The Annals of Probability
1988, Vol. 16, No. 2, 778–792

LOCAL TIME FOR TWO-PARAMETER CONTINUOUS MARTINGALES WITH RESPECT TO THE QUADRATIC VARIATION

BY MARTA SANZ

Universitat de Barcelona

In this paper we give the local time for some continuous two-parameter martingales with respect to the quadratic variation $\langle M \rangle$ and we study some of their sample path properties.

Introduction and notation. In this paper we will deal with processes defined on a complete probability space $(\Omega, \mathscr{F}, P)$, indexed by R_+^2, with the usual partial order $(s, t) \le (s', t')$ if and only if $s \le s'$ and $t \le t'$. Given $z_1, z_2 \in R_+^2$, $z_1 < z_2$, $(z_1, z_2]$ denotes the rectangle $\{z \in R_+^2,\ z_1 < z \le z_2\}$ and $R_z = [0, z]$. If f is a map from R_+^2 to R, the increment of f on a rectangle $(z_1, z_2]$, $z_1 = (s_1, t_1)$, $z_2 = (s_2, t_2)$ is $f((z_1, z_2]) = f(z_2) - f(s_1, t_2) - f(s_2, t_1) + f(z_1)$.

We consider an increasing family of sub-σ-fields of $\mathscr{F}$, $(\mathscr{F}_z)_{z \in R_+^2}$ satisfying conditions (F1)–(F4) of [3]. If $M = \{M_z,\ z \in R_+^2\}$ is a real valued, integrable and $\mathscr{F}_z$-adapted process we recall that M is a martingale if for any $z \le z'$, $E(M_{z'} | \mathscr{F}_z) = M_z$. Any two-parameter martingale gives rise to the martingales $M_{\cdot t} = \{M_{st}, \mathscr{F}_{s\infty},\ s \ge 0\}$ and $M_{s\cdot} = \{M_{st}, \mathscr{F}_{\infty t},\ t \ge 0\}$, where $\mathscr{F}_{s\infty} = \bigvee_{y \ge 0} \mathscr{F}_{sy}$, $\mathscr{F}_{\infty t} = \bigvee_{x \ge 0} \mathscr{F}_{xt}$. For $p \ge 1$, $\mathscr{M}_c^p$ will denote the class of all continuous martingales, vanishing on the axes, such that $E[|M_z|^p] < \infty$ for all z.

Given $M \in \mathscr{M}_c^p$, $p \ge 2$, we can consider the quadratic variations $\langle M \rangle_z$, $\langle M_{\cdot t} \rangle_s$, $\langle M_{s \cdot} \rangle_t$ which, following [10], possess continuous versions. Associated to M we shall consider a martingale $\tilde{M} \in \mathscr{M}_c^{p/2}$ which is defined as the $L^{p/2}$-limit of sums $\sum_{u \in \mathscr{S}_z^n} M(\Delta_u^1) M(\Delta_u^2)$, where $\mathscr{S}_z^n$ denotes an increasing sequence of partitions of R_z whose norm tends to zero, and if $u = (s_i, t_j) \in \mathscr{S}_z^n$, $\Delta_u^1 = ((s_i, 0), (s_{i+1}, t_j)]$, $\Delta_u^2 = ((0, t_j), (s_i, t_{j+1})]$. All these processes appear in Itô's formula given in [4] and [11].

The purpose of this article is to study local time for two-parameter continuous martingales M as a density of the "measure of sojourn time" with respect to the quadratic variation $\langle M \rangle$.

It is well known that Itô's formula provides a suitable tool to study local times. However, in the two-parameter case this method leads to a local time as a density with respect to the measure induced by the quadratic variation $\langle \tilde{M} \rangle$. The first results on local time for two-parameter processes are given in [3] and refer to the Brownian sheet $W = \{W_z,\ z \in R_+^2\}$. These authors prove the existence of a process $\{L(x, s, t), (x, s, t) \in R \times R_+^2\}$ a.s. jointly continuous in x, s and t such that for every bounded and measurable $f\colon R \to R$,

$$\int_{R_{st}} f(W_{uv}) uv \, du \, dv = \int_R L(x, s, t) f(x) \, dx, \quad \text{a.s.}$$

Received May 1986; revised March 1987.
AMS 1980 *subject classification*. 60H99.
Key words and phrases. Local time, two-parameter martingales, quadratic variation.

This is a local time with respect to the measure induced by the increasing process $\langle \tilde{W} \rangle_z$. Imkeller [9] gives a rather complete study of local time for the N-parameter d-dimensional Brownian sheet, also with respect to the measure induced by the analogues of $\langle \tilde{W} \rangle_z$ in his setting. Nualart [11] proposes a local time for $M \in \mathscr{M}_c^4$ with respect to $\langle \tilde{M} \rangle_z$ as an application of Itô's formula.

A different approach is given in [13]. In this reference Walsh proposes a local time for the Brownian sheet with respect to the Lebesgue measure on R_+^2, the measure induced by $\langle W \rangle_z$. This local time is given by integration of the local time on lines normalized in a convenient way. He obtains a process $L(x, s, t)$ a.s. continuous in the three variables, continuously differentiable in s and t, and gives estimates on the moduli of continuity of $L(x, s, t)$ and its partials. See also [5, 1, 6, 12] for related results.

Following the ideas developed in [13] we propose and study in this article a local time for a class of two-parameter continuous martingales with respect to the quadratic variation. In the first part we state the main results, in the second we present some examples.

1. As pointed out in [11], it is not difficult to give examples of two-parameter martingales for which the local time with respect to its quadratic variation exists. However it may have bad behaviour. The next example, closely related to that given in [11], shows this possibility.

EXAMPLE 0. Let $M^1 = \{M_s^1,\ s \geq 0\}$, $M^2 = \{M_t^2,\ t \geq 0\}$ be two continuous independent martingales, bounded in L^2, with respect to some filtrations $\{\mathscr{F}_s^1,\ s \geq 0\}$, $\{\mathscr{F}_t^2,\ t \geq 0\}$, respectively. Consider the martingale $M = \{M_{st} = M_s^1 M_t^2,\ s, t \geq 0\}$ with respect to the product filtration $\mathscr{F}_{st} = \mathscr{F}_s^1 \vee \mathscr{F}_t^2$. Denote by $L_1(x, s)$, $L_2(y, t)$ the local times of M^1 and M^2 at x, y, with respect to the quadratic variation $\langle M^1 \rangle$, $\langle M^2 \rangle$, respectively. Then, for every bounded and measurable $f\colon R \to R$, we have

$$\begin{aligned}\int_{R_{st}} f(M_z)\,d\langle M \rangle_z &= \int_0^t \int_0^s f(M_u^1 M_v^2)\,d\langle M^1 \rangle_u\,d\langle M^2 \rangle_v \\ &= \int_0^t \left(\int_R f(x M_v^2) L_1(x, s)\,dx \right) d\langle M^2 \rangle_v \\ &= \int_R L_2(y, t) \left(\int_R f(x, y) L_1(x, s)\,dx \right) dy = \int_R L(x, s, t) f(x)\,dx,\end{aligned}$$

where $L(x, s, t) = \int_R L_1(y, s) L_2(x/y, t)\,dy/|y|$.

