A CORRELATION INEQUALITY FOR THE SYMMETRIC EXCLUSION PROCESS

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We prove a correlation inequality concerning the occupation of disjoint sets in the symmetric exclusion process. As an application we derive a pointwise ergodic theorem for this same process.

1. Introduction. Let S be a finite or countable set and $p(x, y)$ a probability matrix on S. The exclusion process given by S and p is a Markov process on $X = \{0, 1\}^S$ whose formal generator Ω is given by the expression

$$(\Omega f)(\eta) = \sum_{x, y \in S} p(x, y)\eta(x)(1 - \eta(y))\big(f(\eta_{xy}) - f(\eta)\big),$$

where f is any cylinder real-valued function on X, η is an arbitrary element of X and η_{xy} is obtained from η by interchanging the values of η at x and y. This generator corresponds to the following description: Particles are put on the elements of S in such a way that no two particles occupy the same site. These particles attempt to jump at times given by independent Poisson processes of parameter one. If a particle at x attempts a jump it chooses a site y, independently of the Poisson processes, with probability $p(x, y)$. If the site chosen is occupied the particle stays at x, otherwise it goes to that site.

In this paper we will only consider the case in which $p(x, y)$ is symmetric. In this situation the process is known to satisfy the following correlation inequality: If $x_1, x_2, \ldots, x_n$ are distinct elements of S, then for all $t \geq 0$ and all $\eta \in X$,

$$(1.1) \qquad P\left(\prod_{i=1}^{n} \eta_t(x_i) = 1\right) \leq \prod_{i=1}^{n} P\big(\eta_t(x_i) = 1\big).$$

For the proof of this inequality as well as for the construction of the process the reader is referred to [4]. This correlation inequality has been used to prove a large deviation result ([3]) and a pointwise ergodic theorem ([1]). It also played a role in studying the position of a tagged particle ([2]). In Section 2 we generalize this inequality and in Section 3 we apply the generalization to extend the results of [1]. If one drops from the assumptions the symmetry of $p(x, y)$, simple examples in which even (1.1) fails can be constructed.

2. A correlation inequality. The following theorem extends (1.1). In its proof we will freely identify elements of X with subsets of S in this way:

$\eta \leftrightarrow \{x \in S: \eta(x) = 1\}$. With this in mind A_t will denote the exclusion process starting from $A \subset S$.

THEOREM 2.1. *If A and B are disjoint subsets of S, then for all $\eta \in X$ and all $t \geq 0$,*

$$P\Big(\prod_{x \in A \cup B} \eta_t(x) = 1\Big) \leq P\Big(\prod_{x \in A} \eta_t(x) = 1\Big)P\Big(\prod_{x \in B} \eta_t(x) = 1\Big).$$

PROOF. Using the notation $\eta(D) = \prod_{x \in D}\eta(x)$, $f(D, t) = P(\eta(D_t) = 1)$ and the self-duality property of the symmetric exclusion process (Theorem 1.1, page 363 of [4]), it follows that the inequality to be proved is equivalent to

$$f(A \cup B, t) \leq f(A, t)f(B, t).$$

We will first prove this assuming A and B are finite. Let $|A| = n$ and $|B| = m$, where $|\ |$ denotes the cardinality. We now think of the process $(A \cup B)_t$ in the following way: After an exponentially distributed time of parameter $(n + m)$ an element of $A \cup B$ is chosen at random. Each element has probability $1/(n + m)$ of being selected. The selected element will then attempt to jump according to $p(x, y)$ and the jump will be done or not according to the exclusion rule described in the introduction. Since the probability that the first attempt occurs in the time interval $(s, s + \Delta s)$ is $(n + m)e^{-(n+m)s}\Delta s + o(\Delta s)$, we have

$$f(A \cup B, t) = f(A \cup B, 0)e^{-(n+m)t} + \int_0^t (n + m)e^{-(n+m)s}$$

$$\times \Bigg\{ \frac{1}{n + m}\Bigg[\sum_{x, y \in A} p(x, y) + \sum_{x, y \in B} p(x, y)\Bigg]f(A \cup B, t - s)$$

