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REGULARIZED SELF-INTERSECTION LOCAL TIMES OF PLANAR BROWNIAN MOTION[1]

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Let

$$T_k^\varepsilon(\lambda; t_1,\dots,t_k) = \rho(X_{t_1})q^\varepsilon(X_{t_2} - X_{t_1}) \cdots q^\varepsilon(X_{t_k} - X_{t_{k-1}}),$$

where X_t is a Brownian motion in $\mathbb{R}^2$, $\lambda(dx) = \rho(x)\,dx$ and q^ε converges to Dirac's delta function as $\varepsilon \downarrow 0$. The self-intersection local times of order k are described by a generalized random field

$$T_k(\lambda; t_1,\dots,t_k) = \lim_{\varepsilon\downarrow 0} T_k^\varepsilon(\lambda; t_1,\dots,t_k), \quad \text{for } 0 < t_1 < \cdots < t_k.$$

The field "blows up" as $t_i - t_j \to 0$ for some $i \neq j$. We show that with a proper choice of the coefficients $B_k^l(\varepsilon)$, a generalized random field

$$\mathscr{T}_k(\lambda; t_1,\dots,t_k) = \lim_{\varepsilon\downarrow 0}\left[T_k^\varepsilon(\lambda; t_1,\dots,t_k) + \sum_{l=1}^{k-1}\left[B_k^l(\varepsilon)T_l^\varepsilon\right](\lambda; t_1,\dots,t_k)\right]$$

is well defined for all $0 \le t_1 \le \cdots \le t_k$ and it coincides with $T_k(\lambda; t_1,\dots,t_k)$ for $t_1 < \cdots < t_k$.

1. Main results.

1.1. We denote by (X_t, P_μ) the Brownian motion in $\mathbb{R}^2$ with the initial law μ (which can be any σ-finite measure on $\mathbb{R}^2$). If $0 < t_1 < \cdots < t_n$, then the joint probability density for $X_{t_1},\dots,X_{t_n}$ is given by the formula

$$p_\mu(t,x) = \int \mu(dx_0)p_{t_1}(x_1 - x_0)p_{t_2-t_1}(x_2 - x_1)\cdots p_{t_n-t_{n-1}}(x_n - x_{n-1}). \tag{1.1}$$

Here

$$p_t(x) = t^{-1}p(x/\sqrt{t}), \qquad p(z) = (2\pi)^{-1}e^{-|z|^2/2}. \tag{1.2}$$

We start from a probability density $q(z)$ on $\mathbb{R}^2$ such that

$$\begin{aligned} &\int |\ln|x|\,|^k q(x)\,dx < \infty, \quad \text{for all } k > 0,\\ &\int e^{\beta|x|}q(x)\,dx < \infty, \quad \text{for some } \beta > 0. \end{aligned} \tag{1.3}$$

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Put

$$q^{\varepsilon}(x) = \varepsilon^{-2} q(x/\varepsilon), \tag{1.4}$$

fix a measure $\lambda(dx) = \rho(x)\,dx$ and consider a sequence of functions

$$T_k^{\varepsilon}(\lambda; t_1, \dots, t_k) = \rho(X_{t_1}) q^{\varepsilon}(X_{t_2} - X_{t_1}) \cdots q^{\varepsilon}(X_{t_k} - X_{t_{k-1}}), \tag{1.5}$$
$$(t_1, \dots, t_k) \in D_k, \text{ for } k = 1, 2, \dots .$$

Here

$$D_k = \{0 \le t_1 \le \cdots \le t_k\}. \tag{1.6}$$

1.2. Let D be a region in $\mathbb{R}^k$. A *generalized random field* (g.r.f.) over D is a continuous linear mapping F from a space $\mathscr{D}$ of functions on D (test functions) to a space L of random variables [i.e., measurable functions on $(\Omega, \mathscr{F}, P_\mu)$].

In this paper we use the following test functions. Put

$$\|\varphi\|_\beta = \sup_{t \in D} e^{\beta|t|} |\varphi(t)|, \quad \text{for } \beta > 0, \qquad |t| = |t_1| + \cdots + |t_k|,$$

and denote by $\mathscr{D}^\beta(D)$ the set of all functions φ on D which are infinitely differentiable (including the boundary) and satisfy the condition $\|\mathbb{D}^l \varphi\|_\beta < \infty$ for all $l = (l_1, \dots, l_k)$. Here $\mathbb{D}^l = \mathbb{D}_1^{l_1} \cdots \mathbb{D}_k^{l_k}$ with $\mathbb{D}_i = \partial/\partial t_i$. For every positive integer n we put

$$\|\varphi\|_{\beta, n} = \sup_{|l| < n} \|\mathbb{D}^l \varphi\|_\beta .$$

The space of test functions $\mathscr{D}(D)$ is the union of $\mathscr{D}^\beta(D)$ over all $\beta > 0$. The convergence $\varphi_n \to \varphi$ in $\mathscr{D}(D)$ means that all φ_n and φ belong to the same $\mathscr{D}^\beta(D)$ and $\|\varphi_n - \varphi\|_{\beta, n} \to 0$ for all n.

The space L of random variables is defined as the intersection of $L^p(P_\mu)$ over all $p \ge 2$; the convergence $Y_n \to Y$ in L means that $P_\mu |Y_n - Y|^p \to 0$ for all $p \ge 2$.

The formula

$$T_k^{\varepsilon}(\lambda; \varphi) = \int_{D_k} T_k^{\varepsilon}(\lambda; t) \varphi(t)\, dt, \qquad \varphi \in \mathscr{D}(D_k),$$

defines a g.r.f. if

(a) λ has a bounded density; and
(b) either μ is finite or μ has a bounded density and then λ is finite.

We write $f_1 \simeq f_2$ if $f_1(\varepsilon) - f_2(\varepsilon) = O(|\varepsilon|^\alpha)$ for some $\alpha > 0$ as $|\varepsilon| \to 0$ [in fact, everywhere we use the notation $\simeq$, $f_1(\varepsilon) - f_2(\varepsilon) = O(|\varepsilon|^\alpha)$ for all $0 < \alpha < 1$ and, in many cases, for all $0 < \alpha < 2$].

1.3. Suppose that B is a continuous linear mapping from $\mathscr{D}(D_k)$ to $\mathscr{D}(D_l)$. To every g.r.f. F over D_l there corresponds a g.r.f. BF over D_k defined by the formula $(BF)(\varphi) = F(B\varphi)$.