It is easy to see that $\lim_{x \to 0} L(x, s, t) = \infty$ a.s. Indeed,

$$\begin{aligned}\lim_{x \to 0} L(x, s, t) &= L_2(0, t) \int_R L_1(y, s) \frac{dy}{|y|} \\ &= L_2(0, t) \int_0^s \frac{1}{|M_u^1|}\,d\langle M^1 \rangle_u.\end{aligned}$$

Define $A_t = \inf\{s \geq 0,\ \langle M^1 \rangle_s \geq t\}$; it is well known that $B_t = M_{A_t}^1$ is a Brownian motion with respect to the time-changed filtration $\{\mathscr{F}_{A_t},\ t \geq 0\}$. Therefore,

$$\int_0^s \frac{1}{|M_u^1|} d\langle M^1\rangle_u = \int_0^{\langle M^1\rangle_s} \frac{1}{|B_t|}\, dt = \infty \quad \text{a.s.}$$

THEOREM 1. *Let $M \in \mathscr{M}_c^2$ be a martingale satisfying the hypotheses*:

1. $\langle M\rangle_{st} = \int_{R_{st}} g(u,v)\,du\,dv$ *a.s., where $g(u,v)$ is a continuous measurable and adapted process.*
2. $\langle M_{\cdot t}\rangle_s = \int_0^s f(u,t)\,du$ *a.s.,* $\langle M_{s\cdot}\rangle_t = \int_0^t h(s,v)\,dv$ *a.s., where f and h are jointly continuous measurable and adapted processes, strictly positive on $(0,\infty)^2$. Assume also that for each $s \geq 0$, $t \geq 0$,*

$$\sup_{0\leq u\leq s} E\left[|f(u,t)|^p\right] \leq \infty, \qquad \sup_{0\leq v\leq t} E\left[|h(s,v)|^p\right] \leq \infty,$$

for some real $p > 4$.

Then there exists a process $\{L(x,s,t),\ x \in R - \{0\},\ (s,t) \in R_+^2\}$ a.s. jointly continuous and, for fixed $x \neq 0$, continuously differentiable in s and t on $(s,t) \in (0,\infty)^2$, which is a local time of M with respect to $\langle M\rangle$.

Before giving the proof of this theorem we present a useful inequality of Barlow and Yor [2, 15].

PROPOSITION 1. *Let M, N be local continuous martingales and denote by L^M, L^N their respective local times. For any $p \geq 1$ the following inequality holds*:

$$\sup_{x\in R}\left\|\sup_t |L^M(x,t) - L^N(x,t)|\right\|_{L^p} \leq C_p \|M - N\|_{\mathscr{H}^p}^{1/2}\{\|M\|_{\mathscr{H}^p}^{1/2} + \|N\|_{\mathscr{H}^p}^{1/2}\}.$$

As a consequence we have

PROPOSITION 2. *Let $p \in [1,\infty]$, $\lambda \in (0,1]$ and $\{M^\alpha,\ \alpha \in R^d\}$ be a family of continuous local martingales such that*

$$\|M^\alpha - M^\beta\|_{\mathscr{H}^p} \leq C_p|\alpha - \beta|^\lambda.$$

Then, if $\lambda p > 2(d+1)$ and L^α denotes the local time of M^α, there exists a version of $\{L^\alpha(x,t),\ (\alpha,x,t) \in R^d \times R \times R_+\}$ continuous in the three variables. Moreover, for any $A > 0$, $\rho > 0$ and $\gamma \in (0, \lambda/2 - (d+1)/p)$, there exists a random variable $H_{A\rho\gamma}$, finite a.s., such that for each x, y, $|x|, |y| \leq A$ and each α, β, $|\alpha|, |\beta| \leq \rho$,

$$\sup_t |L^\alpha(x,t) - L^\beta(y,t)| \leq H_{A\rho\gamma}\left[(x-y)^2 + |\alpha - \beta|^2\right]^{\gamma/2}.$$

In the following, unless we specify the contrary, local time will mean local time with respect to the quadratic variation.

PROOF OF THEOREM 1. Let L_1 (resp., L_2) be the local time associated to $M_{\cdot t}$ (resp., $M_{s\cdot}$). Fix $s_0 > 0$, $t, t' \geq 0$. For any $p \geq 2$, we have

$$E\left[\sup_{0\leq s\leq s_0} |M_{st} - M_{st'}|^p\right] \leq C_p E\left[|M_{s_0t} - M_{s_0t'}|^p\right] \leq C_p E\left(\int_t^{t'} h(s_0, v)\, dv\right)^{p/2}$$
$$\leq C_p \sup_{t\leq v\leq t'} E\left[|h(s_0, v)|^{p/2}\right]|t' - t|^{p/2}.$$

The first factor of the last member of this inequality is finite for some $p > 8$; therefore, applying Proposition 2 to the family of continuous martingales $\{M_{\cdot t}, t \in R_+\}$ and $\lambda = \frac{1}{2}$, we obtain the continuity of $\{L_1(x, s, t), (x, s, t) \in R \times R_+^2\}$ in its three variables.

Define

$$L(x, s, t) = \int_{R_{st}} \frac{g(u, v)}{f(u, v)} L_1(x, du, v)\, dv. \tag{1}$$

For any $x \neq 0$ the integral in (1) is finite. In fact, let $\delta > 0$ be such that $|x| \geq \delta > 0$; fix $\omega \notin N$, $P(N) = 0$, N being the set where the continuity of M fails. There exist $\varepsilon > 0$ such that if $(s, t) \in D^{\varepsilon}_{s_0t_0} = \{(s, t) \in R_+^2,\ s \leq \varepsilon,\ t \leq t_0$ or $s \leq s_0,\ t \leq \varepsilon\}$, $|M_{st}(\omega)| < \delta$. Consequently in $D^{\varepsilon}_{s_0t_0}$, $M(\omega)$ has not visited x; therefore, $L_1(x, s, t, \omega) = 0$.

