[0, +\infty)$. The result is that

$$\int_0^\infty (d-1)r^{-1}[u'(r;\theta)]^2\,dr = -(2/3)\theta^3 + \theta^2 u(R;\theta) - u^3(0;\theta)/3;$$

which implies that $u(R;\theta) > 2\theta/3 \to +\infty$ as $\theta \to +\infty$. As in the proof of Lemma (3.4) it follows that $\Delta u = u^2$ in $B(0; R)$. Therefore we can derive inequalities such as (3.11) and (3.13) with equality there replaced by inequality $(\le)$, since $u'(r; R) \ge 0$ for $0 \le r < R$. It follows immediately that $\lim_{r\to R^-} u(r) = +\infty$.

To establish that $(3.16)_R$ admits only one solution, we consider two positive solutions u_1 and u_2 and set $v_2(x) = c^2u_2(cx)$, for $0 < c < 1$. Then $\Delta v_2 = v_2^2$ in $B(0; R)$ and v_2 is finite on $\partial B(0; R)$. If u_1 were ever less than v_2, then at a point where $u_1 - v_2$ assumes its negative absolute minimum, $0 \le \Delta(u_1 - v_2) = u_1^2 - v_2^2 < 0$, which is absurd. Therefore $u_1(x) \ge c^2u_2(cx)$. Letting $c \to 1^-$ yields $u_1(x) \ge u_2(x)$. Since the roles of u_1 and u_2 are interchangeable, we conclude that $u_1 = u_2$.

The radial property follows from the unicity and the invariance of $(3.16)_R$ under rotations. The self-similarity property viz. $u(x; R) = R^{-2}u(R^{-1}x; 1)$ follows from the unicity, since $R^{-2}u(R^{-1}x; 1)$ is clearly a solution of $(3.16)_R$. □

4. A singular elliptic boundary value problem II. In this section we examine the problem of existence and uniqueness for the "exterior" problem:

$$(4.1)_\varepsilon \qquad \begin{aligned} \Delta u(x) &= u^2(x), && x \in B^c(0; \varepsilon), \\ u(x) &\to +\infty, && \text{as } x \to \partial B(0; \varepsilon), \\ u(x) &\to 0, && \text{as } |x| \to +\infty. \end{aligned}$$

As in the previous section, we present the solution in such a way as to support the proof of a theorem in Section 2—Theorem 2. We omit details when they are very similar to those of Section 3.

PROPOSITION (4.2). *The problem $(4.1)_\varepsilon$ admits a unique positive solution $u \equiv u(\cdot; \varepsilon)$. It is strictly positive and radial. Moreover, if, for $n > \varepsilon^{-1}$,*

$$\varphi_n(x) := \begin{cases} 1, & 0 \le |x| \le \varepsilon - n^{-1}, \\ n(\varepsilon - |x|), & \varepsilon - n^{-1} < |x| < \varepsilon, \\ 0, & \varepsilon \le |x|, \end{cases}$$

and $u_n(\cdot; \theta)$ is the solution of the equation, $\Delta u_n - u_n^2 + \theta^2\varphi_n = 0$, then for $x \in B^c(0; \varepsilon)$, $u(x) = \lim_{\theta \to +\infty}\lim_{n \to +\infty}u_n(x; \theta)$, both limits being increasing. Also, $u(x; \varepsilon) = \varepsilon^{-2}u(\varepsilon^{-1}x; 1)$.

PROOF. The unicity for $(4.1)_\varepsilon$ follows as in the proof of Proposition (3.15) with "$c < 1$" replaced by "$c > 1$" and "$c \to 1^-$" replaced by "$c \to 1^+$." The radial property follows from the unicity and the invariance of $(4.1)_\varepsilon$ under rotations. The self-similarity property viz. $u(x; \varepsilon) = \varepsilon^{-2}u(\varepsilon^{-1}x; 1)$ also follows from the unicity.

The existence, positivity, uniqueness and monotonicity of u_n are established by (2.5) and Lemma (3.2); again $u_n(\cdot; \theta)$ is radial. Denote by $u(\cdot; \theta) := \lim_{n \to +\infty}u_n(\cdot; \theta)$. Then as in Lemma (3.4), $u(\cdot; \theta) \in C^1(\mathbb{R}^d) \cap C^2(\mathbb{R}^d \setminus \partial B(0; \varepsilon))$, $0 < u(x; \theta) \le \theta$, and is the unique solution, on $\mathbb{R}^d \setminus \partial B(0; \varepsilon)$, of the equation, $\Delta u - u^2 + \theta^2 \cdot 1_{B(0; \varepsilon)} = 0$. As in Lemma (3.4), since $u(\cdot; \theta)$ is an increasing function of $\theta \in \mathbb{R}_+$, the proposition will be proved once we establish an upper bound for $u(x; \theta)$ with respect to θ; and then the boundary conditions. We switch to the radial notation for the remainder of the proof and suppress the dependence of u on θ when convenient.

