The Annals of Probability
1988, Vol. 16, No. 2, 600–619

TWO-PARAMETER HUNT PROCESSES AND A POTENTIAL THEORY

BY G. MAZZIOTTO

Centre National d'Etudes des Télécommunications

A two-parameter Markov process X with regular trajectories is associated to a pair of commuting Feller semigroups P^1 and P^2 considered on the same space E. A subsequent potential theory is developed with respect to an operator $\mathscr{L}$ which is the product of the generators of P^1 and P^2, respectively. The definition of a harmonic function f on an open subset A is expressed in terms of the hitting stopping line of A^c by X and the stochastic measure generated by $f(X)$. A PDE problem in A with boundary conditions on A^c is studied.

1. Introduction and preliminaries.

1.1. *Introduction.* Given a pair of semigroups P^1 and P^2 defined on two distinct spaces E^1 and E^2, respectively, the notions of bi-excessive or bi-harmonic functions on the Cartesian product space $E = E^1 \times E^2$ are well known: see Cairoli (1966) and Walsh (1968). Moreover, these definitions may be written out in terms of a two-parameter Markov process X, which is simply the tensor product of arbitrary realizations X^1 and X^2 of the semigroups P^1 and P^2, respectively, as in Walsh (1981), Dynkin (1981) and also in Mazziotto (1985). The generators of P^1 and P^2, operating on functions defined on E^1 and E^2, respectively, extend trivially to commuting operators on functions of E, say $\mathscr{L}^1$ and $\mathscr{L}^2$. The Dirichlet problem associated to their product, $\mathscr{L} = \mathscr{L}^1 \times \mathscr{L}^2$, is studied in several recent publications, such as Dynkin (1981), Vanderbei (1983), Doppel and Jacob (1983) or Jacob (1985), using various approaches. It concerns the existence and the uniqueness of a function φ on E which is harmonic with respect to $\mathscr{L}$ in a fixed open subset $A \subset E$, i.e., $\mathscr{L}\varphi = 0$ on A, and which is equal to a given function f on the boundary ∂A. As pointed out in Dynkin (1981) or Vanderbei (1983), the solutions of this problem are not unique in general; the uniqueness requires, for instance, additional conditions on higher-order derivatives of φ on ∂A, or regularity assumptions. Moreover, the domain A must have a rather simple geometric shape.

Consider now a slightly different situation. Let Δ be the Laplacian operator defined on functions on a Euclidean space E. Given a bounded subset A, the biharmonic problem consists in solving the equation

$$\Delta^2\varphi = \Delta(\Delta\varphi) = 0, \quad \text{on } A,$$

subject to boundary conditions, such as φ and $\Delta\varphi$ are equal to fixed functions on ∂A. This problem was treated by Has'minskii (1960) and Helms (1967), and the

Received April 1986; revised December 1986.

AMS 1980 *subject classifications*. Primary 60J45; secondary 31C10.

Key words and phrases. Two-parameter Markov process, biharmonic functions, commuting semigroups, probabilistic potential theory.

solution was expressed in terms of a (one-parameter) Brownian motion. A discrete-time version of this problem is studied in Vanderbei (1984) by means of a two-parameter process. It is clear that the operator Δ^2 is the product of two commuting operators, defined on the same space E.

The preceding examples suggest studying the following general problem. Given an open subset A of a l.c.d. (locally compact with a denumerable basis) space E and a sufficiently smooth function f on E, does there exist a function φ on E such that $(\mathscr{L}^1 - p_1 I)\cdot(\mathscr{L}^2 - p_2 I)\varphi = 0$ on A, and $\varphi = f$ on A^c (the complementary set of A in E)? The operators $\mathscr{L}^1$ and $\mathscr{L}^2$ are assumed to be the generators of two Feller semigroups on E, say P^1 and P^2, respectively, which commute with each other. The real numbers p_1 and p_2 are taken both strictly greater than zero, and I denotes the identity map on E. The case where p_1 and/or p_2 are equal to zero could be treated similarly, under extra technical assumptions which ensure the boundedness of various expressions.

The aim of this paper is to develop a potential theory, expressed in terms of a two-parameter Markov process, which allows a stochastic representation of solutions of the above problem. The organization is as follows.

After the preliminary definitions of Paragraph 1.2, we show in Section 2 that the two one-parameter semigroups P^1 and P^2 associated to $\mathscr{L}^1$ and $\mathscr{L}^2$, respectively, determine a two-parameter semigroup P. We prove by extending the construction made in Blumenthal and Getoor (1968) that, if P enjoys conditions analogous to the Feller conditions of the classical theory, then there exists a realization X of P which is a two-parameter Markov process having regular trajectories; for these reasons, it will be called a Hunt process. This result extends those of Mazziotto (1985) and, partially, those of Michel (1979). In Section 3, we state a notion of harmonicity with respect to the operator $(\mathscr{L}^1 - p_1 I)\cdot(\mathscr{L}^2 - p_2 I)$. The approach follows Dynkin (1965) or Meyer (1967) in the classical theory. To this end, we introduce various elements of a potential theory which recall and generalize notions encountered in Cairoli (1966, 1968), Walsh (1968, 1981), Dynkin (1981), Vanderbei (1983) and Mazziotto (1985). Finally, we exhibit a function which is harmonic in a given open subset A, and equal to a smooth function f on the complementary set A^c. This function is written out in terms of a stochastic measure generated by the function f and the Hunt process X, applied to the stochastic interval determined by the stopping line, where X begins to hit A^c and the point at infinity. Unfortunately, we can only answer the question of uniqueness for solutions when we know that they lie in a set of smooth functions.

1.2. *Preliminaries.* For the main notions and basic definitions of the theory of two-parameter processes used in this paper, we refer to the works of Cairoli and Walsh (1975) and Wong and Zakai (1974). The two-parameter processes are indexed by $\mathbb{R}^2_+$ or by its one-point compactification $\overline{\mathbb{R}^2_+} = \mathbb{R}^2_+ \cup \{\infty\}$. This index set is endowed with the partial order relation,

$$s = (s_1, s_2) \le t = (t_1, t_2), \quad \text{if and only if } s_1 \le t_1 \text{ and } s_2 \le t_2,$$

$$\text{with } 0 = (0,0) \le t \le \infty = (\infty, \infty), \forall\, t.$$

Given $t \in \mathbb{R}_+^2$, set $[0, t] = \{s\colon 0 \le s \le t\}$ and $[t, \infty] = \{s\colon s \ge t\}$; these are called closed intervals. The strict sense partial order,

$$s = (s_1, s_2) \ll t = (t_1, t_2), \quad \text{if and only if } s_1 < t_1 \text{ and } s_2 < t_2,$$

enables us to define open intervals in a similar manner. More generally, for any set $H \subset \overline{\mathbb{R}}_+^2$, we denote by $[H, \infty]$ the closure of the union over $t \in H$ of all the intervals $[t, \infty]$, and by début(H) or $D(H)$ the lower boundary of $[H, \infty]$.

Given s, $t \in \mathbb{R}_+^2$, we set $s \cdot t = s_1 t_1 + s_2 t_2$ and $|t| = t_1 t_2$. Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a complete probability space; a two-parameter filtration is a family $\mathscr{F} = (\mathscr{F}_t;\ t \in \overline{\mathbb{R}}_+^2)$ of sub-σ-fields of $\mathscr{A}$ which satisfies the following condition:

(F1) $\mathscr{F}$ is increasing, i.e., $\mathscr{F}_s \subset \mathscr{F}_t$ for $s \le t$, and $\mathscr{F}_\infty = \mathscr{A}$.

Moreover, $\mathscr{F}$ is said to be $\mathbb{P}$-complete if, in addition to (F1), the following conditions are fulfilled:

(F2) $\mathscr{F}_0$ contains all the $\mathbb{P}$-negligible sets of $\mathscr{A}$.
(F3) $\mathscr{F}$ is right-continuous, i.e., $\forall\ t\colon \mathscr{F}_t = \bigcap_{s \gg t} \mathscr{F}_s$.