The continuity of L_1 in its three variables implies the same property for L. On the other hand, if $\phi\colon R \to R$ is a bounded Borel measurable function we must have

$$\int_0^s \phi(M_{uv}) f(u, v)\, du = \int_R L_1(x, s, v)\phi(x)\, dx \quad \text{a.s.}$$

Consequently,

$$\int_{R_{st}} \phi(M_{uv}) g(u, v)\, du\, dv = \int_R L(x, s, t)\phi(x)\, dx \quad \text{a.s.},$$

which says that $\{L(x, s, t), x \in R - \{0\}, (s, t) \in R_+^2\}$ is a local time for M with respect to its quadratic variation.

Analogously we could have proposed

$$L(x, s, t) = \int_{R_{st}} \frac{g(u, v)}{h(u, v)} L_2(x, u, dv)\, du \tag{2}$$

as local time.

Notice that for $x \neq 0$ fixed, $L(x, s, t)$ is continuously differentiable in s and t on $(0, \infty)^2$ and

$$\frac{\partial}{\partial s} L(x, s, t) = \int_0^t \frac{g(s, v)}{h(s, v)} L_2(x, s, dv),$$
$$\frac{\partial}{\partial t} L(x, s, t) = \int_0^t \frac{g(u, t)}{f(u, t)} L_1(x, du, t).$$

In the case that M_{st} is the Brownian sheet $\{W_{st}, (s, t) \in R^2\}$, our local time coincides with that given by Walsh in [13]. □

In order to study the modulus of continuity of $L(\cdot, s, t)$, we have to make further assumptions on the processes g and f.

THEOREM 2. *Let $M \in \mathscr{M}_c^2$ be a martingale satisfying the same hypotheses as in Theorem 1. Assume that:*

(a) *For each $v \geq 0$, the processes $g(\cdot, v)$, $f(\cdot, v)$ are semimartingales. If $g(\cdot, v) = m_v(\cdot) + a_v(\cdot)$ and $f(\cdot, v) = n_v(\cdot) + b_v(\cdot)$ are their canonical decompositions, we assume that*

$$\langle m_v(\cdot)\rangle_u = \int_0^u \xi_v(\sigma)\, d\sigma,$$

$$\langle n_v(\cdot)\rangle_u = \int_0^u \chi_v(\sigma)\, d\sigma,$$

$$\langle m_v(\cdot), n_v(\cdot)\rangle_u = \int_0^u \eta_v(\sigma)\, d\sigma,$$

where ξ_v, χ_v and η_v are measurable, adapted and jointly continuous processes. The total variation of a_v and b_v on intervals is integrable on $[0, t)$ for any $t \geq 0$.

(b) *For each $\varepsilon > 0$, $s \geq \varepsilon$, $t \geq \varepsilon$ and p as in Theorem 1 we have*

$$\text{(3)}\qquad \begin{aligned} &\int_{R_{st}^\varepsilon} E\left[\left|\frac{\xi_v(u)}{f(u, v)^2}\right|^p\right] du\, dv < \infty, \\ &\int_{R_{st}^\varepsilon} E\left[\left|\frac{\chi_v(u) g(u, v)^2}{f(u, v)^4}\right|^p\right] du\, dv < \infty. \end{aligned}$$

Then, for each $s, t > 0$, $s \leq s_0$, $t \leq t_0$, $0 < T_1 < T_2 < \infty$, there exists a random variable B, finite a.s., such that

$$\text{(4)}\qquad |L(x, s, t) - L(x', s, t)| \leq B|x' - x|^\gamma, \qquad \gamma < \frac{1}{4} - \frac{2}{p},$$

for all $x, x' \in [T_1, T_2]$.

If (b) *holds for any $p \geq 1$, then for every $s, t > 0$, $s \leq s_0$, $t \leq t_0$, $0 < T_1 < T_2 < \infty$, there exists a random variable B having moments of any order such that*

$$\text{(5)}\qquad |L(x, s, t) - L(x', s, t)| \leq B|x' - x|^\gamma, \qquad \gamma < \tfrac{1}{4},$$

for all $x, x' \in [T_1, T_2]$.

PROOF. Fix $\varepsilon > 0$ and if $s, t > \varepsilon$ denote $R_{st}^\varepsilon = [\varepsilon, s] \times [\varepsilon, t]$. Consider the local time of M on the set $[\varepsilon, \infty)^2$, which will be called L^ε. Observe that

$$L^\varepsilon(x, s, t) = \int_{R_{st}^\varepsilon} \frac{g(u, v)}{f(u, v)} L_1^\varepsilon(x, du, v)\, dv,$$

where $L_1^\varepsilon(x, s, t) = L_1(x, s, t) - L_1(x, \varepsilon, t)$.

It suffices to establish inequalities (4) and (5) for L^ε. Indeed, given $T_1 > 0$, ω-a.s., we can find δ which depends on ω such that $\sup_{(s,t) \in D^\delta_{s_0 t_0}} |M_{st}| < T_1$; consequently, $L(x, s, t) = 0$ on $D^\delta_{s_0 t_0}$ for any $|x| \geq T_1$. Let $s, t > \delta'$ and define $\varepsilon = \delta \wedge \delta'$; then $L(x, s, t) = L^\varepsilon(x, s, t)$ for all $|x| \geq T_1$. In the next lemma we prove that, under our hypotheses, an alternative expression of L^ε is

$$
\begin{aligned}
L^\varepsilon(x, s, t) &= \int_\varepsilon^t \frac{g(s, v)}{f(s, v)} L_1^\varepsilon(x, s, v)\, dv \\
&\quad - \int_{R^\varepsilon_{st}} L_1^\varepsilon(x, u, v) \left[\frac{1}{f(u, v)}\, d_1 g(u, v) - \frac{g(u, v)}{f(u, v)^2} d_1 f(u, v) \right. \\
&\qquad \left. + \frac{g(u, v)}{f(u, v)^3} \chi_v(u)\, du - \frac{1}{f(u, v)^2} \eta_v(u)\, du \right] dv.
\end{aligned} \tag{6}
$$