$$+ \frac{1}{n + m}\Bigg[\sum_{x \in A} \sum_{y \notin A \cup B} p(x, y)f(A_{xy} \cup B, t - s)\Bigg]$$

$$+ \frac{1}{n + m}\Bigg[\sum_{x \in B} \sum_{y \notin A \cup B} p(x, y)f(A \cup B_{xy}, t - s)\Bigg]$$

$$+ \frac{1}{n + m}\Bigg[\sum_{x \in A} \sum_{y \in B} p(x, y) + \sum_{x \in B} \sum_{y \in A} p(x, y)\Bigg]$$

$$\times f(A \cup B, t - s)\Bigg\}\, ds,$$

where A_{xy} and B_{xy} denote $A \cup \{y\} \setminus \{x\}$ and $B \cup \{y\} \setminus \{x\}$, respectively.

To obtain a similar expression for $f(A, t)f(B, t)$ we consider a Markov process on $X \times X$ starting from (A, B) whose marginals are independent copies

of the exclusion process. Reasoning as above we get

$$f(A, t)f(B, t) = f(A, 0)f(B, 0)e^{-(n+m)t} + \int_0^t (n + m)e^{-(n+m)s}$$

$$\times \left\{ \frac{1}{n + m} \left[\sum_{x, y \in A} p(x, y) + \sum_{x, y \in B} p(x, y) \right] \right.$$

$$\times f(A, t - s)f(B, t - s)$$

$$+ \frac{1}{n + m} \left[\sum_{x \in A} \sum_{y \notin A \cup B} p(x, y)f(A_{xy}, t - s)f(B, t - s) \right]$$

$$+ \frac{1}{n + m} \left[\sum_{x \in B} \sum_{y \notin A \cup B} p(x, y)f(A, t - s)f(B_{xy}, t - s) \right]$$

$$+ \frac{1}{n + m} \left[\sum_{x \in A} \sum_{y \in B} p(x, y)f(A_{xy}, t - s)f(B, t - s) \right]$$

$$+ \left. \frac{1}{n + m} \left[\sum_{x \in B} \sum_{y \in A} p(x, y)f(A, t - s)f(B_{xy}, t - s) \right] \right\} ds.$$

From these two expressions and the symmetry of p we obtain

$$f(A \cup B, t) - f(A, t)f(B, t)$$

$$(2.2) \quad \begin{aligned} = \int_0^t e^{-(n+m)s} &\left\{ \left[\sum_{x, y \in A} p(x, y) + \sum_{x, y \in B} p(x, y) \right] \right. \\ &\times \left[f(A \cup B, t - s) - f(A, t - s)f(B, t - s) \right] \\ &+ \sum_{x \in A} \sum_{y \notin A \cup B} p(x, y)\left[f(A_{xy} \cup B, t - s) - f(A_{xy}, t - s)f(B, t - s) \right] \\ &+ \sum_{x \in B} \sum_{y \notin A \cup B} p(x, y)\left[f(A \cup B_{xy}, t - s) - f(A, t - s)f(B_{xy}, t - s) \right] \\ &+ \sum_{x \in A} \sum_{y \in B} p(x, y)\left[2f(A \cup B, t - s) - f(A_{xy}, t - s)f(B, t - s) \right. \\ &\qqu\qquad\qquad \left.\left. - f(A, t - s)f(B_{yx}, t - s) \right] \right\} ds. \end{aligned}$$