Concerning the upper bound, define $w(r) := 6(r - \varepsilon)^{-2}$ for $r > \varepsilon$. Then $w'' = w^2$ and $w' < 0$. Therefore for $r > \varepsilon$,

$$(4.3) \quad [u(r) - w(r)]'' + [(d-1)/r][u(r) - w(r)]' \ge u^2(r) - w^2(r).$$

As $r \to \varepsilon^+$, $u(r) - w(r) \to -\infty$; and as $r \to +\infty$, $u(r) - w(r) \to 0$. If $u - w$ were ever positive, then it would assume a positive maximum at some $r > \varepsilon$, at which the left-hand side of (4.3) would be nonpositive, while the right-hand side would be positive, which is absurd. Therefore $u(r;\theta) \le 6(r-\varepsilon)^{-2}$, and so $u(r) \equiv \lim_{\theta\to+\infty} u(r;\theta) \le 6(r-\varepsilon)^{-2} < +\infty$, for $r > \varepsilon$; in particular, $\lim_{r\to+\infty} u(r) = 0$.

Concerning the boundary condition at $r=\varepsilon$, we first establish some estimates for $u \equiv u(\cdot;\theta)$. For $r \ne \varepsilon$,

(4.4) $$u''(r) + [(d-1)/r]u'(r) = u^2(r) - \theta^2 \cdot 1_{[0,\varepsilon]}(r), \qquad 0 < u(r) < \theta.$$

Therefore, for $0 < r < \varepsilon$, $[r^{d-1}u'(r)]' = r^{d-1}[u^2(r) - \theta^2] \le 0$, which implies that $r^{d-1}u'(r)$ is decreasing. Since $u'(0) = 0$, $u'(r) < 0$, i.e., u is decreasing on $[0,\varepsilon)$. Integrating the equation twice we find that for $0 \le r < r_1 < \varepsilon$,

(4.5) $$\begin{aligned} u(r_1) - u(r) &= \int_r^{r_1} s^{1-d} \int_0^s t^{d-1}\left[u^2(t) - \theta^2\right] dt\,ds \\ &\le \int_r^{r_1} s^{1-d} \int_r^s t^{d-1}\left[u^2(t) - \theta^2\right] dt\,ds. \end{aligned}$$

Since $0 < u(r_1) < u(t) < u(r)$, if $\lim_{\theta\to+\infty} u(r;\theta) < +\infty$, then the left-hand side of (4.5) remains bounded as $\theta \to +\infty$, while the right-hand side clearly diverges to $-\infty$. Thus $\lim_{\theta\to+\infty} u(r;\theta) = +\infty$ for all $r \in [0,\varepsilon)$.

For $r > \varepsilon$, $[r^{d-1}u'(r)]' = r^{d-1}u^2(r) > 0$, which implies that $r^{d-1}u'(r)$ is increasing. If u' were ever positive, it would remain so thereafter; but this contradicts the already-established result that $\lim_{r\to+\infty} u(r) = 0$, $u > 0$. Therefore $u'(r) \le 0$; also $-c := \lim_{r\to+\infty} r^{d-1}u'(r)$ exists. Returning to (4.4) we have

$$\begin{aligned} u'' \ge u^2 &\Rightarrow \left[(u'/2)^2\right]' \le \left[(u^3/3)\right]' \\ &\Rightarrow \left[u'(\varepsilon)\right]^2 \ge (2/3)u^3(\varepsilon) \\ &\Rightarrow u'(\varepsilon) \le -\sqrt{(2/3)}\,u^{3/2}(\varepsilon), \end{aligned}$$

since $\lim_{r\to+\infty} u'(r) = 0$ [for $d = 1$, $c = 0$—otherwise $\lim_{r\to+\infty} u(r) = -\infty$]. Therefore if $u'(\varepsilon;\theta)$ remains bounded as $\theta \to +\infty$, then so does $u(\varepsilon;\theta)$; and also $u'(r;\theta)$ for $0 \le r \le \varepsilon$, since $r^{d-1}u'(r;\theta)$ is a decreasing function of r when $0 \le r \le \varepsilon$. By the mean value theorem (on $[r,\varepsilon]$), we would then have that $u(r;\theta)$ would remain bounded as $\theta \to +\infty$, contradicting $\lim_{\theta\to+\infty} u(r;\theta) = +\infty$, for $0 \le r < \varepsilon$. Therefore $u'(\varepsilon;\theta) \to -\infty$ as $\theta \to +\infty$, at least along some sequence.