Call

$$I_1(x) = \int_{R^\varepsilon_{st}} L_1^\varepsilon(x, u, v) \frac{1}{f(u, v)}\, d_1 m_v(u)\, dv$$

and

$$I_2(x) = \int_{R^\varepsilon_{st}} L_1^\varepsilon(x, u, v) \frac{g(u, v)}{f(u, v)^2}\, d_1 n_v(u)\, dv.$$

Fix $T_1 < x < x' < T_2$, p as in hypothesis (b). Then

$$
\begin{aligned}
&E\left[|I_1(x) - I_1(x')|^p \right] \\
&\quad \leq C_{p, \varepsilon, t} \int_\varepsilon^t E\left[\left| \int_\varepsilon^s \frac{1}{f(u, v)^2} \left(L_1^\varepsilon(x, u, v) - L_1^\varepsilon(x', u, v) \right)^2 \xi_v(u)\, du \right|^{p/2} \right] dv \\
&\quad \leq C_{p, \varepsilon, s, t} \left(\int_\varepsilon^t E\left[\sup_{\varepsilon \leq u \leq s} \left| L_1^\varepsilon(x, u, v) - L_1^\varepsilon(x', u, v) \right|^{2p} \right] dv \right)^{1/2} \\
&\qquad \times \left(\int_{R^\varepsilon_{st}} E\left[\left| \frac{\xi_v(u)}{f(u, v)^2} \right|^p \right] du\, dv \right)^{1/2}.
\end{aligned} \tag{7}
$$

By Proposition 1 applied to the one-parameter martingales $M = M_{\cdot v} - x$, $N = M_{\cdot v} - x'$ and $x = 0$, we have

$$
\begin{aligned}
&E\left[\sup_{\varepsilon \leq u \leq s} \left| L_1^\varepsilon(x, u, v) - L_1^\varepsilon(x', u, v) \right|^{2p} \right] \\
&\quad \leq C_p |x - x'|^p \left\{ |x|^p + |x'|^p + C_{p,t} \left\{ E\left[\sup_{0 \leq v \leq t} |h(s, v)|^p \right] \right\}^{1/2} \right\} \\
&\quad \leq C'_{p, t, T_2} |x - x'|^p.
\end{aligned}
$$

Consequently, $E[|I_1(x) - I_1(x')|^p] \leq C|x - x'|^{p/2}$. In a similar way we obtain $E[|I_2(x) - I_2(x')|^p] \leq C|x - x'|^{p/2}$.

Hence, for $i = 1, 2$,

$$E\left[\int_{T_1}^{T_2}\int_{T_1}^{T_2}\frac{|I_i(x) - I_i(x')|^p}{|x - x'|^{p/2}}\,dx\,dx'\right] < \infty.$$

Call

$$B_i(\omega) = \int_{T_1}^{T_2}\int_{T_1}^{T_2}\frac{|I_i(x) - I_i(x')|^p}{|x - x'|^{p/2}}\,dx\,dx' < \infty \quad \text{a.s.}$$

Applying the Garsia–Rodemich–Rumsey lemma [7], we have for every $T_1 < |x| < |x'| \leq T_2$,

$$\begin{aligned}|I_i(x) - I_i(x')| &\leq 8\int_0^{|x-x'|}\left(\frac{4B_i}{u^2}\right)^{1/p}\frac{1}{2\sqrt{u}}\,du \\ &\leq C_p B_i^{1/p}|x' - x|^{1/2-2/p}.\end{aligned}$$

This gives a modulus of continuity for the paths of the stochastic integrals appearing in (6).

The other integrals of (6) are easy to handle. In fact, due to the properties of f, g, a_v and b_v, and Proposition 2 applied to the martingale $M_{\cdot v}$, each term is bounded by

$$H_{T_i,t,\gamma}|x - x'|^{\gamma},$$

where $\gamma < \frac{1}{4} - 2/p$ and $H_{T_i,t,\gamma}$ is a random variable a.s. finite.

The second part of the theorem is obvious. □

LEMMA 1. *With the same assumptions as in Theorem 2, we have*

$$\begin{aligned}L^{\varepsilon}(x, s, t) = &\int_{\varepsilon}^{t}\frac{g(s,v)}{f(s,v)}L_1^{\varepsilon}(x,s,v)\,dv \\ (8)\qquad &-\int_{R_{st}^{\varepsilon}}L_1^{\varepsilon}(x,u,v)\left[\frac{1}{f(u,v)}\,d_1g(u,v) - \frac{g(u,v)}{f(u,v)^2}\,d_1f(u,v)\right. \\ &\left.+\frac{g(u,v)}{f(u,v)^3}d\langle f(\cdot,v)\rangle_u - \frac{1}{f(u,v)^2}d\langle g(\cdot,v), f(\cdot,v)\rangle_u\right]dv.\end{aligned}$$

PROOF. The idea is the following. Formally

$$L^{\varepsilon}(x,s,t) = \int_{\varepsilon}^{t}\frac{g(s,v)}{f(s,v)}L_1^{\varepsilon}(x,s,v)\,dv - \int_{R_{st}^{\varepsilon}}L_1^{\varepsilon}(x,u,v)\frac{\partial}{\partial u}\frac{g(u,v)}{f(u,v)}\,du\,dv.$$

To give a meaning to this expression we replace $\partial/\partial u$ by the stochastic differential obtained by using Itô's formula.

The proof will be done in two steps. In the first one we will see that the right-hand side of (8) is well defined; in the second part we will prove that it defines the local time of M on $[\varepsilon, \infty)^2$.

STEP 1. For any $\delta > 0$ consider a function $\varphi_\delta \in C_b^2(R)$, such that $\varphi_\delta(x) = \delta$ if $|x| < \delta$, $\varphi_\delta(x) = x$ if $|x| > 2\delta$, $|\varphi_\delta(x)| \geq |x|$. By Itô's formula we obtain

$$\frac{g(u,v)}{\varphi_\delta(f(u,v))} = \int_0^u \left[\frac{1}{\varphi_\delta(f(\sigma,v))} d_1 g(\sigma,v) - \frac{g(\sigma,v)\varphi_\delta'(f(\sigma,v))}{\varphi_\delta(f(\sigma,v))^2} d_1 f(\sigma,v) \right.$$
$$- \frac{\varphi_\delta'(f(\sigma,v))}{\varphi_\delta(f(\sigma,v))^2} d\langle g(\cdot,v), f(\cdot,v)\rangle_\sigma$$
$$- \frac{1}{2}\frac{g(\sigma,v)\varphi_\delta''(f(\sigma,v))}{\varphi_\delta(f(\sigma,v))^2} d\langle f(\cdot,v)\rangle_\sigma$$
$$\left. + \frac{g(\sigma,v)\varphi_\delta'(f(\sigma,v))^2}{\varphi_\delta(f(\sigma,v))^3} d\langle f(\cdot,v)\rangle_\sigma \right].$$

