The Annals of Probability
1988, Vol. 16, No. 2, 854–862

INDEPENDENCE OF ORDER STATISTICS

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A sharp bound for the dependence between sets of order statistics is established which gives precise information about the asymptotic independence of order statistics.

1. Introduction and main result. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables ($\equiv$ iid rv's) with common distribution function F and denote by $X_{1:n}, \ldots, X_{n:n}$ the corresponding order statistics.

Starting with the work by Gumbel (1946) on extremes, the asymptotic independence of order statistics has been investigated in quite a few articles. For detailed references we refer to page 118 of the book by Galambos (1978) and to Falk and Kohne (1986).

Let $Y_1, \ldots, Y_n$ and $Z_1, \ldots, Z_n$ be independent rv's with common distribution function F which are independent of $X_1, \ldots, X_n$. Denote the corresponding order statistics by $Y_{i:n}$ and $Z_{i:n}$. Then the dependence between the two sets of order statistics $X_{1:n}, \ldots, X_{k:n}$ and $X_{n-m+1:n}, \ldots, X_{n:n}$ is measured by

$$\begin{aligned}
&\Delta_F(n, k, n-m+1) \\
&\quad := \sup_{B \in \mathbb{B}^{k+m}} |P\{(X_{1:n}, \ldots, X_{k:n}, X_{n-m+1:n}, \ldots, X_{n:n}) \in B\} \\
&\qquad\qquad - P\{(Y_{1:n}, \ldots, Y_{k:n}, Z_{n-m+1:n}, \ldots, Z_{n:n}) \in B\}|,
\end{aligned}$$

where $\mathbb{B}^d$ denotes the Borel σ-algebra of $\mathbb{R}^d$.

Now, let $U_1, \ldots, U_n, V_1, \ldots, V_n, W_1, \ldots, W_n$ be independent rv's which are uniformly distributed on $(0,1)$ and denote the corresponding order statistics in each group by $U_{1:n}, \ldots, U_{n:n}, V_{1:n}, \ldots, V_{n:n}$ and $W_{1:n}, \ldots, W_{n:n}$. It is well known that the distribution of X_1 coincides with that of $F^{-1}(U_1)$ and, if F is continuous, $F(X_1)$ is uniformly distributed on $(0,1)$. Consequently,

$$\begin{aligned}
&\Delta_F(n, k, n-m+1) \\
&\quad \le \sup_{B \in \mathbb{B}^{k+m}} |P\{(U_{1:n}, \ldots, U_{k:n}, U_{n-m+1:n}, \ldots, U_{n:n}) \in B\} \\
&\qquad\qquad - P\{V_{1:n}, \ldots, V_{k:n}, W_{n-m+1:n}, \ldots, W_{n:n}) \in B\}| \\
&\quad =: \Delta(n, k, n-m+1),
\end{aligned} \tag{1.1}$$

where equality holds if F is continuous.

Therefore, in order to investigate the dependence between sets of order statistics, it suffices to consider uniformly on $(0,1)$ distributed rv's. This fact including formula (1.1) was already established in Falk and Kohne (1986), Proposition 2.

Received May 1986; revised November 1986.
AMS 1980 *subject classification*. 62G30.
Key words and phrases. Uniform distribution, conditional expectation, asymptotic expansion.

Under the assumption that $k + m = O(n^{2/3})$ Falk and Kohne (1986) computed exact rates at which $\Delta(n, k, n - m + 1)$ converges to zero. In particular, it follows from their Theorem 3 and formula (4) that if $k + m = O(n^{2/3})$,

$$\Delta(n, k, n - m + 1) = O\Big(\{km/(n(n - k - m + 1))\}^{1/2}\Big). \tag{1.2}$$

However, it is well known that not only are the extremes asymptotically independent from each other but that they are also independent from the central order statistics [see, for example, Tiago de Oliveira (1961), Rosengard (1962), Rossberg (1965) and Ikeda and Matsunawa (1970)].