Given a filtration $\mathscr{F}$ satisfying (F1), (F2) and (F3), we define two one-parameter filtrations $\mathscr{F}^1 = (\mathscr{F}_u^1;\ u \in \mathbb{R}_+)$ and $\mathscr{F}^2 = (\mathscr{F}_u^2;\ u \in \mathbb{R}_+)$ by setting

$$\forall\ u \in \mathbb{R}_+\colon \mathscr{F}_u^1 = \bigvee_{v \in \mathbb{R}_+} \mathscr{F}_{(u, v)} \quad \text{and} \quad \mathscr{F}_u^2 = \bigvee_{v \in \mathbb{R}_+} \mathscr{F}_{(v, u)}.$$

These one-parameter filtrations verify the usual conditions of the classical theory [see Dellacherie and Meyer (1975)]. The filtration $\mathscr{F}$ hereafter considered also enjoys the classical conditional independence property called (F4) in Cairoli and Walsh (1975) or Wong and Zakai (1974):

(F4) For every $t \in \mathbb{R}_+^2$, the σ-fields $\mathscr{F}_{t_1}^1$ and $\mathscr{F}_{t_2}^2$ are conditionally independent given $\mathscr{F}_t$.

Every process $X = (X_t;\ t \in \overline{\mathbb{R}}_+^2)$ is implicitly assumed to be measurable; we say that X is adapted to the filtration $\mathscr{F}$ (or $\mathscr{F}$-adapted) if, for any $t \in \mathbb{R}_+^2$, the random variable X_t is $\mathscr{F}_t$-measurable. A process X is said to be a modification of another process X' if and only if, $\forall\ t \in \mathbb{R}_+^2\colon X_t = X_t'$, $\mathbb{P}$-a.s. We do not distinguish two processes X and X' such that $\mathbb{P}(\exists\ t\colon X_t \ne X_t') = 0$.

On the product space $\Omega \times \overline{\mathbb{R}}_+^2$, the σ-fields of, respectively, the predictable sets, the optional sets and the progressively measurable sets are defined in Merzbach (1980), Meyer (1981) and Bakry (1981), in the same way as in the classical theory. Given a random set $H \subset \Omega \times \overline{\mathbb{R}}_+^2$, we extend similarly the notions of stochastic intervals as $[H, \infty]$, and of début, as $D(H)$. Recall [see Merzbach (1980)] that the début of a progressively measurable (resp. predictable) set is what is called a stopping line (resp. predictable stopping line). A stepped stopping line is a stopping line which has only a countable number of distinct configurations in $\overline{\mathbb{R}}_+^2$ when ω runs over Ω, each of them being composed of a finite number of segments parallel to the coordinate axes, with all the summits in the set $\mathbb{D}$ of the dyadic points in $\mathbb{R}_+^2$ [see Merzbach (1979)]. If L and L' are arbitrary stopping lines, we say that $L \leq L'$ if and only if $[L', \infty] \subset [L, \infty]$; if we identify each $t \in \mathbb{R}^2_+$ with the stopping line $D(t)$, then this relation extends the partial ordering of $\overline{\mathbb{R}^2_+}$. It is proved in Bakry (1981) that a predictable stopping line can be announced by an increasing sequence of stepped stopping lines.

Let $X = (X_t;\ t \in \overline{\mathbb{R}^2_+})$ be a two-parameter process assumed to be integrable and right-continuous in $\mathbb{L}^1(\Omega, \mathscr{A}, \mathbb{P})$. Let $\mathscr{J}$ be the finitely additive algebra generated by the elementary predictable sets of the type $F \times]s, t]$, for $F \in \mathscr{F}_s$ and $s \leq t$; the σ-algebra generated by $\mathscr{J}$ is the predictable σ-field $\mathscr{P}$. Then we define a finitely additive function, called Doleans function, μ^X on $\mathscr{J}$ by setting

$$\mu^X(F \times]s, t]) = E(\mathbb{1}_F \cdot X_{]s,t]}),$$

where $\mathbb{1}_F$ denotes the indicator function of F, and where

$$X_{]s,t]} = X_s + X_t - X_{(s_1, t_2)} - X_{(s_2, t_1)}.$$

The process X will be said to be admissible [see Merzbach (1979), in other works, the terminology is admissible quasimartingale, or sometimes semimartingale] if and only if the function μ^X extends to some σ-additive signed measure on $\mathscr{P}$, again denoted by μ^X. Various conditions ensuring that a process is admissible can be found in Merzbach (1979), Brennan (1979) and Meyer (1981).

A two-parameter process X is said to be a supermartingale if and only if

$$\forall\, s, t \in \mathbb{R}^2_+,\ s \leq t\colon E(X_t | \mathscr{F}_s) \leq X_s \quad \text{a.s.}$$

Moreover, we will say that X is of positive variation if

$$\forall\, s, t \in \mathbb{R}^2_+,\ s \leq t\colon E(X_{]s,t]} | \mathscr{F}_s) \geq 0 \quad \text{a.s.}$$

Let E be a locally compact space countable at infinity (l.c.d. space), and let δ denote the point at infinity of the Alexandrov compactification $E_\delta = E \cup \{\delta\}$. In the sequel, any function defined on E is implicitly extended to E_δ by giving the value zero at infinity. The space of all the Borel bounded (continuous bounded) function on E (or E_δ) is denoted by $b(E)$ [resp. $C_0(E)$].

Given an arbitrary point $t \in \mathbb{R}^2_+$, consider the four following quadrants: $Q^1_t = \{s\colon s \gg t\}$, $Q^2_t = \{s\colon s_1 > t_1,\ s_2 < t_2\}$, $Q^3_t = \{s\colon s \ll t\}$, $Q^4_t = \{s\colon s_1 < t_1,\ s_2 > t_2\}$. A function f from $\mathbb{R}^2_+$ into a topological space U is said to admit quadrantal limits [see Bakry (1980)] if and only if

$$\forall\, t \in \mathbb{R}^2_+, \forall\, i = 1, \ldots, 4\colon \liminf_{s \in Q^i_t;\, s \to t} f(s) = \limsup_{s \in Q^i_t;\, s \to t} f(s);$$

f is said to be cad-lq if it admits quadrantal limits and is right-continuous, that is to say,

$$\forall\, t \in \mathbb{R}^2_+\colon f(t) = \lim_{s \in Q^1_t;\, s \to t} f(s).$$

2. Two-parameter Hunt processes. A two-parameter sub-Markov (resp. Markov) semigroup $P = (P_t;\ t \in \mathbb{R}^2_+)$ on E (resp. E_δ) is a family, indexed by $\mathbb{R}^2_+$, of kernels on E (resp. E_δ) such that

$$\forall\, s, t \in \mathbb{R}^2_+\colon P_{s+t} = P_s \cdot P_t = P_t \cdot P_s \quad \text{and} \quad P_0 = \text{Identity}.$$

As usual, any sub-Markov semigroup N extends to some Markov semigroup P, as follows:

$$\forall\, t \in \mathbb{R}^2_+ : P_t(x; A) = N_t(x; A \cap E) + \mathbb{1}_A(\delta)(1 - N_t(x; E)), \quad \text{if } x \in E,$$
$$= \mathbb{1}_A(\delta), \quad \text{if } x = \delta,$$

for any $x \in E_\delta$ and any $A \in \mathscr{E}_\delta$.

In the sequel we do not distinguish between the sub-Markov semigroup N on E and its Markov extension P on E_δ.

Given a two-parameter semigroup $P = (P_t;\ t \in \mathbb{R}^2_+)$, we obtain two one-parameter semigroups $P^1 = (P^1_u;\ u \in \mathbb{R}_+)$ and $P^2 = (P^2_v;\ v \in \mathbb{R}_+)$ by setting

$$\forall\, u, v \in \mathbb{R}_+ : P^1_u = P_{(u,0)} \quad \text{and} \quad P^2_v = P_{(0,v)}.$$

The semigroup property of P implies that P^1 and P^2 commute each other, i.e.,

$$\forall\, u, v \in \mathbb{R}_+ : P^1_u \cdot P^2_v = P^2_v \cdot P^1_u = P_{(u,v)}.$$

Conversely, the above formula allows us to define a two-parameter semigroup P from two commuting one-parameter semigroups P^1 and P^2.