For every $l \le k$ we consider a mapping B_k^l from $\mathscr{D}(D_k)$ to $\mathscr{D}(D_l)$ given as

$$(1.7) \qquad (B_k^l\varphi)(t_1,\dots,t_l) = \sum_\sigma \varphi(t_{\sigma_1},\dots,t_{\sigma_k}),$$

where the sum is taken over all mappings σ from $\{1,2,\dots,k\}$ onto $\{1,2,\dots,l\}$ such that $\sigma_i \le \sigma_j$ for $i < j$. For instance,

$$\begin{aligned}(B_k^k\varphi)(t_1,\dots,t_k) &= \varphi(t_1,\dots,t_k),\\ (B_k^{k-1}\varphi)(t_1,\dots,t_{k-1}) &= \varphi(t_1,t_1,t_2,\dots,t_{k-1}) + \varphi(t_1,t_2,t_2,\dots,t_{k-1})\\ &\quad + \cdots + \varphi(t_1,t_2,\dots,t_{k-1},t_{k-1}),\\ &\;\;\vdots\\ (B_k^1\varphi)(t_1) &= \varphi(t_1,\dots,t_1).\end{aligned}$$

Our main result is stated in the following theorem.

THEOREM 1.1. *Let measures* μ, λ *satisfy conditions* (a), (b) *in Section* 1.2 *and let* q *satisfy condition* (1.3). *Put*

$$(1.8) \qquad \mathscr{T}_k^\varepsilon(\lambda) = \sum_{l=1}^k h_\varepsilon^{k-l}(B_k^l T_l^\varepsilon)(\lambda),$$

where B_k^l *are given by* (1.7),

$$(1.9) \qquad h_\varepsilon = \frac{1}{\pi}\left\{\ln\varepsilon + \int\left[C + \ln\frac{|y|}{\sqrt{2}}\right]q(y)\,dy\right\}$$

and $C = 0.5772157\dots$ *is Euler's constant.*

There exist generalized random fields $\mathscr{T}_k(\lambda)$ (*independent of* q) *such that*

$$(1.10) \qquad \mathscr{T}_k(\lambda,\varphi) = \lim_{\varepsilon\downarrow 0}\mathscr{T}_k^\varepsilon(\lambda;\varphi), \quad \text{in } L \text{ for all } \varphi \in \mathscr{D}(D_k).$$

Moreover, for every $m \ge 2$ *and each* $\varphi \in \mathscr{D}(D_k)$,

$$(1.11) \qquad P_\mu\left[|\mathscr{T}_k(\lambda,\varphi) - \mathscr{T}_k^\varepsilon(\lambda;\varphi)|^m\right] \simeq 0.$$

REMARK. The limit of the field (1.8) exists with h_ε replaced by $h_\varepsilon + \kappa$ with an arbitrary constant κ. For example, we can take $h_\varepsilon = (1/\pi)\ln\varepsilon$. Our choice of κ is made to get the limit independent of q.

1.4. Put $T_k(\varepsilon,\lambda,u) = T_k^\varepsilon(\lambda;\psi_{ku})$, $\mathscr{T}_k(\varepsilon,\lambda,u) = \mathscr{T}_k^\varepsilon(\lambda;\psi_{ku})$, where $\psi_{ku}(t) = 1_{t_k<u}$, $t \in D_k$, $u > 0$. Since $B_k^l\psi_{ku} = \left[\begin{smallmatrix}k-1\\ l-1\end{smallmatrix}\right]\psi_{lu}$, we have

$$(1.12) \qquad \mathscr{T}_k(\varepsilon,\lambda,u) = \sum_{l=1}^k \begin{bmatrix}k-1\\ l-1\end{bmatrix} h_\varepsilon^{k-l}T_l(\varepsilon,\lambda,u).$$

It is proved in Dynkin (1987) that there exist random variables $\mathscr{T}_k(\lambda,u)$ such that

$$\int_0^\infty du\, e^{-ru}P_\mu|\mathscr{T}_k(\varepsilon,\lambda,u) - \mathscr{T}_k(\lambda,u)|^p \simeq 0,$$

for all $r > 0$ and all $p \geq 2$. This implies

$$P_\mu \left| \int_0^\infty du\, e^{-ru} \mathscr{T}_k(\varepsilon, \lambda, u) - \int_0^\infty du\, e^{-ru} \mathscr{T}_k(\lambda, u) \right|^p \simeq 0.$$

Note that

$$\int_0^\infty du\, e^{-ru} \mathscr{T}_k(\varepsilon, \lambda, u) = \mathscr{T}_k^\varepsilon(\lambda, \varphi_r),$$

where $\varphi_r(t) = r^{-1} e^{-rt_k}$ belongs to $\mathscr{D}(D_k)$. It follows from (1.11) that

$$\mathscr{T}_k(\lambda; \varphi_r) = \int_0^\infty du\, e^{-ru} \mathscr{T}_k(\lambda, u). \tag{1.13}$$

1.5. The expression for $\mathscr{T}_k^\varepsilon(\lambda)$ can be simplified by the change of variables

$$v_1 = t_1, \qquad v_i = t_i - t_{i-1}, \quad \text{for } i = 2, \dots, k.$$

To every function f there corresponds a function $\tilde{f}$ such that $\tilde{f}(v_1, \dots, v_k) = f(t_1, \dots, t_k)$. In particular,

$$\tilde{T}_k^\varepsilon(\lambda, \mathbf{v}) = \rho(X_{v_1}) q^\varepsilon(X_{v_1+v_2} - X_{v_1}) \cdots q^\varepsilon(X_{v_1 + \cdots + v_k} - X_{v_1 + \cdots + v_{k-1}}). \tag{1.14}$$

Let $\tilde{\varphi}$ be a function of $\mathbf{v} = (v_1, \dots, v_k)$ and let $\Lambda = \{i_1, \dots, i_l\} \subset \{1, \dots, k\}$. We put $v_\Lambda = (v_{i_1}, \dots, v_{i_l})$ and we denote by $\tilde{\varphi}(v)_\Lambda$ the function of v_Λ obtained from $\tilde{\varphi}(v)$ by setting $v_j = 0$ for all $j \notin \Lambda$. In this notation we can rewrite (1.7) and (1.8) as

$$\tilde{\mathscr{T}}_k^\varepsilon(\lambda, \tilde{\varphi}) = \sum_\Lambda h_\varepsilon^{k-l} \int dv_\Lambda\, \tilde{\varphi}(v)_\Lambda \tilde{T}_l^\varepsilon(\lambda, v_\Lambda), \tag{1.15}$$

where Λ runs over all subsets of the set $\{1, \dots, k\}$ which contain 1 and $l = |\Lambda|$ is the cardinality of Λ.