Define

$$L^{\varepsilon,\delta}(x,s,t) = \int_\varepsilon^t \frac{g(s,v)}{f(s,v)} L_1^\varepsilon(x,s,v)\,dv - \int_{R_{st}^\varepsilon} L_1^\varepsilon(x,u,v) d_1 \frac{g(u,v)}{\varphi_\delta(f(u,v))}\,dv.$$

We claim that

$$\lim_{\delta\to 0} L^{\varepsilon,\delta}(x,s,t) = \int_\varepsilon^t \frac{g(s,v)}{f(s,v)} L_1^\varepsilon(x,s,v)\,dv$$
$$(9) \qquad - \int_{R_{st}^\varepsilon} L_1^\varepsilon(x,u,v) \left[\frac{1}{f(u,v)} d_1 g(u,v) - \frac{g(u,v)}{f(u,v)^2} d_1 f(u,v) \right.$$
$$\left. + \frac{g(u,v)}{f(u,v)^3} d\langle f(\cdot,v)\rangle_u - \frac{1}{f(u,v)^2} d\langle g(\cdot,v), f(\cdot,v)\rangle_u \right] dv.$$

The convergence of the stochastic integrals can be proved by the following argument. Let $p \geq 1$. We have

$$E\left[\left| \int_{R_{st}^\varepsilon} \left(\frac{1}{\varphi_\delta(f(u,v))} - \frac{1}{f(u,v)} \right) L_1^\varepsilon(x,u,v)\, d_1 m_v(u)\,dv \right|^p \right]$$
$$\leq C_{p,s,t,\varepsilon} E \int_{R_{st}^\varepsilon} \left| \left(\frac{1}{\varphi_\delta(f(u,v))} - \frac{1}{f(u,v)} \right)^2 L_1^\varepsilon(x,u,v)^2 \xi_v(u) \right|^{p/2} du\,dv.$$

But

$$\left| \left(\frac{1}{\varphi_\delta(f(u,v))} - \frac{1}{f(u,v)} \right)^2 L_1^\varepsilon(x,u,v)^2 \xi_v(u) \right|$$
$$\leq C \frac{1}{f(u,v)^2} \sup_{\varepsilon \leq u \leq s} |L_1^\varepsilon(x,u,v)|^2 |\xi_v(u)|$$

and

$$\int_{R_{st}^{\varepsilon}} E\left[\left|\frac{\xi_v(u)}{f(u,v)^2}\right|^{p/2} \sup_{\varepsilon\le u\le s} |L_1^{\varepsilon}(x,u,v)|^p\right] du\,dv$$

$$\le C_{s,\varepsilon}\left(\int_{\varepsilon}^{t} E\left\{\sup_{\varepsilon\le u\le s} |L_1^{\varepsilon}(x,u,v)|^{2p}\right\} dv\right)^{1/2}$$

$$\times\left(\int_{R_{st}^{\varepsilon}} E\left[\left|\frac{\xi_v(u)}{f(u,v)^2}\right|^p\right] du\,dv\right)^{1/2} < \infty.$$

In fact, by Proposition 1 applied to $M = M_{\cdot v}$, $N \equiv 0$,

$$E\left\{\sup_{\varepsilon\le u\le s} |L_1^{\varepsilon}(x,u,v)|^{2p}\right\} \le C_p E\left\{\sup_{\varepsilon\le u\le s} |M_{uv}|^{2p}\right\}$$

$$\le C_p E\langle M_{s\cdot}\rangle_t^p \le C_{p,t} \sup_{0\le v\le t} E\left[|h(s,v)|^p\right] < \infty.$$

Consequently,

$$E\left[\left|\int_{R_{st}^{\varepsilon}}\left(\frac{1}{\varphi_\delta(f(u,v))} - \frac{1}{f(u,v)}\right) L_1^{\varepsilon}(x,u,v)\, d_1 m_v(u)\, dv\right|^p\right] \underset{\delta\to 0}{\to} 0.$$

The same argument applies to the integral

$$\int_{R_{st}^{\varepsilon}} \left(g(u,v)/f(u,v)^2\right) L_1^{\varepsilon}(x,u,v)\, d_1 n_v(u).$$

On the other hand, since $f(u,v)$ is jointly continuous and strictly positive on $(0,\infty)^2$, there exists $\gamma > 0$, which may depend on ω, such that $f(u,v) > \gamma$ on R_{st}^{ε}. Consequently for δ small enough we have

$$\varphi_\delta(f(u,v)) = f(u,v),\ \varphi_\delta'(f(u,v)) = 1$$

and $\varphi_\delta''(f(u,v)) = 0$ for $(u,v) \in R_{st}^{\varepsilon}$. This ensures the desired convergence of the remaining integrals, and hence (9) is proved.

STEP 2. $\lim_{\delta\to 0} L^{\varepsilon,\delta}(x,s,t) = L^{\varepsilon}(x,s,t)$. In order to establish this identity we must check that if $\phi\colon R \to R$ is any bounded Borel function

$$\int_R \lim_{\delta\to 0} L^{\varepsilon,\delta}(x,s,t)\phi(x)\,dx = \int_{R_{st}^{\varepsilon}} \phi(M_z)\,d\langle M\rangle_z \quad \text{a.s.} \tag{10}$$