Let $\mathscr{L}^1$ (resp. $\mathscr{L}^2$) be the generator of the semigroup P^1 (resp. P^2); we denote by $\mathscr{L}^1_r$ (resp. $\mathscr{L}^2_r$) the operator defined for $r \in \mathbb{R}_+$ as to be $\mathscr{L}^1 - rI$ (resp. $\mathscr{L}^2 - rI$), where I is the identity operator. It can be easily checked that, for any $p = (p_1, p_2) \in \mathbb{R}^2_+$, $\mathscr{L}^1_{p_1}$ and $\mathscr{L}^2_{p_2}$ commute with each other. Then we define the operator $\mathscr{L}_p$ by setting

$$\mathscr{L}_p = \mathscr{L}^1_{p_1} \cdot \mathscr{L}^2_{p_2} = \mathscr{L}^2_{p_2} \cdot \mathscr{L}^1_{p_1}.$$

The domain of the operator $\mathscr{L}_p$ is the set $\mathscr{D}(\mathscr{L}_p)$ of all the functions f on E_δ such that $\mathscr{L}^1_{p_1}(\mathscr{L}^2_{p_2} f)$ and $\mathscr{L}^2_{p_2}(\mathscr{L}^1_{p_1} f)$ are well-defined functions which are identical and continuous on E_δ.

The two-parameter resolvent family associated with semigroup $P = (P_t;\ t \in \mathbb{R}^2_+)$ is defined by setting

$$\forall\, p \in \mathbb{R}^2_+,\ p \gg 0, \forall\, f \in b(E) : U_p f = \int_{\mathbb{R}^2_+} e^{-p \cdot t} P_t f\, dt.$$

Under additional integrability assumptions for P, this definition could also be extended to some p belonging to the coordinate axes, as in the classical theory for $p = 0$. We do not study such problems in this paper.

We shall say that the semigroup P is realized by a two-parameter Markov process if there exists a measurable space $(\Omega, \mathscr{A})$ endowed with a family of probability laws $(\mathbb{P}_x;\ x \in E_\delta)$ and a two-parameter process $X = (X_t;\ t \in \mathbb{R}^2_+)$ on $(\Omega, \mathscr{A})$ with values in E_δ such that

$$\forall\, x \in E_\delta, \forall\, f \in b(E), \forall\, t \in \mathbb{R}^2_+ : P_t f(x) = E_x(f(X_t)),$$

where $E_x(\cdot)$ denotes the expectation with respect to $\mathbb{P}_x$.

Given a semigroup P, it is easy to obtain a coarse realization $(\Omega, A, (\mathbb{P}_x;\ x \in E_\delta), X)$. As in the classical theory, the problem consists in finding a realization where the space Ω is not too large and the process X enjoys nice regularity properties. The aim of this section is to generalize the construction made by Blumenthal and Getoor (1968) for a Hunt or a Standard process. The extra assumptions needed for the two-parameter semigroup P are similar to the Feller conditions of the one-parameter situation. More precisely, we shall consider the following hypotheses:

(C1) For any $t \in \mathbb{R}^2_+$, the kernel P_t maps the set $C_0(E)$ into itself.

(C2) For any $f \in C_0(E)$, the functions $(P_t f;\ t \in \mathbb{R}^2_+)$ converge uniformly towards f when t tends to zero.

REMARK 2.1. As in the classical theory, the above conditions can be expressed in terms of the resolvent. It can be proved exactly as in Blumenthal and Getoor (1968), (I-9-4), that (C1) and (C2) imply the following conditions:

(C′1) For any $p \gg 0$, U_p maps $C_0(E)$ into $C_0(E)$.

(C′2) For any $f \in C_0(E)$, $|p|U_p f$ converges uniformly towards f when p_1 and p_2 both increase infinitely.

To begin with, let us construct a coarse realization of the semigroup P. We set $\Omega^0 = (E_\delta)^{\mathbb{R}^2_+}$ and $\mathscr{A}^0 = (\mathscr{E}_\delta)^{\mathbb{R}^2_+}$, the process defined by the coordinates is denoted by $X = (X_t;\ t \in \mathbb{R}^2_+)$, and $\theta(\theta_t;\ t \in \mathbb{R}^2_+)$ is the usual translation semigroup over Ω. Let $\mathscr{F}^0 = (\mathscr{F}^0_t;\ t \in \mathbb{R}^2_+)$ be the natural filtration of the process X. Given $x \in E_\delta$, the probability $\mathbb{P}_x$ is obtained by induction on the finite distributions. For $n \in \mathbb{N}$ and for a pair of subdivisions of $\mathbb{R}_+$: $0 = u^1 < u^2 < \cdots < u^n$ and $0 = v^1 < v^2 < \cdots < v^n$, the probability $\mathbb{P}_x^{u^1,\dots,v^n}$ is defined on the measurable space $(E_\delta^{n^2}, \mathscr{E}_\delta^{\otimes n^2})$ by setting

$$\mathbb{P}_x^{u^1,\dots,v^n}(dx_{1,1},\dots,dx_{n,n}) = \prod_{j=1}^{n-1}\prod_{k=1}^{n} P^1_{u^{j+1}-u^j}(dx_{j+1,k};\ x_{j,k}) \times \prod_{i=1}^{n-1} P^2_{v^{i+1}-v^i}(dx_{1,i+1};\ x_{1,i})\,\varepsilon_x(dx_{1,1}),$$

where ε_x is the Dirac measure on x. It can be easily checked that, since the semigroups P^1 and P^2 commute each other, we could have obtained the same probability law by acting first with P^2 and then with P^1, or by choosing any intermediate strategy. By the way, we get a projective family of probability laws indexed over the double definite subdivisions of $\mathbb{R}_+$, which generates, as an application of the Ionescu–Tulcea theorem [see Neveu (1964)], a Borel kernel $\mathbb{P} = (\mathbb{P}_x;\ x \in E_\delta)$ of probability laws on $(\Omega, \mathscr{A})$. By the definition, for every $x \in E_\delta$: $\mathbb{P}_x(X_0 = x) = 1$. The process X is said to be normal.

It is clear that $(\Omega, \mathscr{A}, (\mathbb{P}_x;\ x \in E_\delta))$ or $(\Omega, \mathscr{A}, \mathbb{P})$ is a realization of the two-parameter semigroup P. The process X of this realization verifies several conditions which rely on already known Markov properties in the plane $\mathbb{R}^2_+$.

Let $x \in E_\delta$ be fixed. For each $t \in \mathbb{R}^2_+$, $\mathscr{F}^x_t$ denotes the σ-field generated by $\bigcap_{h \gg 0} \mathscr{F}^0_{t+h}$ and all the $\mathbb{P}_x$-negligible sets of $\mathscr{A}$. We also consider the σ-field $\mathscr{F}^{1,x}_t$ or $\mathscr{F}^{1,x}_{t_1}$ (resp. $\mathscr{F}^{2,x}_t$ or $\mathscr{F}^{2,x}_{t_2}$) generated by $\bigcup_{u \in \mathbb{R}_+} \mathscr{F}^x_{(t_1,u)}$ [resp. $\bigcup_{u \in \mathbb{R}_+} \mathscr{F}^x_{(u,t_2)}$]. The filtrations $\mathscr{F}^x = (\mathscr{F}^x_t;\ t \in \mathbb{R}^2_+)$, $\mathscr{F}^{1,x} = (\mathscr{F}^{1,x}_t;\ t \in \mathbb{R}^2_+)$ and $\mathscr{F}^{2,x} = (\mathscr{F}^{2,x}_t;\ t \in \mathbb{R}^2_+)$ satisfy the axioms (F1), (F2) and (F3) of the two-parameter process theory as developed by Cairoli and Walsh (1975) and Wong and Zakai (1974).