1.6. We shall see in Section 3 that if φ vanishes near the boundary of D_k, then there exists

$$T_k(\lambda, \varphi) = \lim_{\varepsilon \downarrow 0} T_k^\varepsilon(\lambda, \varphi) \quad \text{in } L \tag{1.16}$$

and

$$P_\mu\left[T_{k_1}(\lambda_1, \varphi_1) \cdots T_{k_n}(\lambda_n, \varphi_n)\right]$$

$$= \int \mu(dz_0) \lambda_1(dz_1) \cdots \lambda_n(dz_n) \times \varphi_1(t^1) \cdots \varphi_n(t^n)\, dt^1 \cdots dt^n \tag{1.17}$$

$$\times m_{k_1 \dots k_n}(z_0, \dots, z_n; t^1, \dots, t^n).$$

The moment functions $m_{k_1 \dots k_n}$ can be described as follows. If $t^a = (t_1^a, \dots, t_{k_a}^a)$ and if

$$0 < t_{b_1}^{a_1} < \cdots < t_{b_N}^{a_N} \tag{1.18}$$

(here $N = k_1 + \cdots + k_n$), then

$$(1.19) \qquad m_{k_1 \cdots k_n}(z_0, \ldots, z_n; t^1, \ldots, t^n) = \prod_{i=1}^{N} p_{u_i}(z_{a_i} - z_{a_{i-1}}),$$

with $a_0 = 0$, and

$$(1.20) \qquad u_1 = t_{b_1}^{a_1}, \qquad u_2 = t_{b_2}^{a_2} - t_{b_1}^{a_1}, \ldots, u_N = t_{b_N}^{a_N} - t_{b_{N-1}}^{a_{N-1}}.$$

If $a_i = a_{i-1}$, then the corresponding factor

$$(1.21) \qquad p_{u_i}(0) = (2\pi u_i)^{-1}$$

is not integrable near the origin and this is the source of the trouble. However, the formula

$$(1.22) \qquad \langle \xi(u_i), \varphi(u_i) \rangle = \int_0^\infty \frac{\varphi(u_i) - e^{-u_i}\varphi(0)}{2\pi u_i}\, du_i$$

defines a generalized function which can be interpreted as a regularization of (1.21).

It turns out that (1.17) remains true with $T_{k_i}(\lambda, \varphi_i)$ replaced by $\mathscr{T}_{k_i}(\lambda, \varphi_i)$ if we replace in (1.19) every function (1.21) with its regularization (1.22).

1.7. We formulate this statement in a more precise way by using the concept of the direct (tensor) product of generalized functions.

A generalized function F of positive variables $u_1, \ldots, u_k$ is a continuous linear functional on $\mathscr{D}(\mathbb{R}_+^k)$ where $\mathbb{R}_+ = [0, \infty)$. Writing $\langle F(u), \varphi(u) \rangle$ means the same as $F(\varphi)$.

Formula (1.22) defines a generalized function of one positive variable u_i. Another example is *the delta function* $\langle \delta(u_i), \varphi(u_i) \rangle = \varphi(0)$. The formula

$$(1.23) \qquad \langle F(u), \varphi(u) \rangle = \int f(u)\varphi(u)\, du, \qquad \varphi \in \mathscr{D}(\mathbb{R}_+^k),$$

defines a generalized function of $u = (u_1, \ldots, u_k)$ if f is a Borel function with the property

$$(1.24) \qquad \int |f(u)| e^{-\beta |u|}\, du < \infty, \quad \text{for every } \beta > 0.$$

Suppose that F_1 is a generalized function of positive variables $\mathbf{u} = (u_1, \ldots, u_k)$ and F_2 is a generalized function of positive variables $\mathbf{v} = (v_1, \ldots, v_l)$. Then there exists a unique generalized function $F(\mathbf{u}, \mathbf{v})$ such that

$$\langle F(\mathbf{u}, \mathbf{v}), \varphi(\mathbf{u}, \mathbf{v}) \rangle = \langle F_1(\mathbf{u}), \langle F_2(\mathbf{v}), \varphi(\mathbf{u}, \mathbf{v}) \rangle \rangle,$$

for all $\varphi \in \mathscr{D}(\mathbb{R}_+^{k+l})$. It is called *the direct product of* F_1 *and* F_2 and it is denoted by $F_1(\mathbf{u}) \times F_2(\mathbf{v})$. [See, e.g., Yosida (1980), Chapter 1, Section 14.] Analogously, we define the direct product $F(\mathbf{u}) \times T(\mathbf{v})$ of a generalized function $F(\mathbf{u})$ and a g.r.f. $T(\mathbf{v})$ and the direct product of two g.r.f.'s.

Suppose that the set $\{1,\dots,N\}$ is partitioned into disjoint subsets $I_1,\dots,I_r$ and let $u_{I_l}=\{u_i,\ i\in I_l\}$. If F_l is generalized function of u_{I_l}, then

$$\prod_{l=1}^{r}F_l(u_{I_l})=F_1(u_{I_1})\times\cdots\times F_r(u_{I_r})$$

is a generalized function of $\mathbf{u}=(u_1,\dots,u_N)$.

Formula (1.15) can be interpreted as

$$\mathscr{T}_k^\varepsilon(\lambda,\mathbf{v})=\sum_{\Lambda}h_\varepsilon^{k-l}\delta(v_B)\times\tilde{T}_l^\varepsilon(\lambda,v_\Lambda), \tag{1.25}$$

where B is the complement of Λ in $\{1,\dots,k\}$ and $\delta(v_\Lambda)=\prod_{i\in\Lambda}\delta(v_i)$. Heuristically, we can rewrite the sum on the right-hand side as a product

$$\rho(X_{v_1})\prod_{i=2}^{k}\left[q^\varepsilon\left(X_{v_1+\cdots+v_i}-X_{v_1+\cdots+v_{i-1}}\right)+h_\varepsilon\delta(v_i)\right]. \tag{1.26}$$

[In general, the product of generalized functions with the same argument is not defined. In our case it can be *defined* as the sum in (1.25).]

1.8. To prove the results stated in Sections 1.3 and 1.6, first, we investigate the moment functions

$$\mathscr{M}_k^\varepsilon(\mu,\lambda,q;t)=P_\mu\prod_{i=1}^{n}T_{k_i}^\varepsilon(\lambda_i,q_i;t^i) \tag{1.27}$$

of the random field (1.5). (We deal here simultaneously with several density functions q and, to avoid confusion, we write q as an extra argument.) Then we study the moments

$$\mathscr{N}_k^\varepsilon(\mu,\lambda,q;\varphi)=P_\mu\prod_{i=1}^{n}\mathscr{T}_{k_i}^\varepsilon(\lambda_i,q_i;\varphi_i). \tag{1.28}$$

Consider the set S of pairs (a,b) which is the union of the disjoint *ordered* sets

$$S_a=\{(a,1),(a,2),\dots,(a,k_a)\}. \tag{1.29}$$

Denote by Γ the set of all orderings

$$\gamma=\{(a_1,b_1),\dots,(a_N,b_N)\}, \tag{1.30}$$

of S which are compatible with the order within each subset S_a. To every $\gamma\in\Gamma$ there corresponds a set D_γ in $\mathbb{R}_+^N$ described by (1.18). The union of these sets coincides with $D=D_{k_1}\times\cdots\times D_{k_n}$. Formula (1.20) defines a 1–1 linear mapping C_γ from D_γ onto $\mathbb{R}_+^N$. Put

$$I_\gamma=\{1\}\cup\{i\colon a_i\neq a_{i-1}\}, \tag{1.31}$$

$$p_\gamma(\mu,\lambda;u_{I_\gamma})=\int\mu(dz_0)\lambda_1(dz_1)\cdots\lambda_n(dz_n)\prod_{i\in I_\gamma}p_{u_i}(z_{a_i}-z_{a_{i-1}}), \tag{1.32}$$

with $a_0=0$.

