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TIME REVERSAL ON LÉVY PROCESSES

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Time reversal of semimartingales defined on a Lévy process framework is considered. Usually semimartingales cannot be time-reversed such that the reversed process is still a semimartingale. An expansion-of-filtrations result for Lévy processes is established and then it is used to give sufficient conditions such that a semimartingale defined on a Lévy process can be time-reversed and still r.main a semimartingale.

1. Introduction. Usually semimartingales, when reversed, are not semimartingales, as Walsh (1982) has pointed out. Nevertheless, since semimartingales are essentially the most general possible stochastic differentials, it is desirable to obtain sufficient conditions such that they be reversible. This type of problem and related questions have recently been considered by a number of authors [e.g., Follmer (1986), Lindquist and Picci (1985), Pardoux (1986), Picard (1986) and Protter (1987)].

Suppose we are given a complete probability space $(\Omega, \mathscr{F}, P)$ with at least two filtrations $\mathbf{F} = (\mathscr{F}_t)_{t\in[0,1]}$ and $\tilde{\mathbf{H}} = (\tilde{\mathscr{H}}_t)_{t\in[0,1]}$. Let Y be a process with paths that are right-continuous and have left limits (hereafter, *càdlàg*), defined on $[0,1]$. We associate to Y the time-reversed process $\tilde{Y} = (\tilde{Y}_t)_{t\in[0,1]}$ [also denoted $(Y)\tilde{}$] given by

$$\tilde{Y}_t = \begin{cases} 0, & \text{if } t = 0, \\ Y_{(1-t)-} - Y_{1-}, & \text{if } 0 < t < 1, \\ Y_0 - Y_{1-}, & \text{if } t = 1, \end{cases} \tag{1.1}$$

where Y_{u-} denotes the left limit at u, $0 < u \le 1$.

(1.2) DEFINITION. Y is called an $(\mathbf{F}, \tilde{\mathbf{H}})$-*reversible semimartingale* if

(i) Y is an $\mathbf{F}$-semimartingale on $[0,1]$, and
(ii) $\tilde{Y}$ is an $\tilde{\mathbf{H}}$-semimartingale on $[0,1)$.

Note that in the above definition $\tilde{Y}$ need not be a semimartingale on the *closed* interval $[0,1]$. This is an important point; see the discussion following Theorem (1.9). By Stricker's theorem if Y is an $\mathbf{F}$-semimartingale, it is also a semimartingale for its natural filtration (i.e., the minimal filtration to which it is

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adapted), and thus one can simply say Y is a *reversible semimartingale* if both Y and $\tilde{Y}$ are semimartingales with respect to their natural filtrations.

We shall be primarily interested here in Lévy processes and we make the convention that Z will always denote a Lévy process. (A Lévy process Z on $[0,1]$ is a càdlàg process with stationary and independent increments, and with $Z_0 = 0$ a.s.)

We let Z^c denote the continuous local martingale part of Z relative to its natural filtration $\mathbf{F}$. [We refer the reader to Jacod (1979) or Dellacherie and Meyer (1982) for all unexplained terms, notation and "well-known" results.] Then either $Z^c \equiv 0$ or Z^c/σ is a standard Wiener process for some $\sigma > 0$. Let $\tilde{\mathscr{F}}$ be the natural filtration of $\tilde{Z}$.

(1.3) PROPOSITION. *The Lévy process Z is a reversible semimartingale. Its continuous local martingale part Z^c is an* $(\mathbf{F}, \tilde{\mathbf{F}})$*-reversible semimartingale.*

PROOF. $\tilde{Z}$ is clearly also a Lévy process, with the same law as $-Z$; thus it is a reversible semimartingale, since Lévy processes are semimartingales [e.g., Jacod (1979), page 63]. It is well known that for $s < t$, $Z_t^c - Z_s^c$ is measurable with respect to the σ-field $\sigma(Z_u - Z_s;\ s \le u \le t)$. Thus $(Z^c)^{\tilde{}}$ is again a Lévy process with respect to $\tilde{\mathbf{F}}$, and we deduce that Z^c is an $(\mathbf{F}, \tilde{\mathbf{F}})$-reversible semimartingale (indeed, it is a reversible martingale). Using that $\mathscr{L}(\tilde{Z}) = \mathscr{L}(-Z)$, where $\mathscr{L}(X)$ denotes the law of the process X on $[0,1]$, one could easily prove as well that $(Z^c)^{\tilde{}} = (\tilde{Z})^c$. $\square$

Next consider semimartingales of the form

$$X_t = \int_0^t f(Z_{s-})\, dZ_s, \qquad Y_t = \int_0^t f(Z_{s-})\, dZ_s^c, \tag{1.4}$$

for a suitable (e.g., locally bounded) Borel function f. These semimartingales will not in general be $(\mathbf{F}, \tilde{\mathbf{F}})$-reversible since $\tilde{X}$ and $\tilde{Y}$ are not even adapted to $\tilde{\mathbf{F}}$. We shall see later, however, that they are adapted to the following filtration:

(1.5) $\tilde{\mathbf{G}} = (\tilde{\mathscr{G}}_t)_{t \in [0,1]}$ denotes the smallest complete (right-continuous) filtration relative to which $\tilde{Z}$ is adapted and Z_1 is $\tilde{\mathscr{G}}_0$-measurable.

Since $\tilde{Z}_1 = Z_0 - Z_{1-}$ equals $-Z_1$ a.s., this is clearly equivalent to:

(1.6) $\tilde{\mathbf{G}}$ is the smallest complete filtration relative to which $\tilde{Z}$ is adapted and $\tilde{Z}_1$ is $\tilde{\mathscr{G}}_0$-measurable.

For convenience, we define as well:

(1.7) $\mathbf{G} = (\mathscr{G}_t)_{t \in [0,1]}$ denotes the smallest complete (right-continuous) filtration relative to which Z is adapted and Z_1 is $\mathscr{G}_0$-measurable.

Our goal is to show that X and Y in (1.4) are $(\mathbf{F}, \tilde{\mathbf{G}})$-reversible for as many functions f as possible. Clearly, the first step is to prove that Z itself is

$(\mathbf{F}, \tilde{\mathbf{G}})$-reversible. Since $\tilde{Z}$ is also a Lévy process, by comparing (1.6) and (1.7), this amounts to the following theorem, due to Kurtz (1986).

(1.8) THEOREM (Kurtz). *A Lévy process Z is a* $\mathbf{G}$*-semimartingale on* $[0, 1]$.

Actually it is possible to describe the situation for all semimartingales defined on a Lévy process:

(1.9) THEOREM. *Let Z be a Lévy process. Then every* $\mathbf{F}$*-semimartingale is a* $\mathbf{G}$*-semimartingale on* $[0, 1)$.

In the usual terminology [cf., e.g, Jeulin (1980)], the filtration satisfies "Hypothesis (H′)" on $[0, 1)$.

In Theorem (1.9) the restriction to $[0, 1)$ instead of $[0, 1]$ is necessary. Jeulin (1980), pages 46–47, has shown that even if Z is a Brownian motion (and hence a Lévy process), then Hypothesis (H′) does not hold for $\mathbf{G}$ on $[0, 1]$.

Returning to the processes X and Y of (1.4), we still need an hypothesis on the function f.

(1.10) HYPOTHESIS. *There is a right-continuous function $\hat{f}$ of finite variation on compacts such that the set $D = \{x\colon f(x) \neq \hat{f}(x)\}$ is at most countable.*

For example, every Borel function of finite variation on finite intervals is of this description.

Our main results for Lévy processes are the following two theorems. Our other primary result is Theorem (3.3) and its consequences.

(1.11) THEOREM. *Let Z be a Lévy process and let f satisfy* (1.10). *Then the process $X_t = \int_0^t f(Z_{s-})\, dZ_s$ is an* $(\mathbf{F}, \tilde{\mathbf{G}})$*-reversible semimartingale.*

(1.12) THEOREM. *Let Z be a Lévy process, and let f satisfy* (1.10). *Then the process $Y_t = \int_0^t f(Z_{s-})\, dZ_s^c$ is an* $(\mathbf{F}, \tilde{\mathbf{G}})$*-reversible semimartingale.*

It is implicit in Theorem (1.12) that Z^c is not identically zero, since otherwise the statement is trivial. Other results similar to that of (1.11), where the process X is a stochastic integral with respect to the jump measure of Z are given in Section 4.

If either of the semimartingales X or Y defined in (1.4) are $(\mathbf{F}, \tilde{\mathbf{G}})$-reversible, then one can add a process A of finite variation provided it is adapted to $\mathbf{F}$, and also $\tilde{A}$ is adapted to $\tilde{\mathbf{G}}$. Since we can consider the Lévy process Z as a Markov process, we shall see in Section 3 that the reversed process $\tilde{A}$ of an *additive functional* A of Z is adapted to $\tilde{\mathbf{G}}$.