Finally, for each $t \in \mathbb{R}_+^2$, we put $\mathscr{F}_t = \bigcap_{x \in E} \mathscr{F}_t^x$. For each $x \in E$, $\mathscr{F}^x$ is called the $\mathbb{P}_x$-augmentation of the filtration $\mathscr{F}^0$, and similarly, $\mathscr{F} = (\mathscr{F}_t;\ t \in \mathbb{R}_+^2)$ is called the $(\mathbb{P}_x;\ x \in E)$-augmentation, or the $\mathbb{P}$-augmentation of $\mathscr{F}^0$.

In fact, it immediately follows from the above construction that, for any $n \in \mathbb{N}$, for any subdivision $0 < u_1 < \cdots < u_n$ of $\mathbb{R}_+$, for any $v, h \in \mathbb{R}_+$, and for any Borel bounded function f on E_δ^n,

$$E_x\Big(f\big(X_{u_1, v+h}, \dots, X_{u_n, v+h}\big) | \mathscr{F}_v^{2,x}\Big)$$
$$= \int_{E^n} f(x_1, \dots, x_n) P_h^2\big(dx_1; X_{u_1, v}\big) \cdots P_h^2\big(dx_n; X_{u_n, v}\big)$$

and similarly,

$$E_x\Big(f\big(X_{v+h, u_1} \dots, X_{v+h, u_n}\big) | \mathscr{F}_v^{1,x}\Big)$$
$$= \int_{E^n} f(x_1, \dots, x_n) P_h^1\big(dx_1; X_{v, u_1}\big) \cdots P_h^1\big(dx_n; X_{v, u_n}\big).$$

These formulae show that the process X is 1-Markov and 2-Markov in the sense of Korezlioglu, Lefort and Mazziotto (1981). It is proved in this reference (Proposition 2.6) that, for any $x \in E_\delta$, the filtration $\mathscr{F}^x$ satisfies the axiom (F4) of conditional independence. Moreover, the process X is Markov also in the sense of Nualart and Sanz (1979) (Proposition 2.4 of the preceding reference).

In what follows, the realization of the semigroup P may be arbitrarily chosen, say $(\Omega, \mathscr{A}, \mathbb{P}, X)$.

Given a Borel function f on E_δ and given $p \in \mathbb{R}_+^2$, we define a two-parameter process $J(p, f)$ on $(\Omega, \mathscr{A})$ by setting

$$\forall\, t \in \mathbb{R}_+^2 :\ J(p, f)_t = e^{-p \cdot t} f(X_t),$$

with the convention that $e^{-p \cdot \infty} = 0$ and $X_\infty = \delta$.

For $p \gg 0$ and $g \in b(E)$, set $f = U_p g$. Then, straightforward computations show that, for any $x \in E_\delta$ and any $t \in \mathbb{R}_+^2$,

$$J(p, U_p g)_t = E_x\left(\int_{[t, \infty]} e^{-p \cdot s} g(X_s)\, ds | \mathscr{F}_t^x \right), \quad \mathbb{P}_x\text{-a.s.}$$

We easily deduce from this relation that the process $J(p, U_p g)$ is an admissible quasimartingale, for any probability $\mathbb{P}_x$ [see Meyer (1981), 2, Théorème 3.1]. The class of all the processes of the form $J(p, U_p g)$ with $p \gg 0$ and $g \in b(E)$ will play a fundamental role in the sequel. One of the reasons for that lays in the following result.

Proposition 2.1. *Given $x \in E_\delta$, $g \in b(E)$ and $p \gg 0$, the process $J(p, U_p g)$ admits a $\mathbb{P}_x$-modification the trajectories of which are cad-lq functions on $\mathbb{R}_+^2 \cup \{\infty\}$.*

Proof. It is based on the important results of Millet and Sucheston (1981) and Bakry (1980, 1981) which say that several kinds of two-parameter martingales admit cad-lq modifications. More precisely, we start from the following decomposition of $J(p, U_p g)$:

$$J(p, U_p g)_t = A_t + E_x(A_{\infty,\infty} | \mathscr{F}_t^x) - E_x\left(A_{t_1,\infty} | \mathscr{F}_{t_2}^{2,x}\right) - E_x\left(A_{\infty,t_2} | \mathscr{F}_{t_1}^{1,x}\right),$$

with

$$A_t = \int_{[0,t]} e^{-p\cdot s} g(X_s)\, ds, \qquad \forall\, t \in (\mathbb{R}_+ \cup \{\infty\})^2.$$

The process A is continuous. Then the fact that the two last terms admit cad-lq modifications follows from Bakry (1980), Theorèmes 4c and 6; and for the second term, it is the main result of Millet and Sucheston (1981). □

REMARK 2.2. Let us stress another important feature of the processes $J(p, U_p g)$. For $p \gg 0$ and for any nonnegative Borel bounded function g, the process $J(p, U_p g)$ is a supermartingale. Then recall that for any law $\mathbb{P}_x$ and for any $t \in \mathbb{R}_+^2$,

$$\mathbb{P}_x\left(J(p, U_p g)_t > 0;\ \inf_{s \in \mathbb{D} \cap [0,t]} J(p, U_p g)_s = 0 \right) = 0,$$

where $\mathbb{D}$ is an arbitrary denumerable dense subset of $\mathbb{R}_+^2 \cup \{\infty\}$.

The next theorem is the main result of this paragraph: It generalizes a classical one under the Feller conditions (C1) and (C2).

THEOREM 2.1. *If the two-parameter semigroup P satisfies the conditions* (C1) *and* (C2), *then there exists a realization* $(\Omega, \mathscr{A}, \mathbb{P}, X)$ *such that the process X has all its trajectories cad-lq on $\mathbb{R}_+^2$.*

PROOF. It is similar to the one of the classical theory, using the results of Proposition 2.1 and Remark 2.1. In what follows, we extend the construction made by Blumenthal and Getoor (1968), Section I-9-4, and we only reproduce the parts which differ. Let us start from the coarse realization of P defined in what precedes, say $(\Omega^0, \mathscr{A}^0, \mathbb{P}^0, X^0)$.

Denote by Λ_b the subset of Ω^0 of all the functions $\omega = (\omega(t);\ t \in \mathbb{R}_+^2)$ such that, for any $t \in \mathbb{D}$, the set $\{\omega(s);\ s \in \mathbb{D} \cap [0,t]\}$ is bounded in E. By using Remark 2.1, it can be proved exactly as in Blumenthal and Getoor (1968) that

$$\Lambda_b \in \mathscr{A}^0 \quad \text{and} \quad \mathbb{P}_x(\Lambda_b) = 1, \qquad \forall\, x \in E_\delta.$$

Now denote by Λ_a the subset of Ω^0 of all the functions $\omega = (\omega(t);\ t \in \mathbb{R}_+^2)$ having a restriction to $\mathbb{D}$ with quadrantal limits at every point of $\mathbb{R}_+^2$. The proof that $\Lambda_a \in \mathscr{A}^0$ is in the same spirit as in Blumenthal and Getoor (1968), with slight differences we stress below.

