THE CUBE OF A NORMAL DISTRIBUTION IS INDETERMINATE

BY CHRISTIAN BERG

University of Copenhagen

It is established that if \( X \) is a stochastic variable with a normal distribution, then \( X^{2n+1} \) has an indeterminate distribution for \( n \geq 1 \). Furthermore, the distribution of \( |X|^\alpha \) is determinate for \( 0 < \alpha \leq 4 \) while indeterminate for \( \alpha > 4 \).

1. Introduction. Let \( \mathcal{M}^* \) denote the set of probability measures on the real axis having moments of all orders. The \( k \)th moment of \( \mu \in \mathcal{M}^* \) is the number

\[
s_k(\mu) = \int x^k \, d\mu(x), \quad k = 0, 1, \ldots .
\]

Two distributions \( \mu, \nu \in \mathcal{M}^* \) are called equivalent if \( s_k(\mu) = s_k(\nu) \) for \( k = 0, 1, 2, \ldots , \) and \( \mu \) is called determinate (in the Hamburger sense), if the equivalence class \([\mu]\) containing \( \mu \) is equal to \( \{\mu\} \), and indeterminate otherwise.

It is well known that there exist indeterminate distributions. This observation goes back to Stieltjes (1894/1895), who proved that the distributions on \((0, \infty)\) with the densities

\[
a \exp(-t^{1/4}) \quad \text{and} \quad b_k t^{k - \log t}
\]

are indeterminate [see Stieltjes (1894), Sections 55 and 56]. Here \( a, b_k > 0 \) are normalization constants and \( k \in \mathbb{Z} \). For \( k = -1 \) we get the density of a log-normal distribution. Heyde (1963) pointed out that distributions commonly used in statistics need not be determinate, and as an example he gave the log-normal distribution.

If \( X \) is a stochastic variable with a normal distribution, then \( \exp(X) \) has a log-normal distribution. The purpose of this note is to point out that even simpler transformations of \( X \) may lead to indeterminate distributions, viz. \( X^3 \) has an indeterminate distribution. Murad Taqqu has observed that the Carleman condition fails for \( X^3 \), and this suggests that \( X^3 \) is not determinate, although it is well known that the Carleman condition is not necessary for determinacy.

More generally, we prove that \( X^{2n+1} \) has an indeterminate distribution for \( n \geq 1 \), and the distribution of \( |X|^\alpha \) is determinate for \( 0 < \alpha \leq 4 \) while indeterminate for \( \alpha > 4 \). We are thus in the strange situation that \( X^3 \) and \( X^5 \) are indeterminate, whereas \( |X^3| \) and \( X^4 \) are determinate.

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2. Statements and proofs.

PROPOSITION 1. If $X$ has a normal distribution, then $X^{2n+1}$ is indeterminate for $n \geq 1$.

PROOF. We may assume that $X$ has the density $(1/\sqrt{\pi}) \exp(-x^2)$, and then $X^3$ has the density
\[
d(x) = \frac{1}{3\sqrt{\pi}} |x|^{-2/3} \exp(-|x|^{2/3}).
\]
The function
\[
d(x) \left\{1 + r(\cos(\sqrt{3}|x|^{2/3}) - \sqrt{3} \sin(\sqrt{3}|x|^{2/3}))\right\}, \quad x \in \mathbb{R},
\]
is easily seen to be $\geq 0$ for $|r| \leq \frac{1}{2}$, and it is a probability density with the same moments as $d$ for these $r$. To see this, it suffices to prove that
\[
s_k = \int_{-\infty}^{\infty} x^kd(x)(\cos(\sqrt{3}|x|^{2/3}) - \sqrt{3} \sin(\sqrt{3}|x|^{2/3})) \, dx = 0, \quad \text{for} \quad k = 0, 1, \ldots.
\]
This is clear for $k$ odd, and we find
\[
s_{2k} = \frac{2}{3\sqrt{\pi}} \int_{0}^{\infty} x^{2k-2/3} \exp(-x^{2/3})(\cos(\sqrt{3}x^{2/3}) - \sqrt{3} \sin(\sqrt{3}x^{2/3})) \, dx
\]
\[
= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} x^{3k-1/2} e^{-x}(\cos(\sqrt{3}x) - \sqrt{3} \sin(\sqrt{3}x)) \, dx.
\]
Using
\[
\int_{0}^{\infty} x^c e^{-xz} \, dx = z^{-c} \Gamma(c), \quad \text{for} \quad c > 0, \text{Re} \, z > 0,
\]
we get for $z = 1 + i\beta$,
\[\ (*) \quad \int_{0}^{\infty} x^c e^{-x} \cos(\beta x) \, dx = (1 + \beta^2)^{-c/2} \Gamma(c) \left(\cos\left(c \frac{\pi}{3}\right) - \sqrt{3} \sin\left(c \frac{\pi}{3}\right)\right).
\]
Putting $c = 3k + \frac{1}{2}$, $\beta = \sqrt{3}$ we see that
\[
s_{2k} = \frac{1}{\sqrt{\pi}} 2^{-c/2} \Gamma(c) \left(\cos\left(c \frac{\pi}{3}\right) - \sqrt{3} \sin\left(c \frac{\pi}{3}\right)\right) = 0. \quad \square
\]