Integrating the equation once on $[\varepsilon, +\infty)$, we obtain (with $c \equiv c_\theta$)

(4.6) $$c_\theta - \varepsilon^{d-1}u'(\varepsilon;\theta) = \int_\varepsilon^\infty r^{d-1}u^2(r;\theta)\,dr.$$

Now it can be shown [see, e.g., Taliaferro (1978); case I after the change of variables $u(t) := tu([t/(d-2)]^{1/(d-2)})$ for $d \ge 3$; and $y(t) := u(\log t)$ for $d = 2$] that any bounded, positive solution v of (4.4) restricted to the interval $[\varepsilon + 1, +\infty)$, say, behaves asymptotically (in ratio: denoted by $\sim$) as

(4.7) $$v(r) \sim \begin{cases} 2(4-d)r^{-2}, & 1 \le d \le 3, \\ 2r^{-2}\log^{-1}(r), & d = 4, \\ [c/(d-2)]\,r^{2-d}, & d \ge 5, \end{cases}$$

where the value of c (consistent with our usage) is also implicitly described by $-c = \lim_{r \to +\infty} r^{d-1} v'(r)$ and is nonzero ($d \geq 5$); moreover, $v(r) \geq [c/(d-2)]r^{2-d}$. [Actually, the analysis continues from our (4.6) with ε replaced by s and one more ds-integration.] In particular, $\int_{\varepsilon+1}^{\infty} r^{d-1} v^2(r)\,dr < +\infty$ for all d. Returning to (4.6), we know that $u(r) \geq u(r; \theta) \geq [c_\theta/(d-2)]r^{2-d}$, $r > \varepsilon$, so that the constants c_θ remain bounded as $\theta \to +\infty$. Letting $\theta \to +\infty$ [along a sequence such that $u'(\varepsilon; \theta) \to -\infty$] we conclude that $\int_\varepsilon^\infty r^{d-1} u^2(r)\,dr = +\infty$, while $\int_{\varepsilon+1}^{\infty} r^{d-1} u^2(r)\,dr < +\infty$. As u is decreasing on $(\varepsilon, +\infty)$, it follows that $\lim_{r \to \varepsilon^+} u(r) = +\infty$. □

5. Extensions to the infinite-variance case. In this section we extend the results presented in Section 1 to the class of measure-valued branching processes considered in Iscoe (1986a) [and in Watanabe (1968) for the case $X_0 \in M_0(\mathbb{R}^d)$], where the branching mechanism has infinite variance. This is reflected in the replacement of the term u^2 in (1.2) by $u^{1+\beta}$ for a choice of $\beta \in (0,1)$. As such, the same change is made in (1.4), (2.2), (2.4), (2.5), etc. Since there is essentially no qualitative changes needed in the proofs presented in Sections 2 and 3, we shall simply indicate the quantitative changes in the statements and proofs of the theorems. If the only change in a lemma is the alteration, $u^2 \mapsto u^{1+\beta}$, we omit any special mention. Note that it is not necessary to require that $\beta < 1$ in Section 3; $\beta > 0$ is sufficient. The principal novelty in considering this class of processes is that their behavior depends strongly on the relationship between β and d, the dimension of the underlying Euclidean space.

PROPOSITION $(3.5)_\beta$. *Replace u^2 by $u^{1+\beta}$ throughout and the value of $u(0)$ in (3.8) by*

$$u(0) = [2(2+\beta)]^{-1/\beta}\{\Gamma(1/2)\Gamma(\beta/[2(2+\beta)])/\Gamma((1+\beta)/(2+\beta))\}^{2/\beta}R^{-2/\beta}.$$

Appropriate changes are made in integration within the proof.