Let d be some metric for E_δ and define for $\varepsilon > 0$, $h_\varepsilon(x, y) = 1$ if $d(x, y) \geq \varepsilon$ and $h_\varepsilon(x, y) = 0$ if $d(x, y) < \varepsilon$. If U is a finite grid of $\mathbb{R}_+^2$ with $n \cdot n$ elements, say $U = \{(u_i, v_j);\ u_1 < \cdots < u_n,\ v_1 < \cdots < v_n;\ i, j = 1, \dots, n\}$, consider the set $\mathscr{U}$ of all the sequences $z = \{z(k) = (u(k), v(k));\ k = 1, \dots, 2n\}$ in U such that either $(u(1), v(1)) = (u_1, v_1), \dots, u(k) = u(k+1)$ and $v(k) < v(k+1)$, or $u(k) < u(k+1)$ and $v(k) = v(k+1), \dots, (u(2n), v(2n)) = (u_n, v_n)$, either $(u(1), v(1)) = (u_1, v_n), \dots, u(k) = u(k+1)$ and $v(k) > v(k+1)$, or $u(k) > u(k+1)$ and $v(k) = v(k+1), \dots, (u(2n), v(2n)) = (u_n, v_1)$. Define

$$H_\varepsilon(U)(\omega) = \sup_{z \in \mathscr{U}} \sum_{i=2}^{2n} h\big(X_{z(k-1)}(\omega), X_{z(k)}(\omega)\big),$$

then $H_\varepsilon(U)$ is $\mathscr{A}^0$-measurable. For each $V \subset \mathbb{R}_+^2$, define $H_\varepsilon(D) = \sup H_\varepsilon(U)$, where the supremum is taken over all the finite grids of the above type which are contained by D. If D is countable, then $H_\varepsilon(D)$ is again in $\mathscr{A}^0$. Therefore, the set Λ'_a defined by

$$\Lambda'_a = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \{H_{1/n}(\mathbb{D} \cap [0, m]) < \infty\}$$

is measurable. It remains to check that $\Lambda_a = \Lambda'_a$. It is clear that $\Lambda_a \subset \Lambda'_a$. Conversely, Λ'_a represents all the functions ω the restriction of which to any increasing or decreasing path in $\mathbb{D}$ is right- and left-limited at every point of $\mathbb{R}_+^2$. It can be easily shown that this property also implies that the restriction of the function ω to $\mathbb{D}$ admits quadrantal limits at every point of $\mathbb{R}_+^2$, therefore $\Lambda'_a \subset \Lambda_a$. This proves that $\Lambda_a \in \mathscr{A}^0$.

The proof that $\mathbb{P}_x(\Lambda_a) = 1, \forall\, x \in E$ is again similar to the one of Blumenthal and Getoor (1968). From conditions (C1) and (C2), we have that for any $f \in C_0(E)$, $|p|U_p f$ converges uniformly in $C_0(E)$ towards f when p decreases to zero. Then we can find a sequence $\{p_n;\ n \in \mathbb{N}\}$ in $\mathbb{R}_+^2$, and a countable dense subset $\{f_k;\ k \in \mathbb{N}\}$ in $C_0(E)$ such that the family $\{|p_n| U_{p_n} f_k;\ n, k \in \mathbb{N}\}$ is dense in $C_0(E)$ for the uniform convergence topology and separates the points of E. Now let us prove that, for every $i = 1, \dots, 4$, we have the inclusion

$$\Big\{\exists\, t \in \mathbb{R}_+^2:\ \liminf_{s \in \mathbb{D} \cap Q_t^i;\ s \to t} X_s \neq \limsup_{S \in \mathbb{D} \cap Q_t^i;\ s \to t} X_s\Big\}$$

$$\subset \bigcup_{n,k} \Big\{\exists\, t \in \mathbb{R}_+^2:\ \liminf_{s \in \mathbb{D} \cap Q_t^i;\ s \to t} U_{p_n} f_k(X_s) \neq \limsup_{s \in \mathbb{D} \cap Q_t^i;\ s \to t} U_{p_n} f_k(X_s)\Big\}.$$

In fact, if for an arbitrary $t \in \mathbb{R}_+^2$, the set $\{X_s;\ s \in \mathbb{D} \cap Q_t^i,\ s \to t\}$ has two distinct adherent values, say X'_t and X''_t, then there exists n, k such that $U_{p_n} f_k(X'_t) \neq U_{p_n} f_k(X''_t)$. Since $U_{p_n} f_k$ is continuous, $U_{p_n} f_k(X'_t)$ and $U_{p_n} f_k(X''_t)$ are adherent values of the set $\{U_{p_n} f_k(X_s);\ s \in \mathbb{D} \cap Q_t^i,\ s \to t\}$, then the above inclusion follows. By Proposition 2.1, each set of the right-countable union is negligible. Therefore, except eventually on some $\mathbb{P}_x$-negligible set, the restriction to $\mathbb{D}$ of the process X admits quadrantal limits at every point of $\mathbb{R}_+^2$, i.e., $\mathbb{P}_x(\Lambda_s) = 1$, $\forall\, x \in E$. The end of the proof is the same as in Blumenthal and Getoor (1968), I-9-4: delete the complementary set of $\Lambda_a \cap \Lambda_b$ in Ω, replace X by its limit in Q^1. This leads to another realization of the semigroup P we may identify with the canonical realization on the space Ω of the cad-1q function ω from $\mathbb{R}_+^2$ into E_d such that $\omega(s) = \delta$ if $s > t$ and $\omega(t) = \delta$, endowed with its Borel σ-field $\mathscr{A}$, where the coordinate process is still denoted by X, and where the probability laws $\mathbb{P} = (\mathbb{P}_x;\ x \in E_\delta)$ are defined as the images of the $\mathbb{P}_x^0$'s on Ω^0. □

Until the end of the paper, the only realization of the two-parameter semigroup P we will consider is the one constructed by Theorem 2.1; it will be denoted by $(\Omega, \mathscr{A}, \mathbb{P}, X)$. The $\mathbb{P}$-augmentation (resp. for every $x \in E$, the $\mathbb{P}_x$-augmentation) of the natural filtration of X is denoted by $\mathscr{F}$ (resp. $\mathscr{F}^x$). Let $\mathscr{C}$ (resp. $\mathscr{C}^x$) be the set of all the stopping points in $\mathbb{R}_+^2 \cup \{\infty\}$ with respect to the filtration $\mathscr{F}$ (resp. $\mathscr{F}^x$).

The following result complements Theorem 2.1; it will justify the name of two-parameter Hunt process for X. We make the convention that $X_\infty = \delta$.

THEOREM 2.2. *Under the hypotheses of Theorem* 2.1, *the process X has the strong Markov property with respect to stopping points, and is quasi-left continuous in the following sense*:

(i) *For any $\mathscr{F}$-stopping point T and any $\mathscr{F}_T$-measurable $\mathbb{R}^2$-valued random variable, for any $f \in b(E)$ and $x \in E$,*

$$E_x(f(X_{T+S}) | \mathscr{F}_T) = E_{X_T}(f(X_s)), \quad \mathbb{P}_x\text{-}a.s.$$

(ii) *For any increasing sequence of stopping points $(T_n;\ n \in \mathbb{N})$ such that $T = \lim_{n\to\infty} T_n$,*

$$\lim_{n\to\infty} X_{T_n} = X_T, \quad \mathbb{P}_x\text{-}a.s., \qquad \forall\, x \in E.$$

The proof is exactly the same as in Blumenthal and Getoor (1968) (see I-8-11 and I-9-4).

The lifetime of the two-parameter Markov process X is defined as being the stopping line ζ which is the debut of the set $\{t \in \mathbb{R}_+^2\colon\ X_t = \delta\}$. From the definition of Ω, it is clear that $X_t = \delta$ for every $t \in [\zeta, \infty[$.

The following result is exactly the same as in Blumenthal and Getoor (1968) (see I-9-3), and therefore its proof is omitted.

PROPOSITION 2.2. *If X is a Hunt process, then for each t the set $A = \{X_s\colon 0 \le s \le t,\ t < \zeta\}$ is almost surely bounded.*

The next result shows that under additional assumptions, the Hunt realization of P has continuous trajectories. It is a nontrivial generalization of a classical result [see Blumenthal and Getoor (1968), I-9-10].

THEOREM 2.3. *Given the Hunt realization of P, assume that, moreover, the following condition holds*:

For any compact set $K \subset E$ and every $\varepsilon > 0$,

$$\lim_{h\in\mathbb{R}_+,\ h\downarrow 0} \left[\frac{1}{h} \sup_{x\in K} \max_{i=1,2} P_h^i(x, E - B(x,\varepsilon)) \right] = 0,$$

where $B(x, \varepsilon)$ denotes the open ball of center x and radius ε. Then, for any $x \in E$, the process X has $\mathbb{P}_x$-a.s. all its trajectories continuous over $[0, \zeta[$.