As a corollary of Theorem (1.11) and the previous remark we then obtain (for example)

(1.13) COROLLARY. *Let L be the local time at 0 of the Brownian motion $Z = W$. Then the process*

$$U_t = \int_0^t f(W_s)\,dW_s + \int_0^t g(W_s)\,dL_s + \int_0^t h(W_s)\,ds$$

is an $(\mathbf{F}, \tilde{\mathbf{G}})$*-reversible semimartingale, for all Borel locally bounded g and h and all functions f satisfying* (1.10).

Corollary (1.13) is a special case of Theorem (6.2).

This paper is organized as follows. In Section 2 we prove Theorems (1.8) and (1.9), and even slightly more. In Section 3 we prove a general result (unrelated to Lévy processes) and show how it yields simple proofs of Theorems (1.11) and (1.12) under a supplementary hypothesis. Theorem (1.11) is proved in full generality in Section 4. In Section 5 we prove a theorem that is useful for reversing purely discontinuous local martingales with paths of infinite variation [Theorem (5.3)], and we then use it to prove Theorem (1.12). In Section 6 we consider the Brownian case and give proofs that are elementary in the sense that they do not use Markov process theory or the results of Çinlar, Jacod, Protter and Sharpe (1980). Our results are then used to give simple interpretations of recent results of Haussmann, Pardoux and Picard.

2. Expansion of filtrations for Lévy processes. In this section we establish results about the expansion of filtrations that have an interest in their own right. All that is needed for the time-reversal results, however, is Theorem (1.8). A simple proof of Theorem (1.8) alone is given following Comment (2.18) for the convenience of the reader who is interested primarily in time reversal.

For all facts about random measures and stochastic integrals with respect to random measures, we refer the reader to Jacod (1979).

Let μ denote the jump measure of Z. That is,

$$\mu(\omega; dt \times dx) = \sum_{s>0,\ \Delta Z_s(\omega) \neq 0} \varepsilon_{(s, \Delta Z_s(\omega))}(dt \times dx), \tag{2.1}$$

where $\Delta Z_s = Z_s - Z_{s-}$, the jump of Z at time s. Since Z is a Lévy process, the $\mathbf{F}$-compensator of μ is given by

$$\nu(\omega; dt \times dx) = dt \otimes F(dx), \tag{2.2}$$

where F is a nonrandom measure on $\mathbb{R}$, which integrates the function $x \mapsto \min(x^2, 1)$. For every $a > 0$ we have a decomposition for the Lévy process Z,

$$Z_t = b_a t + Z_t^c + \int_0^t \int_{|x| \le a} x(\mu - \nu)(ds \times dx) + \sum_{0<s<t} \Delta Z_s 1_{\{|\Delta Z_s| > a\}}, \tag{2.3}$$

where $b_a \in \mathbb{R}$, and the integral above is a stochastic integral.

We introduce still another filtration, which is larger than $\mathbf{G}$:

(2.4) $\mathbf{H} = (\mathscr{H}_t)_{t \geq 0}$ is the smallest complete filtration relative to which Z is adapted, and Z_1^c and $\mu([0,1] \times A) = \sum_{0 < s \leq 1} 1_A(\Delta Z_s)$ are $\mathscr{H}_0$-measurable, for all Borel sets A lying at a positive distance from 0.

That $\mathbf{H}$ is larger than $\mathbf{G}$ (i.e., $\mathscr{G}_t \subseteq \mathscr{H}_t$ for all t) is easily deduced from the following two facts:

(i) $\int \mu((0,1] \times dx) f(x) = \sum_{0 < s \leq 1} f(\Delta Z_s)$ is $\mathscr{H}_0$-measurable for all Borel functions f vanishing on a neighborhood of 0.

(ii) The integral on the right-hand side of (2.3) is the limit in L^2, as n tends to ∞, of

$$\int_0^t \int_{1/n < |x| \leq a} x(\mu - \nu)(ds \times dx) = \sum_{0 < s \leq t} \Delta Z_s 1_{\{1/n < |\Delta Z_s| \leq a\}} - t \int_{1/n < |x| \leq a} x F(dx).$$

As a consequence Z_1 is clearly $\mathscr{H}_0$-measurable.

The main result of this section is the following.

(2.5) THEOREM. *Let Z be a Lévy process. Then every* $\mathbf{F}$*-semimartingale is an* $\mathbf{H}$*-semimartingale on* $[0,1)$, *where* $\mathbf{H}$ *is as defined in* (2.4).

Since any $\mathbf{H}$-semimartingale that is adapted to $\mathbf{G}$ is also a $\mathbf{G}$-semimartingale by Stricker's theorem, this result yields Theorem (1.9) and a fortiori Theorem (1.8) on $[0,1)$. [For a complete proof of (1.8), see (2.19).]

We begin with two preliminary results that have intrinsic interest. The first one is due to Kurtz (1986).

(2.6) THEOREM. *Assume that the Lévy process Z is integrable* [*i.e., $E(|Z_t|) < \infty$ for all t*]. *Then*

$$M_t = Z_t - \int_0^t \frac{Z_1 - Z_s}{1 - s}\, ds \text{ is a } \mathbf{G}\text{-martingale on } [0,1]. \tag{2.7}$$

PROOF. First assume $E(Z_t^2) < \infty$ for all t. Let $0 \leq s < t \leq 1$ be rationals, with $s = j/n$ and $t = k/n$. Next set

$$Y_i = Z_{(i+1)/n} - Z_{i/n}.$$

Then $Z_1 - Z_s = \sum_{i=j}^{n-1} Y_i$ and $Z_t - Z_s = \sum_{i=j}^{k-1} Y_i$. The random variables Y_i are i.i.d. and integrable. Therefore

$$\begin{aligned} E\{Z_t - Z_s | Z_1 - Z_s\} &= E\left\{ \sum_{i=j}^{k-1} Y_i \,\middle|\, \sum_{i=j}^{n-1} Y_i \right\} \\ &= \frac{k-j}{n-j} \sum_{i=j}^{n-1} Y_i \\ &= \frac{t-s}{1-s}(Z_1 - Z_s). \end{aligned} \tag{2.8}$$

The independence of the increments of Z yields $E\{Z_t - Z_s | \mathcal{G}_s\} = E\{Z_t - Z_s | Z_1 - Z_s\}$; thus $E\{Z_t - Z_s | \mathcal{G}_s\} = (t-s)/(1-s)(Z_1 - Z_s)$ for all rationals $0 \le s < t \le 1$. Since $Z_t - E(Z_t)$ is an $\mathbf{F}$-martingale, the random variables $(Z_t)_{0 \le t \le 1}$ are uniformly integrable, whereas the paths of Z are right-continuous. We deduce that (2.8) holds for all reals, $0 \le s < t \le 1$.

Now fix s and t, $0 \le s < t < 1$. Using first Fubini's theorem for conditional expectations and second (2.8) yields

$$\begin{aligned} E\{M_t - M_s | \mathcal{G}_s\} &= E\{Z_t - Z_s | \mathcal{G}_s\} - \int_s^t \frac{1}{1-u} E\{Z_1 - Z_u | \mathcal{G}_s\}\, du \\ &= \frac{t-s}{1-s}(Z_1 - Z_s) - \int_s^t \frac{1}{1-u}\frac{1-u}{1-s}(Z_1 - Z_s)\, du \\ &= 0. \end{aligned}$$

It remains to verify that $E\{\int_0^1 |Z_1 - Z_s|/(1-s)\, ds\} < \infty$. Due to the independence and stationarity of the increments of Z, we have

$$E\{|Z_1 - Z_s|\} \le E\{(Z_1 - Z_s)^2\}^{1/2} \le a(1-s)^{1/2},$$

for some constant a and $0 \le s \le 1$. Therefore

$$E\left\{\int_0^1 \frac{|Z_1 - Z_s|}{1-s}\, ds\right\} \le a \int_0^1 \frac{\sqrt{1-s}}{1-s}\, ds < \infty.$$

Finally, suppose we have only that $E\{|Z_t|\} < \infty$, $0 \le t \le 1$. Let $J_t^1 = \sum_{0 < s \le t} \Delta Z_s 1_{\{\Delta Z_s > 1\}}$, and $J_t^2 = -\sum_{0 < s \le t} \Delta Z_s 1_{\{\Delta Z_s < -1\}}$. Then $Y_t = Z_t - J_t^1 + J_t^2$ is a Lévy process with bounded jumps, hence square integrable. Since Y, J^1 and J^2 are independent, we have that $Y_t - \int_0^t (Y_1 - Y_s)/(1-s)\, ds$ is a $\mathbf{G}$-martingale on $[0,1]$.