THEOREM 1.2. *Suppose that we are given, for every $i = 1, \dots, n$, a pair of measures μ, λ_i subject to conditions* (a), (b), *Section* 1.2, *and a density q_i which satisfies* (1.3). *Let $\mathscr{T}_k^\varepsilon$ be random fields defined by* (1.8). *Then for every $\varphi_i \in \mathscr{D}(D_{k_i})$, $i = 1, \dots, n$,*

$$(1.33) \qquad \mathscr{N}_k^\varepsilon(\mu, \lambda, q; \varphi) \simeq \sum_{\gamma \in \Gamma} m_\gamma(\mu, \lambda)(\tilde{\varphi}_\gamma),$$

where

$$(1.34) \qquad \tilde{\varphi}_\gamma(\mathbf{u}) = \varphi\left(C_\gamma^{-1}\mathbf{u}\right),$$

with $\varphi(t^1, \dots, t^n) = \varphi_1(t^1) \cdots \varphi_n(t^n)$ and

$$(1.35) \qquad m_\gamma(\mu, \lambda; \mathbf{u}) = p_\gamma\left(\mu, \lambda; u_{I_\gamma}\right) \times \prod_{j \notin I_\gamma} \xi(u_j).$$

Theorem 1.2 will be proved in Section 3 with tools developed in Section 2. We also prove that

$$(1.36) \qquad \lim_{\beta \downarrow 0} \lim_{\varepsilon \downarrow 0} P_\mu \mathscr{T}_k^\varepsilon(\lambda, \varphi) \mathscr{T}_k^\beta(\lambda, \varphi) = \lim_{\varepsilon \downarrow 0} P_\mu \mathscr{T}_k^\varepsilon(\lambda, \varphi) \mathscr{T}_k^\varepsilon(\lambda, \varphi).$$

Theorem 1.1 follows easily from Theorem 1.2 and (1.36).

1.9. Let $u = C(t)$, where C is a unimodular matrix. For every generalized function $F(u)$, the formula

$$\langle \hat{F}(t), \varphi(t) \rangle = \langle F(u), \varphi\left(C^{-1}(u)\right) \rangle$$

determines a generalized function $\hat{F}(t)$. We put $\hat{F} = C(F)$ and we say that it is obtained from F *by the change of variables* $u = C(t)$.

Using this notation, we get from (1.33) and (1.34) the formula

$$(1.37) \qquad P_\mu \prod_{i=1}^n \mathscr{T}_{k_i}\left(\lambda_i, t^i\right) = \sum_{\gamma \in \Gamma} C_\gamma \left\{ p_\gamma\left(\mu, \lambda, u_{I_\gamma}\right) \times \prod_{j \notin I_\gamma} \xi(u_j) \right\},$$

which is the promised precise version of the recipe described in Section 1.6.

1.10. Interest in the self-intersections of the Brownian motion has increased considerably in connection with Symanzik's ideas in quantum field theory.

The functionals $\mathscr{T}_2(\lambda, u)$ mentioned in Section 1.4 have been introduced in a pioneering work by Varadhan (1969) published as an appendix to a Symanzik article. They also have been studied by Dynkin (1985, 1987), Le Gall (1985), Rosen (1986a) and Yor (1985a, b, 1986). The functionals $\mathscr{T}_k(\lambda, u)$ for $k > 2$ first appeared in Dynkin (1984a, b) as a tool for a probabilistic representation of $P(\varphi)_2$ fields. In Dynkin (1986b) various families of functionals which converge to $\mathscr{T}_k(\lambda, u)$ have been investigated and the moment functions of $\mathscr{T}_k(\lambda, u)$ have been evaluated. Theorems 1.1 and 1.2 were, first, announced in Dynkin (1986a).

A different renormalization of the self-intersection local times was proposed in Rosen (1986b) where the existence of an L^2-limit

$$(1.38)\quad I^k(B) = \lim_{\varepsilon \downarrow 0} \int_B \prod_{i=1}^{k} \left[\left[p_\varepsilon(X_{t_k} - X_{t_{k-1}}) \right] - \frac{1}{2\pi(t_k - t_{k-1} + \varepsilon)} \right] dt_1, \ldots, dt_k$$

was proved for every bounded Borel set $B \subset D_k$. [For $k = 3$ this is also done in Yor (1985c) by a different method.] Heuristically,

$$\tilde{I}_k(v) = \sum_{\Lambda} (-1)^{k-l} \xi(v_B) \times \tilde{\mathscr{T}}_l(\lambda, v_\Lambda),$$

where $I^k(t) = \tilde{I}^k(v)$, λ is Lebesgue measure, Λ and B have the same meaning as in (1.25) and $\xi(V_B) = \prod_{i \in B} \xi(v_i)$.

We refer to Dynkin (1988) for more bibliographical information.

2. Preliminaries.

2.1. Suppose that a generalized function F^ε is given for every $\varepsilon \in (0, \varepsilon_0)$. We say that F^ε is *bounded* if, for every test function φ, $F^\varepsilon(\varphi)$ is a bounded real-valued function of ε. Let $F^\varepsilon = F_1^\varepsilon \times F_2^\varepsilon$. Standard arguments [see, e.g., Gel'fand and Shilov (1968), Chapter 1, Section 4.4] show that, *if F_1^ε and F_2^ε are bounded, then so is F^ε.*

Let a^ε be an arbitrary real-valued function. We write $F^\varepsilon = O(a^\varepsilon)$ if $F^\varepsilon / a^\varepsilon$ is bounded. We write $F_1^\varepsilon \asymp F_2^\varepsilon$ if $F_1^\varepsilon - F_2^\varepsilon = O(\varepsilon^\alpha)$ for some $\alpha > 0$. (If F^ε is a real-valued function of ε, this is consistent with the notation introduced in Section 1.2.)

2.2. We need some estimates for Green's function,

$$(2.1)\qquad g_\beta(x) = \int_0^\infty e^{-\beta t} p_t(x)\, dt.$$

(We drop the subscript β if it is equal to 1.)