PROPOSITION $(3.15)_\beta$. *Replace u^2 by $u^{1+\beta}$ throughout and amend the self-similarity property to read $u(x; R) = R^{-2/\beta}u(R^{-1}x; 1)$.*

Various exponents are altered when appropriate minor changes are made when integrating; e.g., in (3.13) v^3 is replaced by $v^{2+\beta}$ and $\sqrt{u(0)}$ by $[u(0)]^{\beta/2}$, etc. Also, in the uniqueness argument we set $v_2(x) = c^{2/\beta}u_2(cx)$.

PROPOSITION $(4.2)_\beta$. *Replace u^2 by $u^{1+\beta}$ throughout and θ^2 by $\theta^{1+\beta}$. The self-similarity property is $u(x; \varepsilon) = \varepsilon^{-2/\beta}u(\varepsilon^{-1}x; 1)$.*

The upper bound w is now $w(r) = c_{\beta,d}(r-\varepsilon)^{-2/\beta}$, where $c_{\beta,d} = \{2[2-(d-2)\beta]/\beta^2\}^{1/\beta}$. Various exponents are altered in integration; e.g., the estimate for $u'(\varepsilon)$ is now $u'(\varepsilon) \leq -[2/(2+\beta)]^{1/2}[u(\varepsilon)]^{1+\beta/2}$, etc. The asymptotic description at (4.7) is now

$$v(r) \sim \begin{cases} c_{\beta,d} r^{-2/\beta}, & \text{if } (d-2)\beta < 2, \\ [\sqrt{2}/\beta]^{2/\beta}[r^2 \log r]^{-1/\beta}, & \text{if } (d-2)\beta = 2, \\ c r^{2-d}, & \text{if } (d-2)\beta > 2. \end{cases}$$

Also, $\int_{\varepsilon+1}^{\infty} r^{d-1} v^{1+\beta}(r)\, dr < +\infty$.

THEOREM 1_β. *Replace u with $u^{1+\beta}$ in* (1.4), *R^{-2} with $R^{-2/\beta}$ in* (1.5) *and* (1.6) *by*

$$u(0) = [2(2+\beta)]^{1/\beta}\{\Gamma(1/2)\Gamma(\beta/[2(2+\beta)])/\Gamma((1+\beta)/(2+\beta))\}^{2/\beta}.$$

COROLLARY$_\beta$. *Replace the expression for E_{δ_0} by* $[u(0)]^{\beta/2}\Gamma(1-\beta/2)$.

In the proof we consider $f(t) = \int_0^\infty r^{-2}[1 - \exp(-tr^{2/\beta})]\, dr$ and show that

$$(df/dt)(t) = (\beta/2)t^{\beta/2-1}\int_0^\infty s^{-\beta/2}e^{-s}\, ds \equiv (\beta/2)t^{\beta/2-1}\Gamma(1-\beta/2),$$

where $s = tr^{2/\beta}$.

THEOREM 2_β. *For each $\varepsilon > 0$,*

$$P_{\delta_0}(X_0 \text{ ever charges } B(x;\varepsilon)) \sim \begin{cases} c_{\beta,d}|x|^{-2/\beta}, & \text{if } (d-2)\beta < 2, \\ c[x^2\log|x|]^{-1/\beta}, & \text{if } (d-2)\beta = 2, \\ c'_{\beta,d}\varepsilon^{d-2-2/\beta}|x|^{2-d}, & \text{if } (d-2)\beta > 2, \end{cases}$$

where $c_{\beta,d} = \{2[2-(d-2)\beta]/\beta^2\}^{1/\beta}$, $c = [\sqrt{2}/\beta]^{2/\beta} \equiv [(d-2)/\sqrt{2}]^{d-2}$, *and $c'_{\beta,d}$ is some constant* (*depending only on β and d*).

In the proof we replace u^2 with $u^{1+\beta}$, θ^2 with $\theta^{1+\beta}$ and ε^{-2} by $\varepsilon^{-2/\beta}$.

THEOREM 3_β. *Let $X_0 = \mu$. Then for each $R > 0$ there exists a finite random time τ_R such that with P_μ-probability* 1, *$X_t(B(0;R)) = 0$ for all $t \geq \tau_R$* (*local extinction*) *if $\int_{\mathbf{R}^d}(1+|x|)^{-2/\beta}\, d\mu(x) < +\infty$.*

In case $\mu = \lambda$ we have local extinction iff $\beta d < 2$. Moreover, τ_R is an absolutely continuous random variable such that $0 < \liminf_{t\to\infty} t^\delta P_\lambda(\tau_R > t) \leq \limsup_{t\to\infty} t^\delta P_\lambda(\tau_R > t) < \infty$, *where* $\delta \equiv (2-\beta d)/(2\beta)$. *Consequently,* $E_\lambda[\tau_R] < +\infty$ *iff* $0 < \beta d < 2(1-\beta)$.