The same proof shows that $J_t^i - \int_0^t (J_1^i - J_s^i)/(1-s)\, ds$ is a martingale on $[0,1]$ if $E\int_0^1 |J_1^i - J_s^i|/(1-s)\, ds < \infty$. But

$$\begin{aligned} E\left\{\int_0^1 \frac{|J_1^i - J_s^i|}{1-s}\, ds\right\} &= E\left\{\int_0^1 \frac{J_1^i - J_s^i}{1-s}\, ds\right\} \\ &= \int_0^1 E\left\{\frac{(J_1^i - J_s^i)}{1-s}\right\} ds \\ &= a_i \int_0^1 \frac{1-s}{1-s}\, ds = a_i < \infty, \end{aligned}$$

by the stationarity of the increments. Since Y, J^1 and J^2 are independent, we conclude that M is a $\mathbf{G}$-martingale on $[0,1]$. □

(2.9) THEOREM. *Let Z be a Lévy process.*

(i) *The process*

$$\hat{Z}_t^c = Z_t^c - \int_0^t \frac{Z_1^c - Z_s^c}{1-s}\, ds \tag{2.10}$$

is an $\mathbf{H}$*-martingale on* $[0,1]$ *with quadratic variation* $\langle \hat{Z}^c, \hat{Z}^c \rangle = \langle Z^c, Z^c \rangle$.

(ii) *The* **H**-*compensator* ρ *of the jump measure* μ *on* $[0,1]$ *is given by*

$$\rho(\omega; dt \times dx) = dt \times \frac{\mu(\omega; (t,1] \times dx)}{1-t} 1_{[0,1)}(t). \tag{2.11}$$

PROOF. We indicate the dependence of the various filtrations on the underlying process Z by writing $\mathbf{F}(Z)$, $\mathbf{G}(Z)$ or $\mathbf{H}(Z)$.

(i) Since Z^c is itself an integrable Lévy process, Theorem (2.6) implies that $\hat{Z}^c$ is a $\mathbf{G}(Z^c)$-martingale on $[0,1]$. In this case $\mathbf{G}(Z^c) = \mathbf{H}(Z^c)$, and letting $Z^d = Z - Z^c$ we have that $\mathbf{H}(Z)$ is the filtration generated by $\mathbf{H}(Z^c)$ and $\mathbf{H}(Z^d)$. [That is, $\mathscr{H}(Z)_t = \bigcap_{s>t} \mathscr{H}(Z^c)_s \vee \mathscr{H}(Z^d)_s$.] Note that the filtrations $\mathbf{H}(Z^c)$ and $\mathbf{H}(Z^d)$ are independent, whence $\hat{Z}^c$ is also an $\mathbf{H}(Z)$-martingale on $[0,1]$.

(ii) Let $\mathscr{R}$ denote the class of Borel subsets of $\mathbb{R}$ lying at a positive distance from 0. For $A \in \mathscr{R}_0$ we set

$$Z_t^A = \mu((0,t] \times A),$$

$$\hat{Z}_t^A = Z_t^A - \int_0^t \frac{Z_1^A - Z_s^A}{1-s}\, ds = \mu((0,t] \times A) - \rho((0,t] \times A).$$

For all $A \in \mathscr{R}_0$ the processes $(\rho((0,t] \times A))_{0 \le t < 1}$ are continuous and adapted to $\mathbf{H}$; thus the random measure ρ is $\mathbf{H}$-predictable. Therefore the statement (ii) is equivalent to the claim that for every $A \in \mathscr{R}_0$, the process $\hat{Z}^A$ is an $\mathbf{H}(Z)$-martingale on $[0,1]$. In other words, for all $0 \le s < t < 1$ it suffices to prove that

$$E\{(\hat{Z}_t^A - \hat{Z}_s^A)UV\} = 0, \tag{2.12}$$

where U is bounded and $\mathscr{F}(Z)_s$-measurable, and V is bounded and measurable with respect to $\sigma(Z_1^c - Z_s^c;\ Z_1^B - Z_s^B\colon B \in \mathscr{R}_0)$.

Due to the independence of the increments of Z, the r.v. U is independent of $(\hat{Z}_t^A - \hat{Z}_s^A)V$, and thus it is enough to prove (2.12) when $U \equiv 1$. Furthermore, by a monotone class argument it is enough to consider V of the form

$$V = f(Z_1^c - Z_s^c) \prod_{i=1}^n f_i(Z_1^{A_i} - Z_s^{A_i}) \prod_{j=1}^m g_j(Z_1^{B_j} - Z_s^{B_j}),$$

where f, f_i and g_j are all bounded Borel; where $(A_1, \dots, A_n)$ is a Borel partition of A; and where $B_j \in \mathscr{R}_0$ with $B_j \cap A = \varnothing$.

Next observe that the processes Z^c and Z^{B_j} are independent of $\hat{Z}^A$ and Z^{A_i}. Thus it is enough to prove (2.12) when $U = 1$ and

$$V = \prod_{i=1}^n f_i(Z_1^{A_i} - Z_s^{A_i}).$$

with A_i as before. Since $\hat{Z}^A = \sum_{i=1}^n \hat{Z}^{A_i}$ and since the processes Z^{A_i} are independent, we have in this case

$$E\{(\hat{Z}_t^A - \hat{Z}_s^A)UV\}$$
$$= \sum_{i=1}^n E\{(\hat{Z}_t^{A_i} - \hat{Z}_s^{A_i}) f_i(Z_1^{A_i} - Z_s^{A_i})\} \prod_{j \ne i,\, 1 \le j \le n} E\{f_j(Z_1^{A_j} - Z_s^{A_j})\}.$$

Finally, it suffices to show

$$E\{(\hat{Z}_t^{A_i} - \hat{Z}_s^{A_i}) f_i(Z_1^{A_i} - Z_s^{A_i})\} = 0. \tag{2.13}$$

At this stage we observe that Z^{A_i} is an *integrable* Lévy process (recall that A_i lies away from 0), and hence by Theorem (2.6) we have that $\hat{Z}^{A_i}$ is a $\mathbf{G}(Z^{A_i})$-martingale on $[0,1)$. Since $Z_1^{A_i} - Z_s^{A_i}$ is $\mathscr{G}(Z^{A_i})_s$-measurable, (2.13) follows and the proof is complete. □

PROOF OF THEOREM (2.5). As is well known, it suffices to prove that any square-integrable $\mathbf{F}$-martingale M on $[0,1)$ with $M_0 = 0$, and which is either continuous or purely discontinuous, is an $\mathbf{H}$-semimartingale on $[0,1)$ [cf. Dellacherie and Meyer (1982)].

Case (i). Let M be a continuous square-integrable $\mathbf{F}$-martingale on $[0,1]$ with $M_0 = 0$. The representation theorem for martingales of a Lévy process implies that $M_t = \int_0^t H_s\, dZ_s^c$ for some predictable process H such that

$$E\left\{\int_0^1 H_s^2\, d\langle Z^c, Z^c\rangle_s\right\} = \sigma^2 E\left\{\int_0^1 H_s^2\, ds\right\} < \infty, \tag{2.14}$$

since $\langle Z^c, Z^c\rangle_t = \sigma^2 t$ for some $\sigma \geq 0$. But $\langle \hat{Z}^c, \hat{Z}^c\rangle = \langle Z^c, Z^c\rangle$, and thus (2.14) yields that the stochastic integral $\hat{M}_t = \int_0^t H_s\, d\hat{Z}_s^c$ is well defined and is an $\mathbf{H}$-martingale on $[0,1)$. Moreover, if $C = Z^c - \hat{Z}^c$, then (2.10) together with (2.14) implies that the Stieltjes integral $D_t = \int_0^t H_s\, dC_s$ is well defined on $[0,1)$.

It remains only to observe that

$$M = \hat{M} + D. \tag{2.15}$$

Equality (2.15) is clear if H is bounded by Stricker's theorem, since $\int_0^t H_s\, dZ_s^c$ has the same value in $\mathbf{F}$ and $\mathbf{H}$. If H is not bounded let $H^n = H1_{\{|H| \leq n\}}$, $M_t^n = \int_0^t H_s^n\, dZ_s^c$, $\hat{M}_t^n = \int_0^t H_s^n\, d\hat{Z}_s^c$ and $D_t^n = \int_0^t H_s^n\, dC_s$. Then

$$M^n = \hat{M}^n + D^n,$$

and M^n, $\hat{M}^n$ and D^n all converge in probability to M, $\hat{M}$ and D, respectively; therefore (2.15) holds.

Case (ii). Let M be a purely discontinuous square-integrable $\mathbf{F}$-martingale on $[0,1)$ with $M_0 = 0$. Then there exists a predictable function W on $\Omega \times [0,1) \times R$ such that

$$M_t = \int_0^t \int_{\mathbb{R}} W(s,x)(\mu - \nu)(ds \times dx), \tag{2.16}$$

where W satisfies

$$E\left\{\int_0^1 \int_{\mathbb{R}} W(s,x)^2 \mu(ds \times dx)\right\} = E\left\{\int_0^1 \int_{\mathbb{R}} W(s,x)^2\, ds\, F(dx)\right\} < \infty, \tag{2.17}$$

and where μ, ν and F are given in (2.1) and (2.2) [cf. Jacod (1979)].