PROOF. Let us prove that for any fixed finite t, the Hunt process X is a.s. continuous on $[0, t] \cap [0, \zeta[$. Since the functions $\omega \in \Omega$ such that $\omega(t) \neq \delta$, are bounded over $[0, t]$, we have

$$\{t < \zeta\} \cap \Omega = \bigcup_{n \in \mathbb{N}} \{X_s \in K_n, \forall\, s \in [0, t]\},$$

where $\{K_n;\ n \in \mathbb{N}\}$ is an increasing sequence of compact sets which covers E. Thus, we only need to show that, for an arbitrary compact set K, the process $(X_s;\ s \leq t)$ is a.s. continuous on the set $\{t < \zeta \text{ and } X_s \in K, \forall\, s \leq t\}$. There is no loss of generality in assuming that $t = (1,1)$. Using the same argument as in Mazziotto and Merzbach (1985), Proposition 3–6, we can see that the problem reduces to verify that the restriction of $(X_s;\ s \in [0,1]^2)$ to any dyadic stepped increasing (nonrandom) path $z = (z_u;\ u \in [0,2])$, $(X_{z_u};\ u \in [0,2])$, is a.s. continuous. Namely, let z be an arbitrary increasing path from $(0,0)$ to $(1,1)$ supported by the dyadic grid of order m: $\mathbb{D}_m = \{(i2^{-m}, j2^{-m});\ i, j \in \mathbb{N}\}$, for $m \in \mathbb{N}$, parametrized by its length. Let K be an arbitrary compact set of E, let $\varepsilon > 0$ and for $n \geq m$, consider the inequalities

$$\begin{aligned}
&\mathbb{P}_x\left(\bigcup_{k=0}^{2^{n+1}-1} \left\{ d\left(X_{z_{k2^{-n}}}, X_{z_{(k+1)2^{-n}}}\right) > \varepsilon \right\} \cap \{X_s \in K, \forall\, s \leq t\} \right) \\
&\leq \sum_{k=0}^{2^{n+1}-1} \mathbb{P}_x\left(\left\{ d\left(X_{z_{k2^{-n}}}, X_{z_{(k+1)2^{-n}}}\right) > \varepsilon \right\} \cap \left\{ X_{z_{k2^{-n}}} \in K \right\} \right) \\
&\leq \sum_{k=0}^{2^{n+1}-1} \Big(\mathbb{P}_x\Big(\left\{ d\left(X_{z_{k2^{-n}}}, X_{z_{k2^{-n}+(2^{-n},0)}}\right) > \varepsilon \right\} \\
&\qquad\qquad \cap \left\{ z_{(k+1)2^{-n}} = z_{k_{2^{-n}}+(2^{-n},0)} \right\} \cap \left\{ X_{z_{k2^{-n}}} \in K \right\} \Big) \\
&\qquad + \mathbb{P}_x\Big(\left\{ d\left(X_{z_{k2^{-n}}}, X_{z_{k2^{-n}+(0,2^{-n})}}\right) > \varepsilon \right\} \\
&\qquad\qquad \cap \left\{ z_{(k+1)2^{-n}} = z_{k2^{-n}+(0,2^{-n})} \right\} \cap \left\{ X_{z_{k2^{-n}}} \in K \right\} \Big) \Big) \\
&\leq \sum_{k=0}^{2^{n+1}-1} \Big(E_x\Big[\mathbb{P}_{X_{z_{k2^{-n}}}}\left\{ d\left(X_0, X_{(2^{-n},0)}\right) > \varepsilon \right\}; \\
&\qquad\qquad \left\{ X_{z_{k2^{-n}}} \in K;\ z_{(k+1)2^{-n}} = z_{k2^{-n}} + (0, 2^{-n}) \right\} \Big] \\
&\qquad + E_x\Big[\mathbb{P}_{X_{z_{k2^{-n}}}}\left\{ d\left(X_0, X_{(0,2^{-n})}\right) > \varepsilon \right\}; \\
&\qquad\qquad \left\{ X_{z_{k2^{-n}}} \in K;\ z_{(k+1)2^{-n}} = z_{k2^{-n}} + (2^{-n}, 0) \right\} \Big] \Big) \\
&\leq 2^{n+1} \max_{i=1,2} \sup_{x \in K} P^i_{2^{-n}}(x, E - B(x, \varepsilon)).
\end{aligned}$$

It follows from the condition of the theorem that the last bound can be chosen as small as we want for n sufficiently great and for arbitrary ε. This proves that the process $(X_{z_u};\ 0 \leq u \leq 2)$ is a.s. continuous. Therefore, we get that for any dyadic increasing path z in $[0,1]^2$, the restriction of the process X to z is a.s. continuous. To achieve the proof, recall that it is shown in Mazziotto and Szpirglas (1981) that the set of the discontinuities of a cad-lq process is a.s. composed of a countable number of vertical and horizontal (random) segments. Therefore necessarily, there exists at least one stepped increasing (nonrandom) path which intersects these segments with a nonzero probability, as soon as this set of discontinuities is nonevanescent. That would be in contradiction with what precedes; thus this set of discontinuities must be evanescent. This completes the proof. □

3. A potential theory for two-parameter semigroups. In this section, we express by means of the two-parameter Hunt process various elements for a potential theory associated to two-parameter semigroups. In the first part, we extend several definitions already given by Cairoli (1966) and by the author in Mazziotto (1985). Then we introduce a notion of harmonicity which generalizes those studied in Dynkin (1981) or in Vanderbei (1983) and in Jacob (1985). The main difference lies in the fact that we do not have to work with product spaces and with two processes, and that the sets where a function is harmonic may have general shapes.

Given a Borel set $A \subset E_\delta$, let us define the first hitting line of A by the process X after an arbitrary point t, as being the debut of the random set $\{s \in \mathbb{R}^2_+ :\ s \gg t,\ X_s \in A\}$, denoted by T^t_A and also by T_A for $t = 0$. By the definition, T^t_A is a stopping line. It is easy to verify the relation

$$\forall\, t \in \mathbb{R}^2_+ :\ t + T_A \circ \theta_t = T^t_A,$$

and, moreover,

$$T^t_A = D(t) \vee T_A, \quad \text{on the set } \{t \ll T_A\}.$$

According to the zero–one law, the set $\{T^t_A = D(0)\} = \{0 \geq T_A\}$, which belongs to $\mathscr{F}_0$, is of $\mathbb{P}_x$-probability zero or one, for all $x \in E$. As in the classical theory, the point x is said to be irregular (resp. regular) for A if $\mathbb{P}_x(T_A = D(0)) = 0$ [resp. $\mathbb{P}_x(T^t_A = D(0)) = 1$]. It is clear that, if A is open, then every point of A is regular for A.

The following definitions of excessive functions are similar to those given in Cairoli (1966, 1968) for tensor products of one-parameter semigroups and in Mazziotto (1985) for bi-Markov processes.

Definition 3.1. Let $p \in \mathbb{R}^2_+$ be fixed, and let f be a nonnegative Borel function on E. f is said to be p-excessive (with respect to the two-parameter semigroup P) if and only if

(i) $\forall\, t \in \mathbb{R}^2_+ :\ e^{-p\cdot t} P_t f \leq f$,
(ii) $(P_t f;\ t \in \mathbb{R}^2_+)$ converges pointwise towards f when t tends to zero.

f is said to be p-excessive with positive variation if and only if it is p-excessive and, moreover, it satisfies

(iii)
$$\forall\, t = (t_1, t_2) \in \mathbb{R}^2_+ :$$
$$f + e^{-p\cdot t} P_t f - e^{-p_1 t_1} P^1_{t_1} f - e^{-p_2 t_2} P^2_{t_2} f \geq 0.$$

The functions f such that $f = U_p g$ for some nonnegative $g \in b(E)$ are p-excessive with positive variation; even if g is arbitrary, it may happen that f is p-excessive [see Mazziotto (1985)]. These functions f will be called p-potentials in the sequel. As it was noticed in Cairoli (1968), the minimum of two p-excessive functions with positive variations does not enjoy, in general, the same property.