We remark that when μ is finite it is well known [see, e.g., Jirina (1966)] that we have global extinction. For fixed d, the condition on β and d concerning $E_\lambda[\tau_R]$ can be rewritten as

$$(*) \qquad \begin{aligned} E_\lambda[\tau_R] &< +\infty, & &\text{if } 0 < \beta < 2/(d+2), \\ E_\lambda[\tau_R] &= +\infty, & &\text{if } 2/(d+2) \leq \beta \leq 1. \end{aligned}$$

Since β is inversely related to the size of the tail of the "offspring" distribution, we see, in a very quantitative way, how the small likelihood of having a large progeny (due to criticality) affects the mean local extinction time. The existence of some description, as in $(*)$, is intuitively plausible since the relatively rare large progenies over successive "generations" would quite likely not be able to diffuse to (or back to) a fixed bounded set before eventually expiring.

In the proof of Theorem 3_β we replace u^2 with $u^{1+\beta}$, θ^2 with $\theta^{1+\beta}$, v^2 with $v^{1+\beta}$, $\theta/[1+\theta t]$ with $[\beta t+\theta^{-\beta}]^{-1/\beta}$ [so that $v(t;\cdot,\theta)\le(\beta t)^{-1/\beta}$], $w(r)$ by the solution from Proposition $(4.2)_\beta$ with $\varepsilon\equiv R$ [so that $w(r)\sim c_{\beta,d}r^{-2/\beta}$ as $r\to+\infty$; the latter being integrable when $\beta d<2$], w^2 by $w^{1+\beta}$, (2.11) by $v(t,x)\le w(r)$ and (2.13) by $(w-v)^{\cdot}-\Delta(w-v)=-(w^{1+\beta}-v^{1+\beta})$. The converse "local extinction $\Rightarrow \beta d<2$" is supplied by Theorem (4.3) of Iscoe (1986a).

APPENDIX

In this Appendix we collect some recent results concerning the temporal asymptotic behavior of nonnegative solutions of the semilinear parabolic p.d.e.

$$\text{(A.1)}\qquad \begin{aligned}&\dot v(t,x)=\Delta v(t,x)-v^{1+\beta}(t,x),\\ &v(t,x)>0,\end{aligned}\qquad t>0,\ x\in\mathbb{R}^d.$$

In (A.1) we make the standing assumption that $0<\beta\le 1$ and $\beta d<2$ and will sometimes assume

$$\text{(A.2)}\qquad \begin{aligned}&\exists\, t_0>0 \text{ and } c_1,c_2>0 \text{ such that for all } x\in\mathbb{R}^d:\\ &0\not\equiv v(t_0,x)\le c_1\exp(-c_2x^2).\end{aligned}$$

The first result is taken from Brezis, Peletier and Terman (1986).

THEOREM (A.3). *There is a unique solution to the p.d.e. at* (A.1) *on* $(0,+\infty)\times\mathbb{R}^d$ *of the form*

$$\text{(A.4)}\qquad w(t,x)=t^{-1/\beta}f(|x|/\sqrt{t}),\qquad t>0,\ x\in\mathbb{R}^d,$$

where $f\colon[0,+\infty)\to\mathbb{R}_+$ *is smooth,* $f'(0)=0$, *and such that* f *possesses the bounds*

$$c_1\exp(-c_2r^2)\le f(r)\le c_3\exp(-c_4r^2),\qquad r>0,$$

for some positive constants c_1,c_2,c_3,c_4.

[Note that $w(0^+,x)=0$ for $x\in\mathbb{R}^d\setminus\{0\}$ and that $w(0^+,0)=+\infty$.]
The next two results were obtained by Escobedo and Kavian (1985).

THEOREM (A.5). *Let* v *be a solution of* (A.1) *satisfying* (A.2), *and let* w *be the solution at* (A.4). *Then*

$$\text{(A.6)}\qquad \lim_{t\to+\infty}t^{1/\beta}\|v(t)-w(t)\|_\infty=0,$$

where $\|\ \|_\infty$ *denotes the usual* sup-norm *on* $C_0(\mathbb{R}^d)$.

The next result which we quote appears in Lemma (2.2) of Escobedo and Kavian (1985).