Note that (2.17) implies that the following stochastic integral is well defined and gives an $\mathbf{H}$-martingale on $[0,1)$:

$$\hat{M}_t = \int_0^t \int_{\mathbb{R}} W(s,x)(\mu - \rho)(ds \times dx),$$

where ρ is as given in (2.11). For $n \in \mathbb{N}$ we set

$$M_t^n = \int_0^t \int_{|x|>1/n} W(s,x)(\mu - \nu)(ds \times dx),$$

$$\hat{M}_t^n = \int_0^t \int_{|x|>1/n} W(s,x)(\mu - \rho)(ds \times dx),$$

$$C_t^n = \int_0^t \int_{|x|>1/n} W(s,x)(\rho - \nu)(ds \times dx),$$

where $M^0 = \hat{M}^0 = C^0 = 0$. These processes are all of finite variation on $[0, t]$ for all $t < 1$. Also $M^n = \hat{M}^n + C^n$; M^n is an **F**-martingale on $[0,1]$; and $\hat{M}^n$ is an **H**-martingale on $[0,1)$. Furthermore, a classical convergence theorem for random measures implies that $M_t^n \to M_t$ and $\hat{M}_t^n \to \hat{M}_t$ in L^2 as n tends to ∞. Therefore $C_t^n \to C_t$ in L^2, where C_t is defined to be $M_t - \hat{M}_t$. It remains only to prove that C has paths of finite variation on $[0, t]$, all $t < 1$.

To this end, we observe that in view of (2.2) and (2.14), we have

$$C_t^n = \int_0^t U_s^n \, ds,$$

where

$$U_s^n(\omega) = \int_{|x|>1/n} \frac{1}{1-s} \mu(\omega; (s,1] \times dx) W(\omega, s, x) - \int_{|x|>1/n} F(dx) W(\omega; s, x).$$

Let $n > m \geq 0$, and with the convention $1/0 = +\infty$,

$$N_t^{n,m,s}(\omega) = \frac{1}{1-s} 1_{\{s<t\}} \int_s^t \int_{1/n<|x|\leq 1/m} W(\omega, u, x)(\mu - \nu)(\omega; du \times dx).$$

Note that $N_t^{n,m,s}(\omega)$ is the integral (with respect to $\mu - \nu$) of the function

$$(\omega, u, x) \mapsto W^{n,m,s}(\omega, u, x) = \frac{1}{1-s} 1_{\{s<u\}} W(\omega, s, x) 1_{\{1/n<|x|\leq 1/m\}}.$$

Therefore $N^{n,m,s}$ is an **F**-martingale, and

$$E\{(N_1^{n,m,s})^2\} = E\left\{\int_0^1 \int_{\mathbb{R}} (W^{n,m,s}(u,x))^2 \, du F(dx)\right\}$$
$$= \frac{1}{1-s} E\left\{\int_{1/n<|x|\leq 1/m} W(s,x)^2 F(dx)\right\}.$$

By construction we also have $N_1^{n,m,s} = U_s^n - U_s^m$. Hence for $t < 1$,

$$E\left\{\int_0^t (U_s^n - U_s^m)^2 \, ds\right\} = \int_0^t \frac{ds}{1-s} E\left\{\int_{1/n<|x|\leq 1/m} W(s,x)^2 F(dx)\right\},$$

which tends to 0 as n, m increase to ∞, by (2.17). We deduce that U^n converges to a limit U in $L^2(\Omega \times [0, t], P(d\omega) \otimes du)$, and, moreover, obviously $C_t = \int_0^t U_s\, ds$, which completes the proof. □

(2.18) COMMENT. Theorems (1.9) and (2.5) will not be used for the time-reversal results comprising the rest of this article. Theorem (1.8), however, is fundamental. It is worth noting, therefore, that it has a simple proof, based only on Theorem (2.6).

(2.19) PROOF OF THEOREM (1.8). Let Z be an arbitrary Lévy process. Let $J_t^1 = \sum_{0<s<t} \Delta Z_s 1_{\{|\Delta Z_s|>1\}}$, the last term on the right-hand side of (2.3) with $a = 1$. Set

$$Z_t' = Z_t - J_t^1.$$

Then Z' is a square-integrable Lévy process and hence is a $\mathbf{G}(Z')$-semimartingale by Theorem (2.6). [We write $\mathbf{G}(Z')$ to indicate the dependence of the filtration on the underlying process, as in the proof of Theorem (2.9).] Moreover, Z' and J^1 are independent, hence $\mathbf{G} = \mathbf{G}(Z)$ is contained in the filtration $\mathbf{K}$, which is generated by the two independent filtrations $\mathbf{G}(Z')$ and $\mathbf{G}(J^1)$, and it readily follows that Z' is a $\mathbf{K}$-semimartingale on $[0,1]$. It is therefore a $\mathbf{G}$-semimartingale on $[0,1]$ as well, because of Stricker's theorem [cf., e.g., Dellacherie and Meyer (1982), page 248]. Moreover, since Z has right-continuous paths with left limits, we deduce that J^1 has paths of finite variation on $[0,1]$ and thus it is a $\mathbf{G}$-semimartingale. Therefore $Z = Z' + J^1$ is a $\mathbf{G}$-semimartingale on $[0,1]$. □

(2.20) COMMENT. In the case where the Lévy process is a Brownian motion these results are not new. Theorem (1.8) for Z a Brownian motion is due to Itô (1978). Theorem (1.9) for the Brownian case can be found (along with many other interesting results) in Jeulin (1980), page 46 ff.

3. Reversal of stochastic integrals. With the notation of Definition (1.2), let Y be an $(\mathbf{F}, \tilde{\mathbf{G}})$-reversible semimartingale. We also suppose given a process H with càdlàg paths that:

(3.1) For all t, $0 \le t \le 1$, H_t is $\mathscr{F}_t$- and $\tilde{\mathscr{G}}_t$-measurable.

The quadratic covariation $[H, Y]$ exists in the following sense: Fix t, $0 \le t \le 1$, and let $\tau_t = (t_0, \dots, t_k)$ be a partition of $[0, t]$ with $t_0 = 0$, $t_k = t$. Let

$$S_{\tau_t}(H, Y) = H_0 Y_0 + \sum_i \left(H_{t_{i+1}} - H_{t_i}\right)\left(Y_{t_{i+1}} - Y_{t_i}\right).$$

(3.2) We say the quadratic covariation exists if there exists a càdlàg, adapted process $[H, Y]$ such that

$$\lim_{n\to\infty} S_{\tau_t^n}(H, Y) = [H, Y]_t, \quad \text{in probability},$$

$0 \le t \le 1$, for each sequence τ_t^n of partitions of $[0, t]$ with mesh tending to 0.

(3.3) THEOREM. *Let H and Y be as given previously with H satisfying* (3.1) *and* (3.2). *Then the processes* $[H, Y]$ *and* $X_t = \int_0^t H_{s-}\, dY_s$ *are* $(\mathbf{F}, \tilde{\mathbf{G}})$*-reversible semimartingales. Moreover,*

$$\tilde{X}_t + [H, Y]_t^{\sim} = \int_0^t H_{1-s}\, d\tilde{Y}_s. \tag{3.4}$$

PROOF. First note that the left-continuous process H_{1-s} is $\tilde{\mathbf{G}}$-adapted by (3.1), and hence $\tilde{\mathbf{G}}$-predictable, so the stochastic integral in (3.4) is well defined.

Fix t, $0 < t < 1$, and let τ be a partition: $\{1 - t = s_0 < s_1 < \cdots < s_n = 1\}$ of $[1 - t, 1]$, chosen such that $\Delta Y_{s_i} = 0$ a.s. for all $i = 1, 2, \dots, n - 1$. (For a process V, $\Delta V_t \equiv V_t - V_{t-}$, the jump at t.) Next we define

$$\begin{aligned} A^\tau &= H_{(1-t)-}\, \Delta Y_{1-t} + \sum_{i=0}^{n-2} H_{s_i}\left(Y_{s_{i+1}} - Y_{s_i}\right) + H_{s_{n-1}}\left(Y_{1-} - Y_{s_{n-1}}\right), \\ B^\tau &= -\sum_{i=0}^{n-1} H_{s_{i+1}}\left(Y_{s_{i+1}-} - Y_{s_i-}\right), \\ C^\tau &= \Delta H_{1-t}\, \Delta Y_{1-t} + \sum_{i=0}^{n-2} \left(H_{s_{i+1}} - H_{s_i}\right)\left(Y_{s_{i+1}} - Y_{s_i}\right) \\ &\quad + \left(H_{1-} - H_{s_{n-1}}\right)\left(Y_{1-} - Y_{s_{n-1}}\right). \end{aligned} \tag{3.5}$$

By hypothesis (3.2) we have (limits are in probability)

$$\begin{aligned} \lim_{\text{mesh}(\tau) \to 0} C^\tau &= [H, Y]_{1-} - [H, Y]_{1-t} + \Delta H_{1-t}\, \Delta Y_{1-t} \\ &= [H, Y]_{1-} - [H, Y]_{(1-t)-} \\ &= -[H, Y]_t^{\sim}. \end{aligned} \tag{3.6}$$

By Hypothesis (3.1) and the assumption that Y is $(\mathbf{F}, \tilde{\mathbf{G}})$-reversible, we know that C^τ is $\tilde{\mathscr{G}}_t$-measurable, hence $[H, Y]^{\sim}$ is $\tilde{\mathbf{G}}$-adapted. It is of finite variation by hypothesis. Therefore $[H, Y]$ is an $(\mathbf{F}, \tilde{\mathbf{G}})$-reversible semimartingale.