Consequently, define

$$\begin{aligned}\Delta(n, k, r, s, n - m + 1) \\ := \sup_{B \in \mathbb{B}^{k+s-r+1+m}} &|P\{(U_{1:n}, \dots, U_{k:n}, U_{r:n}, \dots, U_{s:n}, U_{n-m+1:n}, \dots, U_{n:n}) \in B\} \\ &- P\{(U_{1:n}, \dots, U_{k:n}, V_{r:n}, \dots, V_{s:n}, W_{n-m+1:n}, \dots, W_{n:n}) \in B\}|,\end{aligned}$$

which we have to consider if we want to investigate the simultaneous dependence of the extremes and central order statistics. We remark that a relation analogous to (1.1) also holds for three groups of order statistics.

We will prove in the present paper that the bound given in (1.2) is valid for any choice of $k, m \in \{1, \dots, n\}$ and, based on this result, we will establish an analogous bound for $\Delta(n, k, r, s, n - m + 1)$. These results give precise information about the asymptotic independence of order statistics; in particular, they unify the known results on the asymptotic independence of extremes and central order statistics.

Our main result is the following one.

Theorem 1.3. *There exists a universal constant $C > 0$ such that for $1 \leq k < n - m + 1 \leq n$,*

$$\text{(i)} \qquad \Delta(n, k, n - m + 1) \leq C\{km/(n(n - k - m + 1))\}^{1/2},$$

and for $1 \leq k < r \leq s < n - m + 1 \leq n$,

$$\text{(ii)} \quad \Delta(n, k, r, s, n - m + 1) \leq C\Big[\{k(n - r + 1)/(n(r - k))\}^{1/2} + \{sm/(n(n - s - m + 1))\}^{1/2}\Big].$$

To illustrate Theorem 1.3 we formulate two simple consequences.

Corollary 1.4. (i) $\lim_{n \in \mathbb{N}} \Delta(n, k, r, s, n - m + 1) = 0$ *if* $\lim_{n \in \mathbb{N}}(k + m) \div n = 0$ *and* $0 < \liminf_{n \in \mathbb{N}} r/n \leq \limsup_{n \in \mathbb{N}} s/n < 1$, *which entails that the extreme and intermediate order statistics are asymptotically independent from the central order statistics.*

(ii) *If* $\lim_{n \in \mathbb{N}} k/n = 1$ *and* $\lim_{n \in \mathbb{N}} m/(n - k) = 0$, *then*

$$\lim_{n \in \mathbb{N}} \Delta(n, k, n - m + 1) = 0.$$

This yields in particular asymptotic independence of intermediate and extreme order statistics.

2. Auxiliary results and proofs. Our results are based on the well-known fact that the conditional distribution of $U_{1:n}, \dots, U_{n:n}$, given $U_{r:n} = x \in (0,1)$, is equal to the distribution of $V^x_{1:r-1}, \dots, V^x_{r-1:r-1}, x, W^x_{1:n-r}, \dots, W^x_{n-r:n-r}$, where $V^x_{1:r-1}, \dots, V^x_{r-1:r-1}$ and $W^x_{1:n-r}, \dots, W^x_{n-r:n-r}$ are the order statistics of $r-1$ and $n-r$ independent rv's which are uniformly distributed on $(0,x)$ and $(x,1)$, respectively. Moreover, these two sets are independent. Consequently, we may choose $V^x_{i:r-1} = xV_{i:r-1}$, $i = 1, \dots, r-1$, and $W^x_{i:n-r} = (1-x)W_{i:n-r} + x$, $i = 1, \dots, n-r$.

Finally, notice that an analogous result holds for the conditional distribution of $U_{1:n}, \dots, U_{n:n}$ given $U_{r:n} = x$, $U_{s:n} = y$, $r < s$, $x < y$.

The following basic result is immediate from the preceding considerations. In particular, (i) shows that in Theorem 12 of Falk and Kohne (1986) equality actually holds.