LEMMA (A.7). *Let v be a solution of* (A.1) *satisfying* (A.2). *Then there are constants $c > 1$ and $t_2 > 0$ such that*

$$v(t,x) \le cw(t_2 + t, x), \qquad t > t_0,\ x \in \mathbb{R}^d, \tag{A.8}$$

where w is the solution at (A.4).

We now deduce a consequence of these results which is similar to Lemma 5.1 of Dawson, Fleishmann, Foley and Peletier (1986).

LEMMA (A.9). *Let v be a solution of* (A.1) *satisfying* (A.2). *Then*

$$\lim_{t \to +\infty} t^{\delta}\langle v(t), \lambda\rangle = \langle \vec{f}, \lambda\rangle,$$

where $\delta = (2 - \beta d)/(2\beta) > 0$, and $\vec{f}(x) = f(|x|)$, $x \in \mathbb{R}^d$, with f as in Theorem (A.3).

PROOF.

$$\begin{aligned}\langle v(t), \lambda\rangle &= \langle w(t), \lambda\rangle + \langle v(t) - w(t), \lambda\rangle \quad [w \text{ given at (A.4)}]\\ &= t^{-\delta}\langle \vec{f}, \lambda\rangle + \langle v(t) - w(t), \lambda\rangle,\end{aligned}$$

via the change of variables $x \mapsto \sqrt{t}x$ in $\langle \vec{f}, \lambda\rangle \equiv \int_{\mathbb{R}^d} f(|x|)\,dx$.

Therefore $t^{\delta}\langle v(t), \lambda\rangle = \langle \vec{f}, \lambda\rangle + t^{\delta}\langle v(t) - w(t), \lambda\rangle$. Choosing c and t_2 as in (A.8) of Lemma (A.7) and considering $t > t_0$ we can dominate the second integral in absolute value by

$$\begin{aligned}&\int_{\{|x| \ge \sqrt{t_2 + t}R\}} t^{\delta} c w(t_2 + t, x)\,dx + \int_{\{|x| \ge \sqrt{t_2 + t}R\}} t^{\delta} w(t,x)\,dx\\ &\quad + \int_{\{|x| \le \sqrt{t_2 + t}R\}} t^{\delta}\|v(t) - w(t)\|_{\infty}\,dx\\ &\quad \le (2^d c + 1)\int_{\{|x| \ge R\}} f(|x|)\,dx + (4R)^d t^{1/\beta}\|v(t) - w(t)\|_{\infty},\end{aligned}$$

if also $t > t_2/3$.

Then by Theorem (A.5),

$$\begin{aligned}&\limsup_{t \to +\infty} |t^{\delta}\langle v(t) - w(t), \lambda\rangle|\\ &\qquad \le (2^d c + 1)\int_{\{|x| \ge R\}} f(|x|)\,dx \to 0 \quad \text{as } R \to +\infty. \qquad \square\end{aligned}$$

LEMMA (A.10). *Let v be a solution of* (A.1) *such that for some positive constants c, t_0 and R, $v(t,x) \le ct^{-1/\beta}$ for $t > t_0$ and $x \in \mathbb{R}^d$, and $v(t,x) \le c|x|^{-2/\beta}$ for $t > t_0$ and $|x| > R$. Then*

$$(\text{A.11}) \qquad 0 < \liminf_{t\to+\infty} t^{\delta}\langle v(t), \lambda\rangle \le \limsup_{t\to+\infty} t^{\delta}\langle v(t), \lambda\rangle < +\infty,$$

where $\delta = (2 - \beta d)/(2\beta) > 0$.

PROOF. The solution $\tilde{v}$ of (A.1) for $t \ge t_0$, such that $\tilde{v}(t_0, x) := v(t_0, x)1_{B(0;1)}(x)$, minorizes v for $t \ge t_0$. Since Lemma (A.9) applies to $\tilde{v}$, the lower bound in (A.11) is established. For the upper bound we let K denote a generic constant (depending on c, β and d but not on $t > t_0 + R^2$) whose value varies from line to line in the following estimation:

$$\begin{aligned}\langle v(t), \lambda\rangle &= \int_{\{|x| \le \sqrt{t}\}} v(t,x)\,dx + \int_{\{|x| \ge \sqrt{t}\}} v(t,x)\,dx \\ &\le Kt^{-1/\beta}t^{d/2} + K\int_{\sqrt{t}}^{\infty} r^{-2/\beta} r^{d-1}\,dr \\ &\le Kt^{-\delta}. \end{aligned}$$

□

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