LEMMA 2.1. *For every $\beta > 0$ and every integer $k \geq 0$,*

$$(2.2)\qquad \int dz\, g_\beta(z)^k < \infty.$$

Suppose that a random variable Y has a probability density q which satisfies condition (1.3). *Then there exist constants β_k such that*

$$(2.3)\qquad E\left[g(\varepsilon Y)^k \right] \leq \beta_k |\ln \varepsilon|^k,$$

for all sufficiently small ε. We also have

$$(2.4)\qquad Eg(\varepsilon Y) \asymp -h_\varepsilon.$$

PROOF. It is well known [see, e.g., Itô and McKean (1965), page 233] that

$$(2.5)\qquad g_\beta(x) = \frac{1}{\pi} K_0\left(\sqrt{2\beta}\,|x|\right),$$

where K_0 is a modified Bessel function which can be described [see Watson (1952), 3.71.14 and 3.7.2] by the formula

$$K_0(r) = -I_0(r)\ln\frac{r}{2} + B(r). \tag{2.6}$$

Here

$$I_0(r) = \sum_0^\infty a_m r^{2m}/(2m)!, \qquad a_m = \begin{bmatrix} 2m \\ m \end{bmatrix} 2^{2-m}, \tag{2.7}$$

$$B(r) = -C + \sum_i^\infty a_m(1 + 1/2 + \cdots + 1/m - C)r^{2m}/(2m!). \tag{2.8}$$

Formula (2.2) follows from (2.5), (2.6) and (1.3).

Formula (2.6) also implies that

$$\frac{1}{\pi}K_0(2\varepsilon r) = \tilde{h}_\varepsilon \varphi_\varepsilon(r) + \psi_\varepsilon(r), \tag{2.9}$$

with

$$\tilde{h}_\varepsilon = -\frac{1}{\pi}\ln\varepsilon, \qquad \varphi_\varepsilon(r) = I_0(2\varepsilon r), \qquad \psi_\varepsilon(r) = \frac{1}{\pi}[B(2\varepsilon r) - I_0(2\varepsilon r)\ln r]$$

and h_ε given by (1.9).

Since $a_m \to 0$ and $a_m(1 + 1/2 + \cdots + 1/m - C) \to 0$ as $m \to \infty$, there exist constants $\gamma_1, \gamma_2, \gamma_3$ such that

$$\varphi_\varepsilon(r) \le \gamma_1 e^{2\varepsilon r}, \qquad |\psi_\varepsilon(r)| \le (\gamma_2 + \gamma_3|\ln r|)e^{2\varepsilon r}, \quad \text{for all } r > 0. \tag{2.10}$$

Put $N = |Y|/\sqrt{2}$. By (2.5), (2.6) and (2.9),

$$g(\varepsilon Y) = \tilde{h}_\varepsilon \varphi_\varepsilon(N) + \psi_\varepsilon(N) \le \left(\gamma_1 \tilde{h}_\varepsilon + \gamma_2 + \gamma_3|\ln N|\right)e^{2\varepsilon N}. \tag{2.11}$$

The estimate (2.3) follows from (2.11) and (1.3).

By (2.11)

$$Eg(\varepsilon Y) = a(\varepsilon)\tilde{h}_\varepsilon + b(\varepsilon), \qquad a(\varepsilon) = E\varphi_\varepsilon(N), \qquad b(\varepsilon) = E\psi_\varepsilon(N). \tag{2.12}$$

The functions $a(\varepsilon)$ and $b(\varepsilon)$ are even and analytic in a neighborhood of 0. Since $a(0) = 1$, $b(0) = -h_\varepsilon - \tilde{h}_\varepsilon$ [cf. (1.9)], we have $a(\varepsilon) = 1 + O(\varepsilon^2)$, $b(\varepsilon) = -h_\varepsilon - \tilde{h}_\varepsilon + O(\varepsilon^2)$ which implies (2.4). □

REMARK. Using Hölder's inequality, we conclude from (2.2) that, if λ has a bounded density, then

$$\sup_{x_1,\dots,x_n} \int \lambda(dz) \prod_{i=1}^n g_\beta(x_i - z) < \infty, \tag{2.13}$$

for every $\beta > 0$.

2.3.

LEMMA 2.2. *For every Y described in Lemma 2.1 and for every $\varepsilon > 0$,*

$$\langle K^{\varepsilon}(u), \varphi(u)\rangle = E\int_0^{\infty} du\, \varphi(u) p_u(\varepsilon Y) \tag{2.14}$$

is a continuous linear functional on $\mathscr{D}(\mathbb{R}_+)$ and

$$K^{\varepsilon}(u) \simeq -h_{\varepsilon}\delta(u) + \xi(u), \tag{2.15}$$

where h_{ε} is defined by (1.9) *and*

$$\langle \xi(u), \varphi(u)\rangle = \int_0^{\infty} [\varphi(u) - e^{-u}\varphi(0)]/2\pi u\, du. \tag{2.16}$$

PROOF. We note that

$$R^{\varepsilon}(\varphi) = K^{\varepsilon}(\varphi) + h_{\varepsilon}\varphi(0) - \xi(\varphi) = r^{\varepsilon}\varphi(0) - R_1^{\varepsilon}(\varphi), \tag{2.17}$$

where

$$r^{\varepsilon} = Eg(\varepsilon Y) + h_{\varepsilon} \tag{2.18}$$

and

$$R_1^{\varepsilon}(\varphi) = E\int_0^{\infty}\left[1 - e^{-\varepsilon^2 Y^2/2u}\right] f(u)\, du, \tag{2.19}$$

with

$$f(u) = [\varphi(u) - e^{-u}\varphi(0)]/2\pi u. \tag{2.20}$$

Since $1 - e^{-a} \le \sqrt{a}$ for every $a \ge 0$, we have $R_1^{\varepsilon}(\varphi) \le c_{\varphi}\varepsilon$, where

$$c_{\varphi} = \int_0^{\infty} |f(u)|(2u)^{-1/2}\, du\, E|Y| < \infty.$$

Obviously, this implies (2.15). □

2.4.

LEMMA 2.3. *Let Y be the random variable introduced in Lemma 2.1 and let*

$$g_{\beta}^{u}(x) = \int_0^{u} e^{-\beta t} p_t(x)\, dt. \tag{2.21}$$

Then

$$E\int\left[g_{\beta}^{u}(z) - g_{\beta}^{u}(z - \varepsilon Y)\right]^2 dz \simeq 0 \tag{2.22}$$

uniformly in u.

PROOF. The left-hand side in (2.22) is equal to $2E[Q(0) - Q(\varepsilon Y)]$, where

$$Q(y) = \int g_{\beta}^{u}(z) g_{\beta}^{u}(z - y)\, dz.$$

We have

$$0 \le Q(0) - Q(y) \le (2\pi)^{-1} \int_0^\infty \left(1 - e^{-y^2/2t}\right) e^{-\beta t}\, dt$$
$$\le (2\pi)^{-1} |y| \int_0^\infty (2t)^{-1/2} e^{-\beta t}\, dt,$$

which implies (2.22). □

2.5.