PROPOSITION 2.1. *Let* $1 \le k < r \le s < n - m + 1 \le n$. *Then*

(i)
$$\begin{aligned}&\sup_{B \in \mathbb{B}^{k+s-r+1}} |P\{(U_{1:n}, \dots, U_{k:n}, U_{r:n}, \dots, U_{s:n}) \in B\} \\ &\qquad - P\{(V_{1:n}, \dots, V_{k:n}, W_{r:n}, \dots, W_{s:n}) \in B\}| \\ &= \sup_{B \in \mathbb{B}^2} |P\{(U_{k:n}, U_{r:n}) \in B\} - P\{(V_{k:n}, W_{r:n}) \in B\}| \\ &= \Delta(n,k,r);\end{aligned}$$

(ii)
$$\begin{aligned}&\Delta(n,k,r,s,n-m+1) \\ &= \sup_{B \in \mathbb{B}^4} |P\{(U_{k:n}, U_{r:n}, U_{s:n}, U_{n-m+1:n}) \in B\} \\ &\qquad - P\{(U_{k:n}, V_{r:n}, V_{s:n}, W_{n-m+1:n}) \in B\}|;\end{aligned}$$

(iii) $\Delta(n,k,r,s,n-m+1) \le \Delta(n,k,r) + \Delta(n,s,n-m+1)$.

Another key idea for the derivation of our results will be the comparison of second-order normal approximations to distributions of order statistics. To this end we need the following auxiliary results. By $N(\mu, \sigma^2)$ we denote the normal distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$.

LEMMA 2.2. *For any* $\varepsilon > 0$ *there exists* $C(\varepsilon) > 0$ *such that if* $|\mu| < \varepsilon$ *and* $|\sigma^{-1} - 1| \le 1/5$,

$$\begin{aligned}&\left| \int g \, dN(\mu, \sigma^2) - \int g \, dN(0,1) \right| \\ &\qquad \le \sup\{|g(x)| \exp(-x^2/4) \colon x \in \mathbb{R}\} C(\varepsilon) \{|\sigma^{-1} - 1| + |\mu|\},\end{aligned}$$

for any measurable function $g \colon \mathbb{R} \to \mathbb{R}$ *such that the above integrals are finite.*

PROOF. Put $R(x) = -x^2(\sigma^{-1} - 1) - x^2(\sigma^{-1} - 1)^2/2$. Then, by a Taylor expansion,

$$\left|\int g\,dN(\mu,\sigma^2) - \int g\,dN(0,1)\right|$$
$$\le \sup\{|g(x)|\exp(-x^2/4) : x \in \mathbb{R}\}(2\pi)^{-1/2}$$
$$\times \int \exp(-x^2/4)|1 - \sigma^{-1}\exp\{R(x) + (x - \mu/2)\mu\sigma^{-2}\}|\,dx$$
$$\le \sup\{|g(x)|\exp(-x^2/4) : x \in \mathbb{R}\}C(\varepsilon)\{|\sigma^{-1} - 1| + |\mu|\}. \qquad \square$$

Putting $g = 1_B$ for $B \in \mathbb{B}$, we obtain the following estimate from Lemma 2.2.

COROLLARY 2.3. *There exists a universal positive constant C such that*

$$\sup_{B\in\mathbb{B}} |N(\mu_1, \sigma_1^2)(B) - N(\mu_0, \sigma_0^2)(B)|$$
$$\le C\big[(|\mu_1 - \mu_0|/\sigma_0) + |(\sigma_0/\sigma_1) - 1|\big].$$

Now, we are ready to prove Theorem 1.3.

PROOF OF THEOREM 1.3. By C we denote in the following a generic constant. The assertion of Theorem 1.3 follows from Proposition 2.1 if we show that for $1 \le k < n - m + 1 \le n$,

$$\sup_{B\in\mathbb{B}^2} |P\{(U_{k:n}, U_{n-m+1:n}) \in B\} - P\{(V_{k:n}, W_{n-m+1:n}) \in B\}| \le C\{km/(n(n-k-m+1))\}^{1/2}. \tag{2.4}$$

Conditioning with respect to $U_{n-m+1:n} = x \in (0,1)$, we obtain

$$\sup_{B\in\mathbb{B}^2} |P\{(U_{k:n}, U_{n-m+1:n}) \in B\} - P\{(V_{k:n}, W_{n-m+1:n}) \in B\}|$$
$$(2.5) = \sup_{B\in\mathbb{B}^2}\left|\int [P\{(xV_{k:n-m}, x) \in B\} - P\{(V_{k:n}, x) \in B\}](P^*U_{n-m+1:n})(dx)\right|$$
$$\le \int \sup_{B\in\mathbb{B}} |P\{xV_{k:n-m} \in B\} - P\{V_{k:n} \in B\}|(P^*U_{n-m+1:n})(dx),$$

where P^*X denotes the distribution of a random variable X.