As in the classical theory, and as in Mazziotto (1985) for bi-Markov processes, the p-excessive functions can be approximated by means of p-potentials.

PROPOSITION 3.1. *For $p \gg 0$, any p-excessive function f is the limit of a nondecreasing sequence of p-potentials $(U_p g^n;\ n \in \mathbb{N})$. Moreover, if f is p-excessive with positive variation, then the g^n's can be chosen nonnegative.*

PROOF. See Mazziotto (1985), Proposition 2.2.1. □

Recall that in Section 2, we have associated to any $p \in \mathbb{R}_+^2$ and $f \in b(E)$, the two-parameter process $J(p, f)$ defined by

$$\forall\, t \in \mathbb{R}_+^2:\ J(p, f)_t = e^{-p\cdot t} f(X_t) \quad \text{and} \quad J(p, f)_\infty = 0.$$

It is clear that, if f is p-excessive (resp. with positive variation), then $J(p, f)$ is a nonnegative supermartingale (resp. with positive variation) for any probability $\mathbb{P}_x$. Now, we denote by $\mu_x^{p,f}$ the finitely additive Doleans function, on the algebra of elementary predictable sets $\mathscr{J}$, associated to the process $J(p, f)$ and the probability $\mathbb{P}_x$ as in Section 1; for any pair of stepped stopping lines L and L' such that $L \le L'$, we have [see Merzbach (1979)]

$$\mu_x^{p,f}(]L, L']) = J(p, f)_L - J(p, f)_{L'}.$$

Moreover, if $J(p, f)$ is an admissible process with respect to $\mathbb{P}_x$, then $\mu_x^{p,f}$ extends to a σ-additive measure on the predictable σ-field $\mathscr{P}$, also noted $\mu_x^{p,f}$. This is the case if f is a p-potential, say $f = U_p g$, with $p \gg 0$, namely for any pair of stopping lines L and L' such that $L \le L'$, we have

$$\mu_x^{p,f}(]L, L']) = E_x\left(\int_{]L, L']} e^{-p\cdot t} g(X_t)\, dt\right).$$

Unfortunately, this property does not generalize to all the p-excessive functions: It may be some function f such that $J(p, f)$ is not an admissible process. For p-excessive functions with positive variation, we may refer to the discussion made in Meyer (1981), Chapter 2.3.

For these reasons, we distinguish the following class of p-excessive functions.

DEFINITION 3.2. Given $p \in \mathbb{R}_+^2$, a Borel bounded function f is said to be p-admissible if and only if the Doleans function $\mu_x^{p,f}$ on $\mathscr{J}$ extends to a bounded σ-additive measure on the predictable σ-field $\mathscr{P}$, for any $x \in E_\delta$.

The next notion is widely inspired from those of harmonic operators of the classical theory [in what follows we mainly refer to Meyer (1967)], but its domain of definition is restricted to the admissible functions.

DEFINITION 3.3. Given $p \in \mathbb{R}_+^2$ and A a Borel subset of E_δ, the harmonic operator H_A^p associates to the admissible function f, the function $H_A^p f$ defined by

$$\forall x \in E: H_A^p f(x) = -\mu_x^{p,f}(]T_A, \infty[),$$

where T_A is the first hitting line of A by the process X.

The operator H_A^p is linear and maps the vector space of the admissible functions into the set of the Borel functions. In the classical theory, if the function f is p-excessive and A is open, then $H_A^p f$ is also the less p-excessive majorant of f over A [see Meyer (1967)], and $H_A^p f$ is also called the p-reduite of f over A. It is not clear that such a characterization holds in the present situation. However, we have the following result in that direction.

PROPOSITION 3.2. *Let A be a Borel set, and for $p \gg 0$, let $f = U_p g$ be a p-potential. Then the following assertions hold.*

(i) *$H_A^p f(x) = f(x)$, for any point x which is regular for A.*

(ii) *If f has positive variation, i.e., $g \geq 0$, then the function $H_A^p f$ is p-excessive.*

The proof is based on the following lemma.

LEMMA 3.1. *Let A be a Borel set, and for $p \gg 0$ let $f = U_p g$ be a p-potential. Then, for any stopping point $\tau \in \mathscr{C}$ and for any $x \in E$, the following relation holds*:

$$e^{-p\cdot\tau} H_A^p f(X_\tau) = E_x\left(\int_{[T_A^\tau, \infty]} e^{-p\cdot s} g(X_s)\, ds \,|\, \mathscr{F}_\tau\right).$$

The proof of the lemma is a straightforward application of the Markov property when τ has a constant value, and of the strong Markov property of Proposition 2.2 when τ is a stopping point.

PROOF OF PROPOSITION 3.2. (i) Let x be such that $\mathbb{P}_x(T_A = D(0)) = 1$. Then

$$H_A^p f(x) = E_x\left(\int_{[T_A, \infty]} e^{-p\cdot s} g(X_s)\, ds\right) = U_p g(x) = f(x).$$

(ii) For $t \in \mathbb{R}_+^2$, let us compute $e^{-p\cdot t} P_t H_A^p f$. From Lemma 3.1, we have for any $x \in E$,

$$e^{-p\cdot t} P_t H_A^p f(x) = E_x\left(\int_{[T_A^t, \infty]} e^{-p\cdot s} g(X_s)\, ds\right).$$

Since $g \geq 0$ and $T_A^t \geq T_A$, we obtain

$$e^{-p\cdot t} P_t H_A^p f(x) \leq H_A^p f(x).$$

Moreover, since T_A^t decreases to T_A when t decreases to zero, we also have $\lim_{t\to 0} e^{-p\cdot t} P_t H_A^p f(x) = H_A^p f(x)$. This completes the proof. □

Given $p \in \mathbb{R}_+^2$, a Borel set A and a p-potential f, we consider the process $J(p, H_A^p f)$ and the Doleans function $\mu_x^{p, H_A^p f}$ on $\mathcal{J}$ associated to any probability $\mathbb{P}_x$. In what follows, we study the behavior of $J(p, H_A^p f)$ before X hits A.

Proposition 3.3. *Let $p \in \mathbb{R}_+^2$, A a Borel subset of E_δ and f a p-potential. For any stepped stopping line L and for any x such that $\mathbb{P}_x(L \le T_A) = 1$, it holds that*

$$\mu_x^{p, H_A^p f}([0, L]) = 0.$$

Proof. It is well known [see Merzbach (1979)] that for an arbitrary stepped stopping line, we have

$$\mu_x^{p, H_A^p f}([0, L]) = E_x(J(p, H_A^p f)_L) - H_A^p f(x)$$

and

$$J(p, H_A^p f)_L = \int_{\mathbb{R}_+^2} J(p, H_A^p f)_t \, dA_t^L,$$

where A^L is the optional process with an integrable variation defined by

$$\forall\, t \in \mathbb{R}_+^2 \colon A_t^L = \mathbb{1}_{\{t \ge L\}} \quad \text{and} \quad A_\infty^L = 1.$$

Recall that in the proof of Proposition 3.2 we got, for a p-potential $f = U_p g$, the identity for any t,

$$e^{-p\cdot t} H_A^p f(X_t) = E_x\left(\int_{[T_A^t, \infty]} e^{-p\cdot s} g(X_s) \, ds | \mathcal{F}_t \right).$$