LEMMA 2.4. *Let $h_1, \ldots, h_n$ be positive Borel functions and let*

$$H_r^\beta(u) = \int_0^u e^{-\beta t} h_r(t)\, dt. \tag{2.23}$$

Suppose that $H_r^\beta(\infty) < \infty$ for all $\beta > 0$ and all r. Then for every $\varphi \in \mathscr{D}(D_n)$, there exists a $\beta > 0$ such that

$$\begin{aligned} &\int_{D_n} h_1(t_1) h_2(t_2 - t_1) \cdots h_n(t_n - t_{n-1}) \varphi(t)\, dt \\ &\quad = \int_{\mathbb{R}_+^n} H_1^\beta(u_1) \cdots H_n^\beta(u_n) \psi(u)\, du, \end{aligned} \tag{2.24}$$

where

$$\psi(u) = \mathbb{D}_1, \ldots, \mathbb{D}_n \left[e^{\beta(u_1 + \cdots + u_n)} \varphi(u_1, u_1 + u_2, \ldots, u_1 + u_2 + \cdots + u_n) \right]. \tag{2.25}$$

PROOF. We start with $n = 1$. Let $\varphi \in \mathbb{D}^{\beta'}(\mathbb{R}^+)$ and let $0 < \beta < \beta'$. Integration by parts yields

$$\int_0^c \varphi(t) h_1(t)\, dt = e^{\beta c} \varphi(c) H_1^\beta(c) - \int_0^c \left(e^{\beta t} \varphi(t)\right)' H_1^\beta(t)\, dt. \tag{2.26}$$

Since $e^{\beta c} \varphi(c) \to 0$ as $c \to \infty$, we get (2.24). Now using (2.26) we prove that (2.24) holds for $n + 1$ if it holds for n. □

3. Proofs of Theorems 1.1 and 1.2.

3.1. To evaluate the moment functions $\mathscr{M}_k^\varepsilon(\mu, \lambda, q; t)$ we consider N independent random variables Y_b^a, $(a, b) \in S$, where Y_1^a has the distribution λ_a and $Y_2^a, \ldots, Y_{k_a}^a$ are distributed with the density q_a. Let

$$V_1^a(\varepsilon) = Y_1^a, \qquad V_b^a(\varepsilon) = V_{b-1}^a(\varepsilon) + \varepsilon Y_b^a, \quad \text{for } b > 1. \tag{3.1}$$

Assuming that $\lambda_i(dz_i) = \rho_i(z_i)\, dz_i$, we have the following expression for the joint density of $V_b^a(\varepsilon)$,

$$(3.2)\qquad q^\varepsilon(\lambda; v) = \prod_{a=1}^{n} \rho_a(v_1^a) q_a^\varepsilon(v_2^a - v_1^a) \cdots q_a^\varepsilon\left(v_{k_a}^a - v_{k_a-1}^a\right).$$

By substituting $X_{t_b^a}$ for v_b^a we get

$$(3.3)\qquad \prod_{i=1}^{n} T_{k_i}^\varepsilon\left(\lambda_i, q_i; t^i\right) = q^\varepsilon(\lambda; X_t).$$

If $P_\mu\{X_t \in dx\} = p_\mu(t, x)\,dx$, then

$$(3.4)\quad \mathscr{M}_k^\varepsilon(\mu, \lambda, q; t) = P_\mu q^\varepsilon(\lambda, X_t) = \int q^\varepsilon(\lambda, x) p_\mu(t, x)\,dx = Ep_\mu(t, V(\varepsilon)).$$

Note that

$$(3.5)\qquad p_\mu(t, x) = \int \mu(dx_0) \prod_{r=1}^{N} p_{u_r}\left(x_{b_r}^{a_r} - x_{b_{r-1}}^{a_{r-1}}\right),$$

where $\gamma = \{(a_1, b_1), \dots, (a_N, b_N)\}$ is the ordering of S defined by (1.18), $u_1, \dots, u_N$ are given by (1.20) and $x_{b_0}^{a_0} = x_0$.

It follows from (3.4) that

$$(3.6)\qquad \int_D \mathscr{M}_k^\varepsilon(\mu, \lambda, q; t)\varphi(t)\,dt = \sum_{\gamma \in \Gamma} F_\gamma^\varepsilon(\varphi),$$

where

$$(3.7)\qquad F_\gamma^\varepsilon(\varphi) = \int_{D_\gamma} Ep_\mu(t, V(\varepsilon))\varphi(t)\,dt.$$

3.2. Now we investigate the limit behavior of $F_\gamma^\varepsilon(\varphi)$ as $\varepsilon \to 0$. Note that

$$(3.8)\qquad p_\mu(t, V(\varepsilon)) = \int \mu(dx_0) \prod_1^N p_{u_r}(\eta_r),$$

where

$$(3.9)\qquad \eta_r = V_{b_r}^{a_r}(\varepsilon) - V_{b_{r-1}}^{a_{r-1}}(\varepsilon),$$

with $V_{b_0}^{a_0} = 0$. Put

$$(3.10)\qquad \tilde{\eta}_r = Y_1^{a_r} - Y_1^{a_{r-1}}$$

and compare $F_\gamma^\varepsilon(\varphi)$ with

$$(3.11)\qquad \tilde{F}_\gamma^\varepsilon(\varphi) = \int_{D_\gamma} B^\varepsilon(t)\varphi(t)\,dt,$$

where

$$(3.12)\qquad B^\varepsilon(t) = E\int \mu(dx_0) \prod_{i \in I_\gamma} p_{u_i}(\tilde{\eta}_i) \prod_{j \notin I_\gamma} p_{u_j}(\eta_j)$$

and I_γ is defined by (1.31). By Lemma 2.4

$$(3.13)\qquad F_\gamma^\varepsilon(\varphi) = E\int \mu(dx_0)\int_{\mathbb{R}_+^N} du\, \psi(u) \prod_{r=1}^{N} g_\beta^{u_r}(\eta_r),$$

$$(3.14)\qquad \tilde{F}_\gamma^\varepsilon(\varphi) = E\int \mu(dx_0)\int_{\mathbb{R}_+^N} du\, \psi(u) \prod_{i\in I_\gamma} g_\beta^{u_i}(\tilde{\eta}_i) \prod_{j\notin I_\gamma} g_\beta^{u_j}(\eta_j),$$

with g_β^u and ψ given by (2.21) and (2.25).