We now define

$$G(t) = \sup_{\substack{C, D: C \cap D = \phi \\ |C| = n, |D| = m}} \{ f(C \cup D, t) - f(C, t)f(D, t) \}$$

and

$$F(t) = \sup_{0 \le s \le t} G(s).$$

From these definitions we see that for all disjoint A and B of cardinalities n and

m we have

$$f(A \cup B, t) \le f(A, t)f(B, t) + F(t).$$

Note now that if $x \in A$ and $y \in B$, then $A \cup B = A_{xy} \cup B_{yx}$ and $f(A \cup B, t) \le f(A_{xy}, t)f(B_{yx}, t) + F(t)$. Hence

$$f(A \cup B, t) \le F(t) + \min\{f(A, t)f(B, t), f(A_{xy}, t)f(B_{yx}, t)\}.$$

Using the inequality $\min\{ab, cd\} \le (ad + bc)/2$ which holds for any a, b, c and $d \in \mathbb{R}_+$, we see that

$$2f(A \cup B, t) \le 2F(t) + f(A, t)f(B_{yx}, t) + f(A_{xy}, t)f(B, t).$$

From the last inequality, the definition of $F(t)$ and (2.2), we obtain

$$f(A \cup B, t) - f(A, t)f(B, t)$$

$$\le \int_0^t e^{-(n+m)s}\left[\sum_{x, y \in A} p(x, y) + \sum_{x, y \in B} p(x, y) + \sum_{x \in A} \sum_{y \notin A \cup B} p(x, y) \right.$$

$$\left. + \sum_{x \in B} \sum_{y \notin A \cup B} p(x, y) + 2 \sum_{x \in A} \sum_{y \in B} p(x, y) \right] F(t - s) \, ds.$$

Since p is symmetric and F is nondecreasing this implies that

$$f(A \cup B, t) - f(A, t)f(B, t) \le F(t) \int_0^t (n + m)e^{-(n+m)s} \, ds.$$

Since the right-hand side only depends on $|A|$ and $|B|$ we have

$$G(t) \le F(t) \int_0^t (n + m)e^{-(n+m)s} \, ds,$$

and since the right-hand side is nondecreasing we can conclude that

$$F(t) \le F(t) \int_0^t (n + m)e^{-(n+m)s} \, ds.$$

Since $F(t) \ge 0$ and the integral is strictly less than one we must have $F(t) \equiv 0$.

The theorem is now proved for finite A and B; for arbitrary sets A and B consider sequences of finite sets A_n and B_n such that $A_n \uparrow A$ and $B_n \uparrow B$ and take limits on both sides of the inequality $f(A_n \cup B_n, t) \le f(A_n, t)f(B_n, t)$.

3. Application: A pointwise ergodic theorem. In this section we restrict ourselves to the case in which $S = \mathbb{Z}^d$ and $p(x, y)$ is translation invariant, irreducible and symmetric. In this case it is known that the set of extremal invariant measures is $\{\mu_\rho\}_{0 \le \rho \le 1}$, where μ_ρ denotes the unique product measure such that $\mu_\rho\{\eta: \eta(x) = 1\} = \rho$ for all $x \in \mathbb{Z}^d$. For a proof of this result the reader is referred to [4]. We now state, as an application of Theorem 2.1, a pointwise ergodic theorem.

THEOREM 3.1. *Let η be an element of X such that η_t converges in distribution to μ_ρ. Then for any $f \in C(X)$,*

$$\lim_T \frac{1}{T} \int_0^T f(\eta_s) \, ds = \int f(\eta) \, d\mu_\rho(\eta) \quad a.s.$$

SKETCH OF PROOF. First prove the theorem for functions f of the form $f(\eta) = \eta(A)$, where A is a finite subset of $\mathbb{Z}^d$. This can be done following the proof of Theorem 2.1 of [1] and using the extension of (1.1) given in Section 2. Then note that functions of the form $\eta(A)$ generate a dense subspace of $C(X)$.

REMARK. Theorems VIII.1.13 and VIII.1.24 in [4] applied to the case in which the initial distribution is a Dirac measure show that the hypothesis of Theorem 3.1 is equivalent to $\lim_t \Sigma_y p_t(x, y)\eta(y) = \rho$ for all $x \in \mathbb{Z}^d$, where

$$p_t(x, y) = \sum_{n=0}^{\infty} e^{-t} \frac{t^n}{n!} p^{(n)}(x, y).$$

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