We will prove in the following that

$$\int \sup_{B\in\mathbb{B}} |P\{xV_{k:n-m} \in B\} - P\{V_{k:n} \in B\}|(P^*U_{n-m+1:n})(dx)$$
$$\le C\big[\{(n-m)/(k(n-k-m+1))\} + \{n/(k(n-k))\} + \{km/(n(n-k-m+1))\}^{1/2}\big]. \tag{2.6}$$

This implies (2.4) as can be seen as follows. First, we may assume without loss of generality that $k \geq m$; otherwise we interchange k and m in (2.4) by switching over to $1 - U_{k:n}$ etc. Moreover, it follows from (1.2) that it is sufficient for the proof of (2.4) to consider $k \geq n^{2/3}$.

Now, if $m \leq k$ and $k \geq n^{2/3}$, then

$$(n-m)/(k(n-k-m+1)) + n/(k(n-k)) \leq C\{km/(n(n-k-m+1))\}^{1/2}.$$

Hence, it suffices to prove (2.6).

Next, we want to apply second-order normal approximations to the distributions of $xV_{k:n-m}$ and $V_{k:n}$. To this end, write for $x \in (0,1)$ fixed

$$xV_{k:n-m} = \sigma_1\Big[\big((n-m+1)^{1/2}/\eta_1\big)\big(V_{k:n-m} - \mu_1\big)\Big] + \mu_1$$

and

$$V_{k:n} = \sigma_0\Big[\big((n+1)^{1/2}/\eta_0\big)\big(V_{k:n} - \mu_0\big)\Big] + \mu_0,$$

where

$$\mu_1 = kx/(n-m+1), \qquad \mu_0 = k/(n+1),$$
$$\sigma_1^2 = k(n-m-k+1)x^2/(n-m+1)^3, \qquad \sigma_0^2 = k(n-k+1)/(n+1)^3,$$
$$\eta_1^2 = k(n-m-k+1)/(n-m+1)^2, \qquad \eta_0^2 = k(n-k+1)/(n+1)^2.$$

Notice that μ_1 and σ_1^2 depend on x.

Now, application of a second-order normal approximation as given in Proposition 2.10 in Reiss (1981) (with slightly different standardizing constants) yields for $x \in (0,1)$,

$$\sup_{B \in \mathbb{B}} \left| P\{xV_{k:n-m} \in B\} - \int_B (1 + L_1((y-\mu_1)/\sigma_1)) N(\mu_1, \sigma_1^2)(dy) \right| \leq C(n-m)/(k(n-k-m+1))$$

and

$$\sup_{B \in \mathbb{B}} \left| P\{V_{k:n} \in B\} - \int_B (1 + L_2((y-\mu_0)/\sigma_0)) N(\mu_0, \sigma_0^2)(dy) \right| \leq Cn/(k(n-k)),$$

where for $y \in \mathbb{R}$,

$$L_1(y) = \Big[(n-m-2k+1)/\big(k(n-k-m+1)(n-m+1)\big)^{1/2}\Big]\big((y^3/2) - y\big)$$

and

$$L_2(y) = \Big[(n-2k+1)/\big(k(n-k+1)(n+1)\big)^{1/2}\Big]\big((y^3/2) - y\big).$$