For the sake of brevity we denote by M the product of the measure P with respect to which the mathematical expectations are taken in (3.13) and (3.14) and the measure $\mu(dx_0)|\psi(u)|\,du$. We put $A_i = g_\beta^{u_i}(\eta_i)$, $\tilde{A}_i = g_\beta^{u_i}(\tilde{\eta}_i)$ and $\Delta_i = |A_i - \tilde{A}_i|$. Note that

$$|F_\gamma^\varepsilon(\varphi) - F_\gamma^\varepsilon(\tilde{\varphi})| \le \int_L\int \prod_{j\notin I_\gamma} A_j \prod_{l\in L} \Delta_l \prod_{i\in I_\gamma\setminus L} \tilde{A}_i\, dM,$$

where the sum is taken over all nonempty $L \subset I_\gamma$. By Hölder's inequality each integral does not exceed a product of powers of the integrals

$$\int A_j^{k_j}\, dM, \qquad \int \tilde{A}_i^{k_i}\, dM, \qquad \int \Delta_l^{k_l}\, dM,$$

and we can choose $k_l = 2$ for one l. Using estimates (2.3) and (2.22) we prove that

$$(3.15)\qquad \tilde{F}_\gamma^\varepsilon(\varphi) \simeq F_\gamma^\varepsilon(\varphi).$$

By (3.1), $\eta_j = \varepsilon Y_{b_j}^{a_j}$ for $j \notin I_\gamma$. Since the random variables $\tilde{\eta}_i$, $i \in I_\gamma$, and the family $\{\eta_j,\ j \notin I_\gamma\}$ are mutually independent, we have from (3.6), (3.9), (3.10), (3.12) and (3.15) that

$$(3.16)\qquad \begin{aligned}&\int_D \mathscr{M}_k^\varepsilon(\mu, \lambda, q; t)\varphi(t)\, dt\\ &\simeq \sum_{\gamma\in\Gamma} \int_{D_\gamma} dt\, \varphi(t) E\left[\int \mu(dx_0) \prod_{i\in I_\gamma} p_{u_i}(Y_1^{a_r} - Y_1^{a_{r-1}})\right] \prod_{j\notin I_\gamma} Ep_{u_j}(\varepsilon Y_2^{a_j}),\end{aligned}$$

where according to (1.20)

$$(3.17)\qquad u_i = t_{b_i}^{a_i} - t_{b_{i-1}}^{a_{i-1}}.$$

3.3. The limit behavior of

$$(3.18)\qquad \mathscr{M}_k^{\varepsilon\alpha}(\mu, \lambda; t) = P_\mu T_{k_1}^\varepsilon(\lambda_1, t^1) T_{k_2}^\beta(\lambda_2, t^2),$$

as $\varepsilon \downarrow 0$ and β is fixed can be investigated in a similar way. We repeat the arguments in Sections 3.1 and 3.2 with random variables $V_b^2(\varepsilon)$ replaced by $V_b^2(\beta)$ and the set I_γ replaced by J_γ defined by the condition: $i \notin J_\gamma$ if and only if $i = 1$ or $a_{i-1} = a_i = 1$. In this way we get that

$$(3.19)\qquad \mathscr{M}_k^{\varepsilon\alpha}(\mu, \lambda; t) = \sum_{\gamma\in\Gamma} \phi_\gamma^{\varepsilon\beta}(\varphi) + R(\beta, \varepsilon).$$

Here $\varepsilon^{-\alpha}R(\beta, \varepsilon) \to 0$ (with some $\alpha > 0$) as $\varepsilon \downarrow 0$ and β is fixed, and

$$\phi_\gamma^{\varepsilon\beta}(\varphi) = E\left[\int \mu(dx_0) \prod_{i \in J_\gamma} p_{u_i}\left(\tilde{V}_{b_i}^{a_i} - \tilde{V}_{b_{i-1}}^{a_{i-1}}\right)\right] \prod_{j \notin J_\gamma} Ep_{u_j}\left(\varepsilon Y_2^1\right), \tag{3.20}$$

where $\tilde{V}_b^1 = Y_1^1$ and $\tilde{V}_b^2 = V_b^2(\beta)$.

3.4. We use (1.25) to express $\mathscr{N}_k^\varepsilon(\mu, \lambda, q; \varphi)$ through the moment functions $\mathscr{M}_l^\varepsilon(\mu, \lambda, q; t)$. To use (1.25) as it is stated we need to introduce a new set of variables $v_b^a = t_b^a - t_{b-1}^a$ besides two sets $\{u_i\}$ and $\{t_b^a\}$ we deal with. To avoid cumbersome notation we prefer to return in (1.25) to original variables $t_i = v_i + \cdots + v_i$ and to use the following rules in manipulating with the δ's:

$$f(v)\delta(v) = f(0)\delta(v), \tag{3.21}$$

$$F(s)\delta(t-s) = F(t)\delta(t-s) \tag{3.22}$$

For instance,

$$\begin{aligned}\delta(v_2)q\left(X_{v_1+v_3} - X_{v_1}\right) &= \delta(v_2)q\left(X_{v_1+v_2+v_3} - X_{v_1+v_2}\right)\\ &= \delta(t_2 - t_1)q\left(X_{t_3} - X_{t_2}\right).\end{aligned}$$

Following these rules, we rewrite (1.25) in the form

$$\mathscr{T}_k^\varepsilon(\lambda, t) = \sum_\Lambda \prod_{i=2}^k \left[h_\varepsilon \delta(t_i - t_{i-1})\right] T_l^\varepsilon(\lambda, t_\Lambda). \tag{3.23}$$

This implies

$$\mathscr{N}_k^\varepsilon(\mu, \lambda, q; \varphi) = \sum_{\Lambda_1} \cdots \sum_{\Lambda_n} E \prod_{a=1}^n \left\{ T_{l_a}^\varepsilon\left(\lambda_a, t_{\Lambda_a}^a\right) \prod_{b=2}^{k_a} \left[h_\varepsilon \delta\left(t_b^a - t_{b-1}^a\right)\right]\right\}, \tag{3.24}$$

where Λ_a runs over all subsets of the set $\{1, \dots, k_a\}$ which contain 1.

3.5. To simplify the presentation we say that the elements of $S_a = \{(a, 1), \dots, (a, k_a)\}$ have color a. By identifying $b \in \Lambda_a$ with the pair (a, b) we imbed Λ_a into S_a. The terms on the right-hand side of (3.24) are in a 1–1 correspondence with subsets $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_n$ of S which contain the first elements of each color. Denote the collection of all such sets by $\mathscr{L}$ and consider, for every $\Lambda \in \mathscr{L}$, the set Γ_Λ of all orderings of Λ which agree with the given order within each set S_a. Define *the characteristic set* for a pair (Λ, γ), $\Lambda \in \mathscr{L}$, $\gamma \in \Gamma_\Lambda$, as the set of all $s \in \Lambda$ whose color is different from the color of the left neighbor (in Λ relative to γ). We say that an ordering $\tilde{\gamma} \in \Gamma = \Gamma_S$ is *the standard continuation* of $\gamma \in \Gamma_\Lambda$ if both orderings agree on Λ and if the characteristic sets for $(S, \tilde{\gamma})$ and (Λ, γ) coincide.