To show X is also an $(\mathbf{F}, \tilde{\mathbf{G}})$-reversible semimartingale, it will suffice to show the validity of formula (3.4). To that end, since H is càdlàg, we know that

$$\lim A^\tau = \int_{[1-t, 1)} H_{s-}\, dY_s = X_{1-} - X_{(1-t)-} = -\tilde{X}_t, \tag{3.7}$$

where the limit is in probability as mesh(τ) tends to 0. Equation (3.7) is the Riemann approximation theorem for stochastic integrals [e.g., Dellacherie and Meyer (1982)]; alternatively, it can be shown directly by the dominated convergence theorem for stochastic integrals [cf. Jacod (1979), page 57]. Also, since $Y_{s_{i+1}-} - Y_{s_i-} = -(\tilde{Y}_{1-s_i} - \tilde{Y}_{1-s_{i+1}})$, analogously we have

$$\lim B^\tau = \int_0^t H_{1-s}\, d\tilde{Y}_s. \tag{3.8}$$

From (3.5) we have

$$A^\tau + B^\tau + C^\tau = H_{1-t}\Delta Y_{1-t} + \sum_{i=0}^{n-2} H_{s_{i+1}}\big(Y_{s_{i+1}} - Y_{s_i}\big)$$

$$+ H_{1-}\big(Y_{1-} - Y_{s_{n-1}}\big) - \sum_{i=0}^{n-1} H_{s_{i+1}}\big(Y_{(s_{i+1})-} - Y_{s_{i-}}\big)$$

$$= H_{1-t}\Delta Y_{1-t} + \sum_{i=0}^{n-2} H_{s_{i+1}}\big(\Delta Y_{s_{i+1}} - \Delta Y_{s_i}\big)$$

$$- \Delta H_1\big(Y_{1-} - Y_{s_{n-1}}\big) - H_1\,\Delta Y_{s_{n-1}}.$$

However, we chose τ so that $\Delta Y_{s_i} = 0$ for $1 \le i \le n-1$, and thus

$$A^\tau + B^\tau + C^\tau = \big(H_{1-t} - H_{s_1}\big)\Delta Y_{1-t} + \Delta H_1\big(Y_{s_{n-1}} - Y_{1-}\big),$$

which clearly tends to 0, since s_1 decreases to $1-t$ and s_{n-1} increases to 1. Therefore formula (3.4) follows from (3.6)–(3.8). □

(3.9) COMMENT. If $\tilde{Y}$ is a $\tilde{\mathbf{G}}$-semimartingale on the closed interval $[0,1]$, the same proof shows that $\tilde{X}$ is a $\tilde{\mathbf{G}}$-semimartingale on $[0,1]$.

(3.10) COMMENT. Let f be a $\mathscr{C}^1$ function and suppose Z is a Lévy process. If we take $H = f(Z)$, then Theorem (1.11) follows trivially from Theorem (3.3); one need only check that (3.2) holds, which has been shown by Meyer (1976), page 359. The same argument establishes Theorem (1.12) whenever f is $\mathscr{C}^1$. In Section 6 we apply Theorem (3.3) to stochastic differential equations.

4. Time reversal and additive functionals. In this section we prove Theorem (1.11). It is convenient (and involves no loss of generality) to use the Dynkin realization for our Lévy process Z. That is, we take Ω to be the path space $\Omega = \mathbb{D}([0,\infty),\mathbb{R})$; Z to be the canonical process $Z_t(\omega) = \omega(t)$ for $\omega \in \Omega$; $\mathbf{F}$ to be the canonical filtration; $(\theta_t)_{t\ge 0}$ to be the canonical shifts (so that $Z_{t+s} = Z_t \circ \theta_s$); and we assume given a family of measures $(P^x)_{x\in\mathbb{R}}$ under which Z is a Lévy process with $Z_0 = x$, P^x-a.s. Therefore $\Xi = (\Omega, \mathscr{F}_t, \theta_t, Z_t, P^x)$ is a strong Markov process. These are the standard notational conventions of Blumenthal and Getoor (1968). Note that the measure P of Theorems (1.11) and (1.12) is the measure P^0 restricted to $\mathscr{F}_1$ in this context.

An adapted, càdlàg process A is an *additive functional* (AF) of Ξ if $A_{t+s} = A_t + A_s \circ \theta_t$ a.s., all $s, t > 0$, where the null set does not depend on s or t. Note that we drop the traditional requirement that the paths of A be increasing.

For a given process Y, let $\tilde{Y}$ be as defined in (1.1), and let $\tilde{\mathbf{G}}$ be as defined in (1.5).

(4.1) LEMMA. *If A is an additive functional of Ξ, then $\tilde{A}$ is adapted to $\tilde{\mathbf{G}}$.*

PROOF. It suffices to prove that $\tilde{A}_{t-}$ is $\tilde{\mathscr{G}}_t$-measurable. Since A is additive we have $\tilde{A}_{t-} = A_{t-} \circ \theta_{1-t}$; also A_{t-} is $\mathscr{F}_{t-}$-measurable, so it is enough to show that $\theta_{1-t}^{-1}(\mathscr{F}_t) \subseteq \tilde{\mathscr{G}}_t$. Since $\mathbf{F}$ was defined to be minimal, it is enough to show that $Z_s \circ \theta_{1-t} = Z_{1-t+s}$ is $\tilde{\mathscr{G}}_t$-measurable for all $s < t$. But $Z_{1-t+s} = \tilde{Z}_{(t-s)-} + Z_0 - \tilde{Z}_s$, and also $Z_0 = 0$, P-a.s. Since $\tilde{\mathbf{G}}$ is P-complete, we are done. □

PROOF OF THEOREM (1.11). First we observe that $Z - Z_0$ is an AF, and hence $X_t = \int_0^t f(Z_{s-})\, dZ_s$ [cf. (1.4)] is also an AF of Ξ. [See Çinlar, Jacod, Protter and Sharpe (1980) for proofs of this and related statements.]

We next recall that if Z is a pure step process (that is, a compound Poisson process), then its paths are of finite variation on $[0,1]$. Therefore X and $\tilde{X}$ are also of finite variation on $[0,1]$ and the result follows from Lemma (4.1). Therefore it remains to consider the case where Z is not a pure step process.

Let $\hat{f}$ be the right-continuous function of finite variation on compacts associated to f in (1.10), and let $\hat{X}_t = \int_0^t \hat{f}(Z_{s-})\, dZ_s$. We will first prove that $\hat{X}$ is an $(\mathbf{F}, \tilde{\mathbf{G}})$-reversible semimartingale. Note that $\hat{f}$ is the right derivative of a function F, which is the difference of two convex functions. Let η be the Radon (signed) measure, which is the derivative of $\hat{f}$ taken in the generalized functions sense.

Next we recall the construction of the local time L^a of the $\mathbf{F}$-semimartingale Z at a level a. [This is the *semimartingale local time* as introduced by Meyer (1976), page 365; it is *not* the Markov local time as found, for example, in Blumenthal and Getoor (1968). The latter need not even exist.] Set

$$\operatorname{sign}|x| = \begin{cases} -1, & \text{if } x \le 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Then

$$L_t^a \equiv |Z_t - a| - |Z_0 - a| - \int_0^t \operatorname{sign}(Z_{s-} - a)\, d(Z - Z_0)_s - \sum_{s \le t} \{|Z_s - a| - |Z_{s-} - a| - \operatorname{sign}(Z_{s-} - a)\,\Delta Z_s\}$$

defines the local time. As is well known there exists a jointly measurable version, and we use this one by convention. Since $|Z_t - a| - |Z_0 - a|$ is an AF, L^a is also an AF, which is indeed continuous and nondecreasing in t. Then the Meyer–Tanaka–Itô change-of-variables formula yields

$$F(Z_t) - F(Z_0) = X_t + \frac{1}{2}\int L_t^a \eta(da) + \sum_{s \le t} \{F(Z_s) - F(Z_{s-}) - f(Z_{s-})\,\Delta Z_s\}. \tag{4.2}$$

Denote by S_t the second two terms on the right-hand side of (4.2). Then S_t is an AF with paths of finite variation on $[0,1]$; thus $\tilde{S}$ is a $\tilde{\mathbf{G}}$-semimartingale by Lemma (4.1). Moreover, if we set $V_t = F(Z_t) - F(Z_0)$ for $t \in (0,1)$, we have

$$\tilde{V}_t = F(Z_{(1-t)-}) - F(Z_{1-}) = F(\tilde{Z}_t - \tilde{Z}_1) - F(-\tilde{Z}_1),$$

P-a.s., since $Z_0 = 0$ a.s. However, by Theorem (1.8) we know that $\tilde{Z}_t - \tilde{Z}_1$ is a $\tilde{\mathbf{G}}$-semimartingale on $[0,1)$. Therefore $\tilde{V}$ is also a $\tilde{\mathbf{G}}$-semimartingale on $[0,1)$ since F is the difference of convex functions. Equation (4.2) then yields that $\hat{X}^{\sim}$ is a $\tilde{\mathbf{G}}$-semimartingale on $[0,1)$, and thus $\hat{X}$ is an $(\mathbf{F}, \tilde{\mathbf{G}})$-reversible semimartingale.