Using a Fubini type theorem for stochastic integrals and the identity

$$\int_R L_1^{\varepsilon}(x,s,t)\phi(x)\,dx = \int_{\varepsilon}^{s} \phi(M_{uv})\,d\langle M_{\cdot v}\rangle_u \quad \text{a.s.,}$$

the left-hand side of (10) equals

$$
\begin{aligned}
&\int_\varepsilon^t \frac{g(s,v)}{f(s,v)}\left(\int_\varepsilon^s \phi(M_{uv})\,d\langle M_{\cdot v}\rangle_u\right) dv \\
&\quad - \int_{R_{st}^\varepsilon} \frac{1}{f(u,v)}\left(\int_\varepsilon^u \phi(M_{\sigma v})\,d\langle M_{\cdot v}\rangle_\sigma\right) d_1 g(u,v)\,dv \\
(11)\qquad &\quad + \int_{R_{st}^\varepsilon} \frac{g(u,v)}{f(u,v)^2}\left(\int_\varepsilon^u \phi(M_{\sigma v})\,d\langle M_{\cdot v}\rangle_\sigma\right) d_1 f(u,v)\,dv \\
&\quad - \int_{R_{st}^\varepsilon} \frac{g(u,v)}{f(u,v)^3}\left(\int_\varepsilon^u \phi(M_{\sigma v})\,d\langle M_{\cdot v}\rangle_\sigma\right) d\langle f(\cdot,v)\rangle_u\,dv \\
&\quad + \int_{R_{st}^\varepsilon} \frac{1}{f(u,v)^2}\left(\int_\varepsilon^u \phi(M_{\sigma v})\,d\langle M_{\cdot v}\rangle_\sigma\right) d\langle g(\cdot,v), f(\cdot,v)\rangle_u.
\end{aligned}
$$

The following equality holds:

$$
\begin{aligned}
&\frac{g(s,v)}{f(s,v)}\int_\varepsilon^s \phi(M_{uv})\,d\langle M_{\cdot v}\rangle_u \\
&\quad = \int_\varepsilon^2 \frac{\left(\int_\varepsilon^u \phi(M_{\sigma v})\,d\langle M_{\cdot v}\rangle_\sigma\right)}{f(u,v)}\,d_1 g(u,v) \\
&\qquad + \int_\varepsilon^s \frac{g(u,v)}{f(u,v)}\phi(M_{uv})\,d\langle M_{\cdot v}\rangle_u - \int_\varepsilon^s \frac{g(u,v)}{f(u,v)^2} \\
(12)\qquad &\qquad\qquad \times\left(\int_\varepsilon^u \phi(M_{\sigma v})\,d\langle M_{\cdot v}\rangle_\sigma\right) d_1 f(u,v) \\
&\qquad - \int_\varepsilon^s \frac{\int_\varepsilon^u \phi(M_{\sigma v})\,d\langle M_{\cdot v}\rangle_\sigma}{f(u,v)^2}\,d\langle g(\cdot,v), f(\cdot,v)\rangle_u \\
&\qquad + \int_\varepsilon^s \frac{g(u,v)}{f(u,v)^3}\left(\int_\varepsilon^u \phi(M_{\sigma v})\,d\langle M_{\cdot v}\rangle_\sigma\right) d\langle f(\cdot,v)\rangle_u.
\end{aligned}
$$

In fact, this follows from Itô's formula applied to G_δ: $R^3 \to R$, $G_\delta(x, y, z) = xy/[\varphi_\delta(z)]$, letting $\delta \to 0$ and using the same arguments as in Step 1. Substituting (12) in (11) we get

$$
\begin{aligned}
\int_R L_1^\varepsilon(x,s,t)\phi(x)\,dx &= \int_{R_{st}^\varepsilon} \frac{g(u,v)}{f(u,v)}\phi(M_{uv})\,d\langle M_{\cdot v}\rangle_u\,dv \\
&= \int_{R_{st}^\varepsilon} \phi(M_{uv})\,d\langle M\rangle_{uv} \quad \text{a.s.} \qquad \square
\end{aligned}
$$

2. There are two important classes of two-parameter martingales to which the results proved in Section 1 may apply: (a) martingales with respect to the filtration generated by a Brownian sheet and (b) martingales with respect to the product filtration generated by independent multidimensional Brownian motions. In this section we will present some examples in these classes.

(a) Let $W = \{W_z,\ z \in R_+^2\}$ be the Wiener sheet and $(\mathscr{F}_z)_{z\in R_+^2}$ its associated filtration. Let $M = \{M_z,\ z \in R_+^2\}$ be a two-parameter martingale, with respect to $(\mathscr{F}_z)_{z\in R_+^2}$, vanishing on the axes and bounded in L^2. By Wong and Zakai's representation theorem [14]

$$M_{st} = \int_{R_{st}} \phi_z\, dW_z + \iint_{R_{st}\times R_{st}} \psi(z, z')\, dW_z\, dW_{z'},$$

where $\phi = \{\phi_z,\ z \in R_+^2\}$ is a measurable and adapted process such that $E(\int_{R_{z_0}} \phi_z^2\, dz) < \infty$ for all $z_0 \in R_+^2$, and $\psi = \{\psi(z; z'), (z, z') \in R_+^2 \times R_+^2\}$ is a measurable and $\mathscr{F}_{zvz'}$-adapted process, vanishing unless

$$(z, z') \in D = \{(z, z');\ z = (x, y),\ z' = (x', y');\ x \le x',\ y \ge y'\}$$

and such that $E(\iint_{R_{z_0}\times R_{z_0}} \psi(z;z')^2\, dz\, dz') < \infty$ for all $z_0 \in R_+^2$.

By standard calculations it follows that

$$\langle M\rangle_{st} = \int_{R_{st}} g(u, v)\, du\, dv,$$

with

$$g(u, v) = \phi^2(u, v) + \int_{R_{uv}} \psi^2(x, v; u, y)\, dx\, dy,$$

$$\langle M_{\cdot t}\rangle_s = \int_0^s f(u, t)\, du,$$

where

$$f((u, t)) = \int_0^t \left(\phi(u, v) + \int_{R_{ut}} \psi(z'; u, v)\, dW_{z'} \right)^2 dv,$$

and analogously

$$\langle M_{s\cdot}\rangle_t = \int_0^t h(s, v)\, dv,$$

with

$$h(s, v) = \int_0^s \left(\phi(u, v) + \int_{R_{sv}} \psi(u, v; z')\, dW_{z'} \right)^2 du.$$

In view of this expression for the quadratic variations associated to M, it would be possible to exhibit Brownian martingales for which the results of Theorems 1 and 2 apply. We next give two possible examples.

EXAMPLE 1. Let $M_{st} = \int_{R_{st}} W_z\, dW_z$. This is a strong martingale with $\langle M\rangle_{st} = \langle M_{\cdot t}\rangle_s = \langle M_{s\cdot}\rangle_t = \int_{R_{st}} W_z^2\, dz$. That is, $g(u, v) = W_{uv}^2$, $f(u, t) = \int_0^t W_{uv}^2\, dv$ and $h(s, v) = \int_0^s W_{uv}^2\, du$.