We claim that every $\gamma \in \Gamma_\Lambda$ has a unique standard continuation. Indeed, suppose that the characteristic set of (Λ, γ) is I and that $\alpha_1 < \cdots < \alpha_r$ are all red elements of I. The set of all red elements in Λ is the union of disjoint intervals $[\alpha_1, \tilde{\alpha}], \dots, [\alpha_r, \tilde{\alpha}_r]$. The right neighbor β_i of $\tilde{\alpha}_i$ in Λ, obviously, belongs to I. Since the first red elements of S belongs to Λ, each red element s of S satisfies the relation $\alpha_i \le s \le \alpha_{i+1}$ for some i or the relation $\alpha_r \le s$. The only way to avoid expanding the characteristic set is to put s between α_i and β_{i+1} in the first case or after α_r in the second case. The exact place for s in now uniquely determined by the ordering of red elements.

3.6. It follows from (3.24) and (3.16) that

$$\mathscr{N}_k^\varepsilon(\mu, \lambda, q; \varphi) \simeq \sum_{\Lambda\in L} \sum_{\gamma\in\Gamma_\Lambda} \int_{D_\gamma} \Psi_{\gamma\Lambda}^\varepsilon(t)\varphi(t)\,dt, \tag{3.25}$$

where

$$\Psi_{\gamma\Lambda}^\varepsilon(t) = A(t)C^\varepsilon(t) \prod_{i\in\Lambda} [h_\varepsilon\delta(t_{i-1}^a, t_i^a)], \tag{3.26}$$

with

$$\begin{aligned} A(t) &= \int \mu(dx) p_{t(a_1, b_1)}(Y_1^a - x) \prod_{a\ne\tilde a} p_{t(a,i)-t(\tilde a, j)}(Y_1^{\tilde a} - Y_1^a), \\ C^\varepsilon(t) &= \prod_{a=\tilde a} E p_{t(a,i)-t(\tilde a, j)}(\varepsilon Y_2^a) \end{aligned} \tag{3.27}$$

[for typographical reasons we write $t(a, i)$ instead of t_i^a]. In (3.26) and (3.27) $(\tilde a, j)$ is the left neighbor of (a, i) in Λ relative to γ. By the rule (3.22) we can replace $t(\tilde a, j)$ by $t(\tilde a, \tilde j)$, where $(\tilde a, \tilde j)$ is the left neighbor of (a, i) in S relative to the standard continuation $\tilde\gamma$ of γ. [Note that the left neighbors of (a, i) in Λ and S have the same color.]

The set of variables $\{t(a, i) - t(\tilde a, \tilde j),\ a \ne \tilde a\}$ coincides with the set $\{u_i,\ I \in I_\gamma\}$ and the set $\{(t(a, i) - t(\tilde a, \tilde j),\ a = \tilde a\}$ coincides with $\{u_i, i \notin I_\gamma\}$. Thus the sum of terms in (3.25) corresponding to the pairs (Λ, γ) with a given characteristic set is equal to

$$p_\gamma(\mu, \lambda; u_{I_\gamma}) \prod_{i\notin I_\gamma} E\left[p_{u_i}(\varepsilon Y_2^a) - h_\varepsilon\delta(u_i)\right], \tag{3.28}$$

where the first factor is defined by (1.32). By the criterion (1.24), this factor can be interpreted as a generalized function in u_{I_γ}. Indeed it follows from Hölder's inequality and (2.13) that under the conditions of Theorem 1.2,

$$\int \mu(dz_0)\lambda_1(dz_1)\cdots\lambda_n(dz_n) \prod_{i\in I_\gamma} g_\beta(z_{a_i} - z_{a_{i-1}}) < \infty,$$

for every $\beta > 0$.

Formulas (3.25) and (3.28) and Lemma 1.2 imply Theorem 1.2.

3.7. Using the expression for $P_\mu T_{k_1}^\varepsilon(\lambda_1, \varphi_1)T_{k_2}^\varepsilon(\lambda_2, \varphi_2)$ given in Section 3.3, we prove that

$$P_\mu \mathscr{T}_{k_1}^\varepsilon(\lambda_1, \varphi_1)T_{k_2}^\varepsilon(\lambda_2, \varphi_2) \simeq \sum_{\gamma\in\Gamma} \int f_\gamma(\mu, \lambda; u_{J_\gamma}) \prod_{j\notin J_\gamma} \xi(u_j)\tilde\varphi_\gamma(u)\,du + R',$$

where $R' = o(\varepsilon^{\alpha})$ for some $\alpha > 0$ and

$$(3.29) \qquad f_{\gamma}(\mu, \lambda; u_{J_{\gamma}}) = E\left[\int u(dx_0) \prod_{i \in J_{\gamma}} p_{u_i}\left(\tilde{V}_{b_i}^{a_i} - \tilde{V}_{b_{i-1}}^{a_{i-1}}\right)\right].$$

Using this formula we prove (1.36).

3.8.

PROOF OF THEOREM 1.1. Note that

$$(3.30) \qquad \mathscr{T}_1(\lambda, \varphi) = T_1(\lambda, \varphi) = \int_0^{\infty} \rho(X_t)\varphi(t)\,dt$$

do not depend on ε and q. Consider the set $\mathscr{A}$ of all products $A = \mathscr{T}_1(\lambda_1, \varphi_1) \cdots \mathscr{T}_n(\lambda_n, \varphi_n)$, where $n = 1, 2, \ldots, \varphi_i \in \mathscr{D}(D_1)$ and λ_i are finite measures with continuous densities. It is easy to check that the span of $\mathscr{A}$ is everywhere dense in $L^2(P_{\mu})$.

Let $F_{\varepsilon} = \mathscr{T}_k^{\varepsilon}(\lambda, q; \varphi)$ with λ, μ, q subject to the conditions of Theorem 1.1. By Theorem 1.2, for every $A \in \mathscr{A}$, $P_{\mu}AF_{\varepsilon}$ converges to a finite limit as $\varepsilon \downarrow 0$; besides $P_{\mu}F_{\varepsilon}^2$ is bounded. Hence F_{ε} converges weakly to an element F of $L^2(P_{\mu})$. By (1.36)

$$\lim_{\varepsilon \downarrow 0} P_{\mu}F_{\varepsilon}^2 = \lim_{\beta \downarrow 0} \lim_{\varepsilon \downarrow 0} P_{\mu}F_{\varepsilon}F_{\beta} = \lim_{\beta \downarrow 0} P_{\mu}FF_{\beta} = P_{\mu}F^2.$$

Hence $P_{\mu}(F_{\varepsilon} - F)^2 \to 0$. By Theorem 1.2, $P_{\mu}(F - F_{\varepsilon})^p$ is bounded for every $p \geq 1$, and by the Schwarz inequality

$$P_{\mu}|F - F_{\varepsilon}|^m \leq \text{const}\left[P_{\mu}|F - F_{\varepsilon}|^2\right]^{1/2}$$

for every $m \geq 2$, which proves (1.11). For every $A \in \mathscr{A}$, $P_{\mu}FA$ does not depend on q. Therefore the same is true for F. □

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