Consequently,

$$\begin{aligned}
(2.7)\quad &\int \sup_{B\in\mathbb{B}} |P\{xV_{k:n-m} \in B\} - P\{V_{k:n} \in B\}|(P^*U_{n-m+1:n})(dx) \\
&\le \int \sup_{B\in\mathbb{B}} |N(\mu_1, \sigma_1^2)(B) - N(\mu_0, \sigma_0^2)(B)|(P^*U_{n-m+1:n})(dx) \\
&\quad + \iint |L_1((y-\mu_1)/\sigma_1) - L_2((y-\mu_1)/\sigma_1)| \\
&\qquad\qquad \times N(\mu_1, \sigma_1^2)(dy)(P^*U_{n-m+1:n})(dx) \\
&\quad + \iint |L_2((y-\mu_1)/\sigma_1) - L_2((y-\mu_0)/\sigma_0)| \\
&\qquad\qquad \times N(\mu_1, \sigma_1^2)(dy)(P^*U_{n-m+1:n})(dx) \\
&\quad + \int \sup_{B\in\mathbb{B}} \left| \int_B L_2((y-\mu_0)/\sigma_0) N(\mu_1, \sigma_1^2)(dy) \right. \\
&\qquad\qquad \left. - \int_B L_2((y-\mu_0)/\sigma_0) N(\mu_0, \sigma_0^2)(dy) \right| (P^*U_{n-m+1:n})(dx) \\
&\quad + C[(n-m)/(k(n-k-m+1)) + n/(k(n-k))] \\
&= \int \sup_{B\in\mathbb{B}} |N(\mu_1, \sigma_1^2)(B) - N(\mu_0, \sigma_0^2)(B)|(P^*U_{n-m+1:n})(dx) \\
&\quad + \iint |L_1(y) - L_2(y)| N(0,1)(dy)(P^*U_{n-m+1:n})(dx) \\
&\quad + \iint |L_2(y) - L_1((\sigma_1/\sigma_0)x + ((\mu_1-\mu_0)/\sigma_0))| \\
&\qquad\qquad \times N(0,1)(dy)(P^*U_{n-m+1:n})(dx) \\
&\quad + \int \sup_{B\in\mathbb{B}} \left| \int_B L_2(y) N((\mu_1-\sigma_0)/\sigma_0, (\mu_1/\sigma_0)^2)(dy) \right. \\
&\qquad\qquad \left. - \int_B L_2(y) N(0,1)(dy) \right| (P^*U_{n-m+1:n})(dx) \\
&\quad + C[(n-m)/(k(n-k-m+1)) + n/(k(n-k))] \\
&=: A_1 + \cdots + A_5.
\end{aligned}$$

We will show in the following that $A_i \le C\{km/(n(n-k-m+1))\}^{1/2}$ for $i = 1, \dots, 4$.

First, we deal with A_1. Application of Corollary 2.3 yields

$$A_1 \le C\int (|\mu_1 - \mu_0|/\sigma_0) + |(\sigma_0/\sigma_1) - 1|(P^*U_{n-m+1:n})(dx).$$

Put

$$a_n := \{(n-m+1)/(n+1)\}^{3/2}\{(n-k+1)/(n-k-m+1)\}^{1/2}$$

and

$$b_n := \{k(n+1)/(n-k+1)\}^{1/2}.$$

Then

$$\int(|\mu_1-\mu_0|/\sigma_0) + |(\sigma_0/\sigma_1)-1|(P^*U_{n-m+1:n})(dx)$$
$$= b_n((n+1)/(n-m+1))E(|U_{n-m+1:n}-(n-m+1)/(n+1)|)$$
$$+E(|(a_n/U_{n-m+1:n})-1|).$$

Now, the first integral is bounded by [see formula (3.1.7) of David (1981)]

$$\begin{aligned} & b_n((n+1)/(n-m+1)) \\ & \quad \times E\big([U_{n-m+1:n}-((n-m+1)/(n+1))]^2\big)^{1/2} \\ & \leq b_n((n+1)/(n-m+1))(n-m+1)^{1/2}m^{1/2}/(n+1)^{3/2} \\ & \leq \{km/((n-k+1)(n-m+1))\}^{1/2} \\ & \leq \{km/((n+1)(n-k-m+1))\}^{1/2}. \end{aligned} \tag{2.8}$$

Moreover [see formula (2.1.6) of David (1981)],

$$\begin{aligned} & E(|(a_n/U_{n-m+1:n})-1|) \\ & \quad = [n!/((n-m)!(m-1)!)]\int_0^1 |a_n-x|x^{n-m-1}(1-x)^{m-1}\,dx \\ & \quad = (n/(n-m))E(|a_n-U_{n-m:n-1}|) \\ & \quad \leq (n/(n-m))E\big((a_n-U_{n-m:n-1})^2\big)^{1/2} \\ & \quad \leq (n/(n-m))\big\{(n-m+1)^{1/2}m^{1/2}n^{-3/2}+|a_n-((n-m)/n)| \\ & \quad \leq C\{km/(n(n-k-m+1))\}^{1/2}. \end{aligned} \tag{2.9}$$

(2.8) and (2.9) now imply that $A_1 \leq C\{km/(n(n-k-m+1))\}^{1/2}$.