By the scaling property of Brownian motion we have that the processes $\{f(u,v),\ (u,v)\in R^2_+\}$ and $\{h(u,v),\ (u,v)\in R^2_+\}$ have the same law as $\{uv^2\int_0^1 W_{1\sigma}^2\,d\sigma,\ (u,v)\in R^2_+\}$ and $\{u^2v\int_0^1 W_{1\sigma}^2\,d\sigma,\ (u,v)\in R^2_+\}$, respectively. Using Kolmogorov's continuity criterion, it is easy to check that f and h are jointly continuous. Also the properties of Brownian motion imply that f and h are different from zero in $(0,\infty)^2$, and for each $s\ge 0$, $t\ge 0$, $p\ge 1$,

$$\sup_{0\le u\le s} E\left[|f(u,t)|^p\right]<\infty,\qquad \sup_{0\le v\le t} E\left[|h(s,v)|^p\right]<\infty.$$

Therefore, Theorem 1 applies; Theorem 2 in its second part version also does. In fact, using Itô's calculus we obtain

$$m_v(u)=2\int_0^u W_{\sigma v}\,d_1W_{\sigma v},\qquad a_v(u)=uv,$$

$$n_v(u)=2\int_0^v\int_0^u W_{\sigma\tau}\,d_1W_{\sigma\tau}\,d\tau,\qquad b_v(u)=u\frac{v^2}{2},$$

$$\langle m_v(\cdot)\rangle_u=4\int_0^u W_{\sigma v}^2 v\,d\sigma,\qquad \xi_v(\sigma)=4vW_{\sigma v}^2,$$

$$\langle n_v(\cdot)\rangle_u=4\int_0^u\left(\int_0^v\int_0^v W_{\sigma\tau}W_{\sigma\tau'}(\tau\wedge\tau')\,d\tau\,d\tau'\right)d\sigma,$$

$$\chi_v(\sigma)=4\int_0^v\int_0^v W_{\sigma\tau}W_{\sigma\tau'}(\tau\wedge\tau')\,d\tau\,d\tau',$$

$$\langle m_v(\cdot),n_v(\cdot)\rangle_u=\int_0^u\left(4\int_0^v W_{\sigma v}W_{\sigma\tau}\tau\,d\tau\right)d\sigma,$$

$$\eta_v(\sigma)=4\int_0^v W_{\sigma v}W_{\sigma\tau}\tau\,d\tau,$$

and the processes ξ_v, χ_v, η_v, a_v and b_v satisfy the required properties. On the other hand the random variable $(\int_0^1 W_{1\sigma}^2\,d\sigma)^{-1}$ has moments of any order, due to the inequality $P\{\int_0^1 W_{1\sigma}^2\,d\sigma<\varepsilon\}\le\sqrt{2}\exp[-1/(2^7\varepsilon^2)]$ (see, for instance, Lemma 5.8.6 [8]). Therefore,

$$E\left[\left|\frac{\xi_v(u)}{f(u,v)^2}\right|^p\right]\le E\left[|\xi_v(u)|^{2p}\right]^{1/2}E\left[\frac{1}{f(u,v)^{4p}}\right]^{1/2}$$
$$\le Cv^{-2p}u^{-p}.$$

Analogously, $E[|(\chi_v(u)g(u,v)^2)/(f(u,v)^4)|^p]$ is bounded by a monomial in u and v, and conditions (3) are fulfilled. Hence $L(\cdot,s,t)$ is a.s. γ-Hölder continuous with γ arbitrarily closed to $\frac{1}{4}$.

EXAMPLE 2. Let $J_{st}=\int\int_{R_{st}\times R_{st}}1_D(z,z')\,dW_z\,dW_{z'}$. This process is a martingale which appears in the decomposition of W_{st}^2 given by Itô's formula.

We have

$$\langle J\rangle_{st} = \frac{s^2t^2}{4} = \int_{R_{st}} uv\,du\,dv.$$

Therefore

$$g(u,v) = uv,\ m_v(u) = 0 = \xi_v(\sigma) = \eta_v(\sigma),$$

$$\langle J_{\cdot t}\rangle_s = \int_{R_{st}} (W_{ut} - W_{uv})^2\,du\,dv;$$

thus

$$f(u,t) = \int_0^t (W_{ut} - W_{uv})^2\,dv.$$

Analogously

$$\langle J_{s\cdot}\rangle_t = \int_0^t h(s,v)\,dv, \quad \text{with} \quad h(s,v) = \int_0^s (W_{sv} - W_{uv})^2\,du.$$

Using Itô's calculus we get

$$n_v(u) = 2\int_0^v \int_0^u (W_{\sigma v} - W_{\sigma\tau})\,d_1(W_{\sigma v} - W_{\sigma\tau})\,d\tau,$$

$$b_v(u) = u\frac{v^2}{2},$$

$$\langle n_v(\cdot)\rangle_u = 4\int_0^u \chi_v(\sigma)\,d\sigma,$$

with

$$\chi_\sigma(v) = \int_0^v \int_0^v (W_{\sigma v} - W_{\sigma\tau})(W_{\sigma v} - W_{\sigma\tau'})(v - \tau \vee \tau')\,d\tau\,d\tau'.$$

The process $\{f(u,v) = \int_0^v (W_{uv} - W_{u\tau})^2\,d\tau,\ u, v \geq 0\}$ has the same law as $\{uv^2\int_0^1 W_{1\sigma}^2\,d\sigma,\ u,\ v \geq 0\}$. Therefore, as in the preceding example, $E[|[\chi_v(u)g(u,v)^2]/[f(u,v)^4]|]^p$ is bounded by something like $Cu^r v^s$, with $r, s \in R$, and conditions (3) are satisfied. The remaining hypothesis of Theorem 2 is easily checked, as in Example 1.