In order to finish the proof of the theorem it is then enough to show that $X_t = \hat{X}_t$ a.s. for all $t \in [0,1]$. That is, letting $D = \{x\colon f(x) \neq \hat{f}(x)\}$, it is enough to prove that

$$\int_0^t 1_D(Z_{s-})\, dZ_s = 0 \quad \text{a.s.,} \qquad 0 \le t \le 1. \tag{4.3}$$

First note that since Z is not a pure step process, by Theorem 1 of Blum and Rosenblatt (1959), we have $P(Z_s = x) = 0$ for all x, s. By hypothesis the set D is at most countable, hence by Fubini's theorem we have

$$\int_0^1 1_D(Z_s)\, ds = 0 \quad \text{a.s.} \tag{4.4}$$

Recalling (2.3), for every $a > 0$ the process Z has the decomposition

$$Z_t = M_t^a + b_a t + J_t^a, \tag{4.5}$$

where $J_t^a = \sum_{s \le t} \Delta Z_s 1_{\{|\Delta Z_s| > a\}}$ $b_a \in \mathbb{R}$ and M^a is a martingale such that $\langle M^a, M^a \rangle_t = K_a t$ for some constant K_a. Then (4.4) implies

$$\begin{aligned} E\left\{\left(\int_0^t 1_D(Z_{s-})\, dM_s^a\right)^2\right\} &= E\left\{\int_0^t 1_D(Z_{s-})\, d\langle M^a, M^a\rangle_s\right\} \\ &= K_a E\left\{\int_0^t 1_D(Z_{s-})\, ds\right\} \\ &= 0. \end{aligned}$$

Thus if $Y_t^a = M_t^a + b_a t$ we obtain

$$\int_0^t 1_D(Z_{s-})\, dY_s^a = 0 \quad \text{a.s.,} \qquad 0 \le t \le 1. \tag{4.6}$$

Moreover, since $\lim_{a \to \infty} J_t^a = 0$, $0 \le t \le 1$, combining this with (4.6) and using the decomposition (4.5) yields (4.3). □

(4.7) COMMENT. Let $\tilde{\mathbf{H}}$ be the filtration associated to the Lévy process $\tilde{Z}$ by (2.4). Due to Theorem (2.5), we can obtain more than Theorem (1.11); namely, that X is an $(\mathbf{F}, \tilde{\mathbf{H}})$-reversible semimartingale. We state this as a theorem in the next section [Theorem (5.16)].

5. Time reversal and enlargement of filtrations. In this section we prove Theorem (1.12). We begin, however, with a theorem that has intrinsic interest. We need an additional hypothesis.

(5.1) HYPOTHESIS. For every t, $0 < t \le 1$, the law of the random variable Z_t has a density ρ_t (with respect to Lebesgue measure). Moreover, $\sup_{|y| \le n,\, \varepsilon \le t \le 1} \rho_t(y) < \infty$ for all $n \in \mathbb{N}$, $\varepsilon > 0$.

(5.2) COMMENT. If Z^c is not identically zero, then (5.1) holds.

(5.3) THEOREM. *Assume Hypothesis* (5.1) *holds. Let k be a Borel function on $\mathbb{R}^2$ such that*

(i) *k is bounded and* $\sup_{|x| \le n, |y| \le 1} k(x, y)/|y| < \infty$ *for all* $n \in \mathbb{N}$;

(ii) *for each y, $|y| \le 1$, the function $k(\cdot, y)$ is either right-continuous or left-continuous, and it admits a Radon measure η_y as its generalized function sense derivative; moreover, there is a positive Radon measure η such that $|\eta_y| \le |y|\eta$, all $|y| \le 1$, where $|\eta_y|$ denotes the total variation measure of η_y.*

Then the $\mathbf{F}$*-martingale*

$$V_t = \int_0^t \int_{\mathbb{R}} k(Z_{s-}, y)(\mu - \nu)(ds \times dy) \tag{5.4}$$

is an $(\mathbf{F}, \tilde{\mathbf{G}})$*-reversible semimartingale.*

PROOF. (a) Let $\tilde{\mu}$ denote the jump measure of the reversed process $\tilde{Z}$. Since $\mathscr{L}(\tilde{Z}) = \mathscr{L}(-Z)$, the $\tilde{\mathbf{F}}$-compensator (where $\tilde{\mathbf{F}}$ is the natural filtration of $\tilde{Z}$) of $\tilde{\mu}$ is clearly

$$\tilde{\nu}(dt \times dy) = dt \otimes \hat{F}(dy), \tag{5.5}$$

where $\hat{F}$ is the symmetric analog of F given in (2.2). By virtue of Hypothesis (5.1) and Jacod (1985), pages 28–29, there is a nonnegative $\tilde{\mathbf{G}}$-predictable function U on $\Omega \times [0,1] \times \mathbb{R}$ such that the $\tilde{\mathbf{G}}$-compensator of $\tilde{\mu}$ is

$$\tilde{\tau}(\omega; dt \times dy) = U(\omega, t, y)\tilde{\nu}(dt \times dy) \tag{5.6}$$

and

$$\int_0^t \int_{|y| \le 1} |U(\omega, s, y) - 1|\,|y|\tilde{\nu}(ds \times dy) < \infty, \tag{5.7}$$

for all $t < 1$, $\omega \in \Omega$.

(b) Next we set for $n \in \mathbb{N}$,

$$V_t^n = \int_0^t \int_{|y| > 1/n} k(Z_{s-}, y)(\mu - \nu)(ds \times dy). \tag{5.8}$$

This is an AF with paths of finite variation [cf. Çinlar, Jacod, Protter and Sharpe (1980)], and thus it is an $(\mathbf{F}, \tilde{\mathbf{G}})$-reversible semimartingale by Lemma (4.1). Also since Z is a Lévy process $\Delta Z_1 = 0$ a.s. and we have

$$\begin{aligned} \tilde{V}_t^n &= \int_0^t \int_{|y| > 1/n} k(Z_{(1-s)-}, y)\nu(ds \times dy) \\ &\quad - \int_0^t \int_{|y| > 1/n} k(Z_{(1-s)-}, y)\mu(ds \times dy) \\ &= \int_0^t \int_{|y| > 1/n} k(Z_{1-s}, -y)\tilde{\nu}(ds \times dy) \\ &\quad - \int_0^t \int_{|y| > 1/n} k(Z_{1-s} + y, -y)\tilde{\mu}(ds \times dy). \end{aligned} \tag{5.9}$$

Since k is bounded, both V^n and $\tilde{V}^n$ have bounded jumps; in particular, $\tilde{V}^n$ is a special semimartingale, and its $\tilde{\mathbf{G}}$-canonical decomposition

$$\tilde{V}^n = \tilde{M}^n + \tilde{A}^n$$

is given by [using (5.5), (5.6) and (5.9)]

$$\begin{aligned}
\tilde{M}_t^n &= -\int_0^t \int_{|y|>1/n} k(Z_{1-s}+y, -y)(\tilde{\mu}-\tilde{\tau})(ds \times dy), \\
\tilde{A}_t^n &= \int_0^t \int_{|y|>1/n} \{k(Z_{1-s}, -y) - k(Z_{1-s}+y, -y)U(s,y)\}\tilde{\nu}(ds \times dy) \\
&= \tilde{B}_t^n + \tilde{C}_t^n,
\end{aligned} \tag{5.10}$$

where

$$\begin{aligned}
\tilde{B}_t^n &= \int_0^t \int_{|y|>1/n} k(Z_{1-s}+y, -y)\{1 - U(s, y)\}\tilde{\nu}(ds \times dy), \\
\tilde{C}_t^n &= \int_0^t ds \int_{|y|>1/n} \{k(Z_{1-s}, y) - k(Z_{1-s}-y, y)\}F(dy).
\end{aligned} \tag{5.11}$$

(c) The next step is to let n increase to ∞. Using hypothesis (i), a classical convergence theorem for stochastic integrals with respect to random measures yields that $V_t^n \to V_t$ in probability, uniformly in t. Therefore we also have $\tilde{V}_t^n \to \tilde{V}_t$ in probability for all t, $0 \le t \le 1$. Analogously, by the same theorem $\tilde{M}_t^n \to \tilde{M}_t$ in probability, where

$$\tilde{M}_t = -\int_0^t \int_{\mathbb{R}} k(Z_{1-s}+y, -y)(\tilde{\mu}-\tilde{\tau})(ds \times dy).$$

Using hypothesis (i) again together with (5.7) and (5.11), we have $\tilde{B}_t^n \to \tilde{B}_t$ in probability, where

$$\tilde{B}_t = \int_0^t \int_{\mathbb{R}} k(Z_{1-s}+y, -y)[1 - U(s, y)]\tilde{\nu}(ds \times dy),$$

which is a process with paths of finite variation. We can thus deduce that $\tilde{C}_t^n$ converges in probability to $\tilde{C}_t = \tilde{V}_t - \tilde{M}_t - \tilde{B}_t$. It remains only to prove that $\tilde{C}_t$ is a continuous process of finite variation, since that will imply that $\tilde{V}$ is a semimartingale.