Next, we deal with A_2. Obviously, it suffices to show that

$$\begin{aligned} & \Big|\big[(n-m-2k+1)/(k(n-k-m+1)(n-m+1))^{1/2}\big] \\ & \qquad -\big[(n-2k+1)/(k(n-k+1)(n+1))^{1/2}\big]\Big| \\ & \quad \leq C\{km/(n(n-k-m+1))\}^{1/2}. \end{aligned} \tag{2.10}$$

To this end write

$$\begin{aligned} & \big[(n-m-2k+1)/(k(n-k-m+1)(n-m+1))^{1/2}\big] \\ & \quad -\big[(n-2k+1)/(k(n-k+1)(n+1))^{1/2}\big] \\ & \quad = \{(n-k-m+1)/(k(n-m+1))\}^{1/2} - \{(n-k+1)/(k(n+1))\}^{1/2} \\ & \qquad + \{k/((n-k+1)(n+1))\}^{1/2} \\ & \qquad - \{k/((n-k-m+1)(n-m+1))\}^{1/2}. \end{aligned}$$

Now, it is easy to see that each of the last two terms in the above sum is bounded by $C\{km/(n(n-k-m+1))\}^{1/2}$.

Moreover, elementary computations yield that also

$$\left|\{(n-k-m+1)/(k(n-m+1))\}^{1/2} - \{(n-k+1)/(k(n+1))\}^{1/2}\right|$$
$$\leq \{km/(n(n-k-m+1))\}^{1/2},$$

which completes the proof of (2.10).

In analogy to the proof that $A_1 \leq C\{km/(n(n-k-m+1))\}^{1/2}$ one shows that this bound is also valid for A_3. Thus, it remains to deal with A_4.

First, notice that the coefficient of the polynomial L_2, i.e., $(n-2k+1)/(k(n-k+1)(n+1))^{1/2}$, is uniformly bounded. Moreover, application of Proposition 2.2 to $g = 1_B L_2$ with $\mu = (\mu_1 - \mu_0)/\sigma_0$, $\sigma = \sigma_1/\sigma_0$ yields

$$\left|\int_B L_2\, dN(\mu, \sigma^2) - \int_B L_2\, dN(0,1)\right|$$
$$\leq C\big[|(\sigma_0/\sigma_1) - 1| + (|\mu_1 - \mu_0|/\sigma_0)\big],$$

if $|(\sigma_0/\sigma_1) - 1| \leq 1/5$ and $|\mu_1 - \mu_0|/\sigma_0 \leq 1$. Otherwise,

$$\left|\int_B L_2\, dN(\mu, \sigma^2) - \int_B L_2\, dN(0,1)\right|$$
$$\leq C\left[\int (|x|^3 + |x|) N(\mu, \sigma^2)(dx) + 1\right]$$
$$\leq C(\sigma^3 + |\mu|^3).$$

Now, notice that from the assumption $m \leq k \leq n - m + 1$ we obtain $m \leq (n+1)/2$ and thus, σ remains bounded, i.e., $\sigma^3 \leq C|(\sigma_0/\sigma_1) - 1|^3$ if $|(\sigma_0/\sigma_1) - 1| \geq 1/5$.

Together we obtain

$$A_4 \leq C\int |(\sigma_0/\sigma_1) - 1| + (|\mu_1 - \mu_0|/\sigma_0)$$
$$+ (|\mu_1 - \mu_0|/\sigma_0)^3 (P^* U_{n-m+1:n})(dx).$$

In analogy to the proof that $A_1 \leq C\{km/(n(n-k-m+1))\}^{1/2}$ one shows that this bound is also valid for the above integral. This completes the proof of Theorem 1.3. □

Acknowledgments. The authors would like to thank the referees for their careful reading of the manuscript and an Associate Editor for pointing out an error in an earlier version of the manuscript.

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