(b) Let $\{\mathscr{F}_s^1,\ s \geq 0\}$ and $\{\mathscr{F}_t^2,\ t \geq 0\}$ be two families of increasing σ-fields generated by two independent multidimensional Brownian motions $\{(B_s^1, \ldots, B_s^n),\ s \geq 0\}$, $\{\hat{B}_s^1, \ldots, \hat{B}_s^m),\ s \geq 0\}$, respectively. Define $W_{st}^{ij} = B_s^i \hat{B}_t^i$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, and $\mathscr{F}_{st} = \mathscr{F}_s^1 \vee \mathscr{F}_t^2$. It is known that, if M is a martingale with respect to the product filtration, vanishing on the axes and bounded in L^2, there exists a family of processes $\{h_{ij}(u,v),\ (u,v) \in R_+^2\}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, $\mathscr{F}_{st}$-previsible and $E(\int_{R_{s_0t_0}} h_{ij}^2(u,v)\,du\,dv) < \infty$ for any s_0, t_0, i, j, such that

$$M_{st} = \sum_{i=1}^n \sum_{j=1}^m \int_{R_{st}} h_{ij}(u,v)\,dW_{uv}^{ij}.$$

For such martingales, which are called "bi-Brownian"-martingales, we have

$$\langle M\rangle_{st} = \int_{R_{st}} \sum_{i=1}^{n} \sum_{j=1}^{m} h_{ij}^2(u, v)\, du\, dv,$$

$$\langle M_{\cdot t}\rangle_s = \int_0^s \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \int_0^t h_{ij}(u, v)\, d\hat{B}_v^j \right)^2 du,$$

$$\langle M_{s\cdot}\rangle_t = \int_0^t \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \int_0^s h_{ij}(u, v)\, dB_v^i \right)^2 dv.$$

Consequently, we can obtain in this class examples of processes which satisfy Theorems 1 and 2, putting suitable conditions on the kernels occurring in the representation of M.

EXAMPLE 3. Let $n = m = 1$ and $M_{st} = \int_{R_{st}} h_1(u) h_2(v)\, dB_u\, d\hat{B}_v$ with h_1 (h_2) continuous and $\mathscr{F}_u^1$- ($\mathscr{F}_v^2$-) adapted processes such that $E(\int_0^{s_0} h_1^2(u)\, du) < \infty$ [resp., $E(\int_0^{t_0} h_2^2(v)\, dv) < \infty$] for any s_0 (t_0). This is a particular case of Example 0. If h_1, h_2 are bounded in L^p for some $p > 8$ and $p > 16$, respectively, $h_1(u) \neq 0$ for $u > 0$, and $\int_0^v h_2(\tau)\, d\hat{B}_\tau \neq 0$ for $v > 0$, by Theorem 1 we have

$$L(x, s, t) = \int_0^t \frac{h_2^2(v)}{\left(\int_0^v h_2(\tau)\, d\hat{B}_\tau\right)^2} L_1(x, s, v)\, dv,$$

and, according to Example 0, we must have

$$\int_0^t \frac{h_2^2(v)}{\left(\int_0^v h_2(\tau)\, d\hat{B}_\tau\right)^2} L_1(0, s, v)\, dv = \infty \quad \text{a.s.}$$

REMARK. Comparing the local time $\tilde{L}(x, s, t)$ of M with respect to $\langle \tilde{M}\rangle$ [11] with ours we observe that the former is smoother in x, while the latter is smoother in s, t. The reason for this is essentially because $\tilde{L}$ is derived by means of Itô's formula. It is difficult to exhibit in general the relation between L and $\tilde{L}$; however, in some particular cases we can guess it. Coming back to Example 3,

$$\langle M\rangle_{st} = \int_{R_{st}} h_1^2(u) h_2^2(v)\, du\, dv,$$

$$\langle \tilde{M}\rangle_{st} = \int_{R_{st}} (M_u^1)^2 (M_v^2)^2 h_1^2(u) h_2^2(v)\, du\, dv, \quad \text{where} \quad M_u^1 = \int_0^u h_1(\sigma)\, dB_\sigma,$$

$$M_v^2 = \int_0^v h_2(\tau)\, d\hat{B}_\tau.$$

Consequently, formally, $L(x, s, t) = \int_{R_{st}} 1/[(M_u^1)^2 (M_v^2)^2] \tilde{L}(x, du, dv)$. In view of this expression one can expect worse properties for L than for $\tilde{L}$ near the axes.

Acknowledgment. I would like to thank the referee for a helpful comment concerning the proof of Lemma 1.

REFERENCES

[1] ADLER, R. J. (1980). A Hölder condition for the local time of the Brownian sheet. *Indiana Univ. Math. J.* **29** 793–798.

[2] BARLOW, M. T. and YOR, M. (1981). Semi-martingale inequalities and local times. *Z. Wahrsch. verw. Gebiete* **55** 237–254.

[3] CAIROLI, R. and WALSH, J. B. (1975). Stochastic integrals in the plane. *Acta Math.* **134** 111–183.

[4] CHEVALIER, L. (1982). Martingales continues à deux paramètres. *Bull. Sci. Math.* (2) **106**(2) 19–62.

[5] DAVYDOV, YU. A. (1978). Local times for multiparameter random processes. *Theory Probab. Appl.* **23** 573–583.

[6] EHM, W. (1981). Sample function properties of multi-parameter stable processes. *Z. Wahrsch. verw. Gebiete* **55** 195–228.

[7] GARSIA, A. (1972). Continuity properties of Gaussian processes with a multi-dimensional time parameter. *Proc. Sixth Berkeley Symp. Math. Statist. Probab.* **2** 368–374. Univ. California Press.

[8] IKEDA, N. and WATANABE, S. (1981). *Stochastic Differential Equations and Diffusion Processes.* North-Holland, Amsterdam and Kodansha, Tokyo.

[9] IMKELLER, P. (1984). Stochastic analysis and local times for (N, d)-Wiener processes. *Ann. Inst. H. Poincaré Sect. B (N.S.)* **20** 75–101.

[10] NUALART, D. (1984). On the quadratic variation of two-parameter continuous martingales. *Ann. Probab.* **12** 445–457.

[11] NUALART, D. (1984). Une formule d'Itô pour les martingales continues à deux indices et quelques applications. *Ann. Inst. H. Poincaré Sect. B (N.S.)* **20**(3) 251–275.

[12] TRAN, L. T. (1976). On a problem posed by Orey and Pruitt related to the range of the N-parameter Wiener process in R^d. *Z. Wahrsch. verw. Gebiete* **37** 27–33.

[13] WALSH, J. B. (1978). The local time of the Brownian sheet. *Astérisque* **52–53** 47–61.

[14] WONG, E. and ZAKAI, M. (1974). Martingales and stochastic integrals for processes with a multi-dimensional parameter. *Z. Wahrsch. verw. Gebiete* **29** 109–122.

[15] YOR, M. (1980). Sur la transformée de Hilbert des temps locaux browniens, et une extension de la formule d'Itô. *Séminaire de Probabilités XVI, 1980 / 1981. Lecture Notes in Math.* **920** 238–247. Springer, Berlin.

FACULTAT DE MATEMÀTIQUES
UNIVERSITAT DE BARCELONA
GRAN VIA 585
08007 BARCELONA
SPAIN