(d) Actually we will show that

$$D_t \equiv \int_0^t ds \int_{|y|\le 1} |k(Z_{1-s}, y) - k(Z_{1-s}-y, y)|F(dy) < \infty \quad \text{a.s.} \tag{5.12}$$

If (5.12) holds we can use Lebesgue's dominated convergence theorem to conclude

$$\tilde{C}_t = \int_0^t ds \int_{\mathbb{R}} \{k(Z_{1-s}, y) - k(Z_{1-s}-y, y)\}F(dy),$$

and $\tilde{C}$ will have continuous paths of finite variation. To show (5.12), define

$K_n = \{|Z_s| \le n, \text{ all } s \le 1\}$. Then

$$E\left[1_{K_n} D_t\right] \le \int_0^t ds \int_{|y| \le 1} F(dy) E\left[|k(Z_{1-s}, y) - k(Z_{1-s} - y, y)| 1_{\{|Z_{1-s}| \le n\}}\right]$$

$$= \int_0^t ds \int_{|y| \le 1} F(dy) \int_{|u| \le n} \rho_{1-s}(u) |k(u, y) - k(u - y, y)|\, du,$$

where ρ_t is the density defined in Hypothesis (5.1). Next we use hypothesis (ii) of the theorem to obtain for $|y| \le 1$,

$$|k(u, y) - k(u - y, y)| \le \int_{|u-v| \le y} |\eta_y|(dv) \le |y| \int_{|u-v| \le y} \eta(dv)$$

and thus, if $L_t^n = \sup_{|y| \le n,\ s \in [1-t, 1]} \rho_s(y)$, it follows that

$$E\left[1_{K_n} D_t\right] \le \int_0^t ds \int_{|y| \le 1} F(dy) \int_{|v| \le n+1} \eta(dv) |y| \int_{|u-v| \le y} \rho_{1-s}(u)\, du$$

$$\le 2t L_t^{n+2} \eta([n-1, n+1]) \int_{|y| \le 1} F(dy) |y|^2 < \infty.$$

Since $\bigcup_n K_n = \Omega$, we have established (5.12), and thus the theorem as well. □

(5.13) COMMENT. Hypothesis (5.1) is based on "Condition (A)" of Jacod (1985), page 15. However, Condition (A) is shown to be equivalent to Condition (A′), which gives rise to a weaker condition than Hypothesis (5.1). We can therefore replace Hypothesis (5.1) by a weaker statement:

Let η_t denote the law of the random variable Z_t, and let ζ_t denote the "potential" $\zeta_t(A) = \int_t^1 \eta_s(A)\, ds$. Assume that for each t, $0 < t < 1$, ζ_t has a density $\tilde{\rho}_t$ with respect to Lebesgue measure, and that $\sup_{|y| \le n} \tilde{\rho}_t(y) < \infty$, all $n \in \mathbb{N}$.

(5.14) COMMENT. In (ii) of the previous theorem, the assumption that $k(\cdot, y)$ is either right-continuous or left-continuous is clearly too strong a requirement. Indeed, the property that it admits a Radon measure for its derivative implies that at each point x it has a right and left limit, say $k_+(x, y)$ and $k_-(x, y)$; thus it would be enough to assume only that $k(x, y)$ lies in the interval having $k_+(x, y)$ and $k_-(x, y)$ as its endpoints.

(5.15) COMMENT. As in Comment (4.7), let $\tilde{\mathbf{H}}$ be the filtration associated to the Lévy process $\tilde{Z}$ by (2.4). Then V is an $(\mathbf{F}, \tilde{\mathbf{H}})$-reversible semimartingale.

In fact, we could obtain this result directly by using the method of Section 2 instead of the results of Jacod (1985). More precisely, let $\tilde{\rho}$ be the $\tilde{\mathbf{H}}$-compensator of $\tilde{\mu}$ on $[0,1)$. Then we define $\tilde{M}^n$ and $\tilde{M}$ as before, using $\tilde{\rho}$ instead of $\tilde{\tau}$, so that $\tilde{M}^n$ and $\tilde{M}$ are $\tilde{\mathbf{H}}$-local martingales on $[0,1)$. We still have

$$\tilde{V}^n = \tilde{M}^n + \tilde{A}^n$$

and

$$\tilde{A}^n = \tilde{B}^n + \tilde{C}^n,$$

with $\tilde{C}^n$ unchanged but with $\tilde{B}^n$ given by [instead of (5.11)]

$$\tilde{B}_t^n = \int_0^t \int_{|y|>1/n} k(Z_{1-s} + y, -y)(\tilde{\nu} - \tilde{\rho})(ds \times dy).$$

Thus $\tilde{B}_t^n = \int_0^t \tilde{U}_s^n \, ds$, with

$$\tilde{U}_s^n = \int_{|y|>1/n} \frac{1}{1-s} \tilde{\mu}((s,1] \times dy) k(Z_{1-s} + y, -y)$$
$$- \int_{|y|>1/n} F(dy) k(Z_{1-s} + y, -y)$$

and, exactly as in the proof of (2.5), we deduce from assumption (i) that $\tilde{B}_t^n$ converges to $\tilde{B}_t = \int_0^t \tilde{U}_s \, ds$ in L^2 for a suitable process $\tilde{U}_s$.

The rest of the proof remains unchanged. Observe, however, that, although we do not use the results of Jacod (1985) with this method, we are unable to remove Hypothesis (5.1), which seems necessary to obtain that $\tilde{C}^n$ converges to a process with paths of finite variation.

Proof of Theorem (1.12). Exactly as in the proof of Theorem (1.11), it is enough to show that the process $\hat{Y}_t = \int_0^t \hat{f}(Z_{s-}) \, dZ_s^c$ is an $(\mathbf{F}, \tilde{\mathbf{G}})$-reversible semimartingale, where $\hat{f}$ is the function associated to f in (1.10). In other words, we can and do assume that f is a right-continuous function of finite variation on compacts. We let

$$X_t = \int_0^t f(Z_{s-}) \, dZ_s, \qquad Y_t = \int_0^t f(Z_{s-}) \, dZ_s^c,$$

as in (1.4).

By Theorem (1.11) we know that X is an $(\mathbf{F}, \tilde{\mathbf{G}})$-reversible semimartingale. Also since Z^c is not identically zero, by Comment (5.2) we have that Hypothesis (5.1) holds.

Consider next the decomposition (4.5) of Z, $Z_t = M_t^a + b_a t + J_t^a$ with $a = 1$. The martingale M_t^1 can be written as

$$M_t^1 = Z_t^c + \int_0^t \int_{|y|\le 1} y(\mu - \nu)(ds \times dy).$$

Hence if $A_t = b_1 t + J_t^1$ [the last two terms on the right-hand side of (4.5)], we have

$$X_t = Y_t + \int_0^t \int_{|y|\le 1} f(Z_{s-}) y(\mu - \nu)(dy \times ds) + \int_0^t f(Z_{s-}) \, dA_s. \tag{5.16}$$

Then $C_t = \int_0^t f(Z_{s-}) \, dA_s$ is an AF of Ξ with paths of finite variation, and hence it is an $(\mathbf{F}, \tilde{\mathbf{G}})$-reversible semimartingale by Lemma (4.1). It remains only to show that the middle term on the right-hand side of (5.16) is $(\mathbf{F}, \tilde{\mathbf{G}})$-reversible.

To this end, we use Theorem (5.3) with $k(x, y) = f(x)y$. Note that such a k clearly satisfies the hypotheses (5.3)(i), (ii), and the proof is complete. □

Actually, due to Lemma (4.1) and Theorem (2.5), we can obtain more than Theorems (1.11) and (1.12). Let $\tilde{\mathbf{H}}$ be the filtration associated to the Lévy process $\tilde{Z}$ by (2.4).