EXTENSION. The density of $X^{2n+1}$ is
\[
d_n(x) = \frac{1}{(2n+1)\sqrt{\pi}} |x|^{-2n/(2n+1)} \exp(-|x|^{2/(2n+1)}),
\]
and in this case we consider

$$d_n(x)\left\{1 + r\left(\cos\left(\beta_n|x|^{2/(2n+1)}\right) - \gamma_n\sin\left(\beta_n|x|^{2/(2n+1)}\right)\right)\right\},$$

with

$$\beta_n = \tan \frac{\pi}{2n+1}, \quad \gamma_n = \cot \frac{\pi}{2(2n+1)}.$$

**Proposition 2.** If $X$ has a normal distribution and $\alpha > 0$, then $|X|^\alpha$ is determinate for $\alpha \leq 4$ and indeterminate for $\alpha > 4$.

**Proof.** If $X$ has the density $(1/\sqrt{\pi})\exp(-x^2)$, then $|X|^\alpha$ has the density

$$d\alpha(x) = \frac{2}{\alpha\sqrt{\pi}} x^{1-\alpha} \exp(-x^{2/\alpha}),$$

with respect to Lebesgue measure on $[0, \infty[$. For $\alpha > 4$ we consider

$$d\alpha(x)\left\{1 + r\left(\cos\left(\beta_{\alpha}x^{2/\alpha}\right) - \gamma_{\alpha}\sin\left(\beta_{\alpha}x^{2/\alpha}\right)\right)\right\},$$

where

$$\beta_{\alpha} = \tan \frac{2\pi}{\alpha}, \quad \gamma_{\alpha} = \cot \frac{\pi}{\alpha},$$

and this is a nonnegative function on $[0, \infty[$ for $|r| \leq \sin \pi/\alpha$ with the same moments as $d\alpha$ for these $r$, since

$$s_k = \int_0^\infty x^k d\alpha(x)\left(\cos\left(\beta_{\alpha}x^{2/\alpha}\right) - \gamma_{\alpha}\sin\left(\beta_{\alpha}x^{2/\alpha}\right)\right) dx = 0, \quad k = 0, 1, \ldots .$$

This shows that $|X|^\alpha$ is indeterminate for $\alpha > 4$. Notice that $|X|^\alpha$ is indeterminate even in the Stieltjes sense, i.e., there exist different probabilities on $[0, \infty[$ with the same moments.

To see that $d\alpha$ is determinate for $0 < \alpha \leq 4$ we use

$$s_k(d\alpha) = \int_0^{\infty} x^k d\alpha(x) dx = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{ak + 1}{2}\right), \quad k = 0, 1, \ldots ,$$

and by Stirling’s formula $s_k(d\alpha)^{1/k} \sim ck^{a/2}$ for $k \to \infty$, where $c$ is a suitable constant. This shows that

$$\sum_{k=1}^{\infty} s_k(d\alpha)^{-1/2k} = \infty,$$

for $\alpha \leq 4$, so by a theorem of Carleman [see Shohat and Tamarkin (1943), page 20] $d\alpha$ is determinate in the Stieltjes sense. That $d\alpha$ is the only measure on the whole real line with the same moments as $d\alpha$ is then a consequence of a result of Chihara (1968), page 481, stating: If $\mu$ is determinate in the Stieltjes sense and indeterminate in the Hamburger sense, then $\mu$ is a Nevanlinna extremal measure and in particular discrete. □
REFERENCES