(5.17) THEOREM. *Let Z be a Lévy process and let f, g satisfy* (1.10). *Let A be an additive functional of Z. If*

$$U_t = \int_0^t f(Z_{s-})\,dZ_s + \int_0^t g(Z_{s-})\,dZ_s^c + A_t,$$

then U is an $(\mathbf{F}, \tilde{\mathbf{H}})$-*reversible semimartingale.*

PROOF. By Lemma (4.1), we know that $\tilde{A}$ is $\tilde{\mathbf{G}}$-adapted, and it has paths of finite variation. Since $\tilde{\mathscr{H}}_t \supseteq \tilde{\mathscr{G}}_t$ [as shown in the remark following (2.4)], we have that $\tilde{A}$ is an $\tilde{\mathbf{H}}$-semimartingale. Letting $X_t = \int_0^t f(Z_{s-})\,dZ_s$ and $Y_t = \int_0^t g(Z_{s-})\,dZ_s^c$ as in (1.4), we have by Theorems (1.11) and (1.12) that $\tilde{X}$ and $\tilde{Y}$ are $\tilde{\mathbf{G}}$-semimartingales. But then it follows from Theorem (2.5) that $\tilde{X}$ and $\tilde{Y}$ are each $\tilde{\mathbf{H}}$-semimartingales. Finally, it suffices to note that $\tilde{U} = \tilde{X} + \tilde{Y} + \tilde{A}$ to complete the proof. □

6. The Brownian case and applications. In the Brownian case the situation is particularly simple, since any additive functional A of a standard Brownian motion B has a representation

$$A_t = \int_{\mathbb{R}} L_t^x \mu(dx), \tag{6.1}$$

for some signed measure μ, where L_t^x is a (jointly continuous) version of the local times of B at levels x. The relation (6.1) allows us to use only martingale stochastic integration theory, and, in particular, we can avoid Lemma (4.1). In the Brownian case Theorem (1.8) was first treated in the context of expansion of filtrations by Itô (1978) on $[0,1]$. Theorems (1.11) and (1.12) become in this case

(6.2) THEOREM. *Let f satisfy* (1.10). *Suppose*

$$V_t = \int_0^t f(B_s)\,dB_s + \int_{\mathbb{R}} L_t^x \mu(dx),$$

where μ is a signed measure on $\mathbb{R}$. Then V is an $(\mathbf{F}, \tilde{\mathbf{G}})$-*reversible semimartingale.*

PROOF. Although the proof is a corollary of Theorem (1.11) and Lemma (4.1) (with $Z = B$), we give an autonomous proof.

Let $\hat{f}$ and D be associated with f as in (1.10). It is well known that B spends a.s. zero time in the at most countable set D. Therefore

$$E\left\{\left(\int_0^t 1_D(B_s)\,dB_s\right)^2\right\} = E\left\{\int_0^t 1_D(B_s)\,ds\right\} = 0,$$

and so $\int_0^t f(B_s)\,dB_s = \int_0^t \hat{f}(B_s)\,dB_s$ a.s. Hence it is no restriction to assume that f itself is right-continuous and of finite variation on compacts.

Note that $f = F'_+$, the right derivative of a function F, which is the difference of two convex functions. Letting η be the (generalized function sense) derivative of f, the Meyer–Tanaka–Itô formula yields

$$F(B_t) - F(B_0) = \int_0^t f(B_s)\,dB_s + \frac{1}{2}\int_{\mathbb{R}} L_t^a \eta(da). \tag{6.3}$$

Letting $U_t = F(B_t) - F(B_0)$, an $\mathbf{F}$-semimartingale, we have

$$\tilde{U}_t = F(B_{(1-t)}) - F(B_1) = F(\tilde{B}_t - \tilde{B}_1) - F(-\tilde{B}_1),$$

and since $\tilde{B}_t - \tilde{B}_1$ is a $\tilde{\mathbf{G}}$-semimartingale by (1.8), we have that $\tilde{U}$ is one as well.

It remains to show that $A_t = \int_{\mathbb{R}} L_t^x \mu(dx)$ is $(\mathbf{F}, \tilde{\mathbf{G}})$-reversible. Since it has continuous paths of finite variation, however, it suffices to show that $\tilde{A}_t$ is $\mathscr{G}_t$-measurable. We do this using local time theory instead of using Lemma (4.1).

First, note that $\tilde{B}_t = B_{1-t} - B_1$ is an $\tilde{\mathbf{F}}$-Brownian motion. Let l_t^x be its (jointly continuous) local time. Then well-known results about Brownian local time [see, e.g., Yor (1978), page 32] state

$$L_t^x = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\int_0^t 1_{[x, x+\varepsilon]}(B_s)\,ds \quad \text{a.s.},$$

where the exceptional set can be taken independent of x and t. But then

$$\tilde{A}_t = A_{1-t} - A_1 = \int (L_{1-t}^x - L_1^x)\mu(dx); \tag{6.4}$$

and

$$\begin{aligned} L_{1-t}^x - L_1^x &= \lim_{\varepsilon\to 0} -\frac{1}{\varepsilon}\int_{1-t}^1 1_{[x, x+\varepsilon]}(B_s)\,ds \\ &= \lim_{\varepsilon\to 0} -\frac{1}{\varepsilon}\int_{1-t}^1 1_{[x-B_1, x-B_1+\varepsilon]}(B_s - B_1)\,ds \\ &= \lim_{\varepsilon\to 0} -\frac{1}{\varepsilon}\int_0^t 1_{[x-B_1, x-B_1+\varepsilon]}(\tilde{B}_u)\,du \\ &= -l_t^{x-B_1}. \end{aligned}$$

Combining this with (6.4) yields

$$\tilde{A}_t = -\int_{\mathbb{R}} l_t^{x-B_1}\mu(dx).$$

Since l_t^x are the local times of $\tilde{B}$, they are $\tilde{\mathbf{F}}$-adapted, and thus $\tilde{A}$ is $\tilde{\mathbf{G}}$-adapted. □

Note that since one can take $\mu(dx) = g(x)\varepsilon_{\{0\}}(dx) + h(x)\,dx$, where $\varepsilon_{\{0\}}$ is point mass at 0, Corollary (1.12) is a special case of Theorem (6.2).

An interesting application of these results is to stochastic differential equations. Here our general result, Theorem (3.3), is particularly useful. Let B be a

Brownian motion and X the solution of

$$X_t = X_0 + \int_0^t \sigma(s, X_s)\, dB_s + \int_0^t b(s, X_s)\, ds. \tag{6.5}$$

The filtration $\mathbf{F}$ is that of B, and we define

(6.6) $\tilde{\mathbf{J}} = (\tilde{\mathscr{I}}_t)_{t \in [0,1]}$ denotes the smallest complete (right-continuous) filtration relative to which $\tilde{B}$ is adapted and X_1 is $\tilde{\mathscr{I}}_0$-measurable.

It is a well-known result in the theory of flows [see Kunita (1984), page 227] that if σ and b in (6.5) are of class $\mathscr{C}^1$ with derivatives that are globally Hölder continuous (of any positive index), then the flow $x \to \varphi(s, t; x)$ of (6.5) is a $\mathscr{C}^1$-diffeomorphism. [Here $\varphi(s, t; x)$ represents the value of X_t when $X_s = x$ and $s \le t$.] Moreover, $\varphi(s, t; x)$ is measurable with respect to $\sigma(B_r - B_s;\ s \le r \le t)$. If, furthermore, X_t has a density with respect to Lebesgue measure for all $t \in [0, 1]$, we deduce that the conditional law of $X_1 = \varphi(t, 1; X_t)$ with respect to $\sigma(B_r - B_t\colon t \le r \le 1) = \sigma(\tilde{B}_u;\ 0 \le u \le 1 - t)$ also has a density. In this case the results of Jacod (1985) imply that $\tilde{B}$ is a $\tilde{\mathbf{J}}$-semimartingale on $[0, 1)$, and therefore by Theorem (3.3) we have that $\tilde{X}$ is a $\tilde{\mathbf{J}}$-semimartingale on $[0, 1)$. Haussmann and Pardoux (1986) have studied this type of question for systems and they obtained sufficient conditions for X_t, $t \in (0, 1]$ to have a density. [See also Pardoux (1986).]

By combining a Girsanov technique [as in Protter (1987)] with the former, one can consider a more general stochastic differential equation of the form

$$Y_t = Y_0 + \int_0^t h_s\, ds + \int_0^t \sigma(s, Y_s)\, dB_s, \tag{6.7}$$

where h is $\mathbf{F}$-adapted and jointly measurable. If, for example, h is bounded and σ is bounded away from 0, then the process

$$W_t = B_t - \int_0^t \frac{1}{\sigma(s, Y_s)} h_s\, ds \tag{6.8}$$

is an $\mathbf{F}$-Brownian motion for a probability Q equivalent to P, and the process Y of (6.7) is a solution of

$$Y_t = Y_0 + \int_0^t \sigma(s, Y_s)\, dW_s;$$

the preceding discussion shows that $\tilde{Y}$ is then a reversible semimartingale under Q, if σ is at least $\mathscr{C}^1$ with Hölder continuous derivatives and also if Y_t has a density for all $t \in (0, 1]$. Since Q is equivalent to P, $\tilde{Y}$ is also a P-semimartingale. Picard (1986) has used basically this approach for the case of systems, which, of course, is technically more complicated.

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