Therefore, the following relations hold:

$$\begin{aligned}
E_x(J(p, H_A^p f)_L) &= E_x\left(\int_{\mathbb{R}_+^2} J(p, H_A^p f)_t \, dA_t^L \right) \\
&= E_x\left(\int_{\mathbb{R}_+^2} e^{-p\cdot t} H_A^p f(X_t) \, dA_t^L \right) \\
&= E_x\left(\int_{\mathbb{R}_+^2} E_x\left(\int_{[T_A^t, \infty]} e^{-p\cdot s} g(X_s) \, ds | \mathcal{F}_t \right) dA_t^L \right) \\
&= E_x\left(\int_{\mathbb{R}_+^2} \int_{\mathbb{R}_+^2} e^{-p\cdot s} g(X_s) \mathbb{1}_{\{s \ge T_A^t\}} \, ds \, dA_t^L \right) \\
&= E_x\left(\int_{\mathbb{R}_+^2} e^{-p\cdot s} g(X_s) \left(\int_{\mathbb{R}_+^2} \mathbb{1}_{\{s \ge T_A^t\}} \, dA_t^L \right) ds \right).
\end{aligned}$$

Let us compute separately the term

$$I(s, L) = \int_{\mathbb{R}_+^2} \mathbb{1}_{\{s \ge T_A^t\}} \, dA_t^L.$$

For a stepped stopping line, we have

$$A_t^L = \sum_{i=1}^{l} (-1)^{l-1} \mathbb{1}_{\{t \geq L_i\}},$$

where the collection of random points (L_i; $i = 1, \ldots, l$) denotes the summits of the line L(l is also random, but a.s. finite). We now use the fact that $L \leq T_A$. It follows that $L_i \leq T_A$, $\forall\ i = 1, \ldots, l$. From this, we deduce that

$$\forall\ i = 1, \ldots, l\colon T_A^{L_i} = T_A \vee D(L_i).$$

Thus, we obtain another expression for $I(s, L)$,

$$\begin{aligned} I(s, L) &= \sum_{i=1}^{l} (-1)^{l-1} \mathbb{1}_{\{s \geq T_A^{L_i}\}} = \sum_{i=1}^{l} (-1)^{l-1} \mathbb{1}_{\{s \geq T_A;\ s \geq D(L_i)\}} \\ &= \mathbb{1}_{\{s \geq T_A\}} \sum_{i=1}^{l} (-1)^{l-1} \mathbb{1}_{\{s \geq D(L_i)\}} = \mathbb{1}_{\{s \geq T_A;\ s \geq L\}} \\ &= \mathbb{1}_{\{s \geq T_A\}}. \end{aligned}$$

Finally, coming back to the main expression, we get

$$E_x(J(p, H_A^p f)_L) = E_x\left(\int_{\mathbb{R}_+^2} e^{-p \cdot s} g(X_s) \mathbb{1}_{\{s \geq T_A\}}\, ds \right) = H_A^p f(x),$$

and therefore,

$$\mu_x^{p, H_A^p f}([0, L]) = 0. \qquad \square$$

At this stage, we may introduce our definition of harmonicity; it will be seen further that it generalizes the already known notions.

DEFINITION 3.4. Let $p \gg 0$ and let A be a Borel subset of E_δ, then a Borel function f on E_δ is said to be p-harmonic on A if and only if for any stepped stopping line L and for any x such that $L \leq T_{A^c}$, $\mathbb{P}_x$-a.s.

$$\mu_x^{p, f}([0, L]) = 0.$$

Using this definition, we deduce from Proposition 3.3 that the function $H_{A^c}^p f$, where $f = U_p g$ is a p-potential and A^c is the complementary set of A, is p-harmonic on the set A.

In order to connect this definition with already known notions of harmonicity, we need some more assumptions on the process X and on the functions f and g. Recall that Theorem 2.2 stated conditions on the semigroup P ensuring that X is continuous.

PROPOSITION 3.4. *For $p \gg 0$, let $f = U_p g$ be a p-potential and let A be a Borel subset of E_δ. Then*:

(i) *If g is null over the set A, then f is p-harmonic on A.*

Assume, in addition, that the process X is continuous and that the set A is open.

Then:

(ii) *If f is p-harmonic on A, then f and $H^p_{A^c} f$ coincide.*

Assume, in addition, that the function g is continuous. Then:

(iii) *If f is p-harmonic on A, then g is null over A.*

PROOF. (i) Consider an arbitrary stepped stopping line L. For x such that $L \le T_{A^c}$, $\mathbb{P}_x$-a.s., we also have $\mathbb{P}_x(X_t \in A;\ t \le L) = 1$. Therefore, the function $(t \to e^{-p\cdot t} g(X_t);\ 0 \le t \le L)$ is identically zero $\mathbb{P}_x$-a.s., and it follows that

$$\mu_x^{p,f}([0, L]) = E_x\left(\int_{[0,L]} e^{-p\cdot t} g(X_t)\, dt\right) = 0.$$

This proves that f is p-harmonic on A.

(ii) Since the process X is continuous and A^c is closed, the set $\{t \in \mathbb{R}^2_+:\ X_t \in A^c\}$ is predictable. It follows [see Merzbach (1980) and Bakry (1981)] that the stopping line T_{A^c} is predictable, and then announcable. Moreover, it is proved in Bakry (1981), Théorème 14, that there exists a sequence $(L_n;\ n \in \mathbb{N})$ of stepped stopping lines which announce T_{A^c}. For such a stopping line L, we have from Definition 3.4 that $\mu_x^{p,f}([0, L]) = 0$. Since f is a p-potential, $\mu_x^{p,f}$ is a σ-additive measure; hence, $\mu_x^{p,f}([0, T_{A^c}[) = 0$. This proves that, for every x,

$$f(x) = \mu_x^{p,f}([0,\infty]) = \mu_x^{p,f}([T_{A^c},\infty]) = H^p_{A^c} f(x).$$

(iii) We first assume that g has a constant sign on A. Let $x \in A$ be fixed. Since A is open and $g(X)$ is a continuous two-parameter process, we have $\mathbb{P}_x(T_{A^c} \ne D(0)) = 1$. As in (ii), there exists a sequence $(L_n;\ n \in \mathbb{N})$ of stepped stopping lines which announce T_{A^c}. Therefore, one can find a stepped stopping line L such that $\mathbb{P}_x(L \ne D(0)) > 0$ and $\mathbb{P}_x(L \le T_{A^c}) = 1$. By the definition of the p-harmonicity, we get

$$\mu_x^{p,f}([0, L]) = E_x\left(\int_{[0,L]} e^{-p\cdot t} g(X_t)\, dt\right) = 0.$$

Thus, the continuous process $(g(X_t);\ t \in [0, L])$ is $\mathbb{P}_x$-a.s. identically null. Combining these two results, we obtain that $g(x) = 0$. Now, if g is arbitrary, we replace the set A by either the set $A \cap \{g > 0\}$ or the set $A \cap \{g < 0\}$, and we repeat the above proof. □

PROPOSITION 3.5. *For $p \gg 0$, let $f = U_p g$ be a p-potential with $g \in b(E)$, and let A be a Borel subset of E. Then, for any $x \in E$ fixed, for any stopping point τ such that $\mathbb{P}_x(\tau \in T_A) = 1$ and which can be announced by a sequence of stopping points $(\tau_n;\ n \in \mathbb{N})$ satisfying $\tau_n \ll T_A$, $\forall\ n$ and $\lim_{n\in\mathbb{N}} \tau_n = \tau$, the following equality holds, $\mathbb{P}_x$-a.s.*:

$$\lim_{n\in\mathbb{N}} e^{-p\cdot\tau_n} H^p_A f(X_{\tau_n}) = e^{-p\cdot\tau} f(X_\tau).$$

PROOF. For $x \in E$, consider τ and a sequence $(\tau_n;\ n \in \mathbb{N})$ satisfying the above conditions. Since $\tau_n \ll T_A$, it holds that

$$T_A^{\tau_n} = D(\tau_n) \vee T_A, \quad \mathbb{P}_x\text{-a.s.}, \quad \forall\ n \in \mathbb{N}.$$

From Lemma 3.1, we have $\forall\ n$,

$$e^{-p\cdot\tau_n}H_A^p f(X_{\tau_n}) = E_x\left(\int_{[T_A^{\tau_n},\infty]} e^{-p\cdot s}g(X_s)\,ds|\mathscr{F}_{\tau_n}\right).$$

Using the preceding identity, we get

$$e^{-p\cdot\tau_n}H_A^p f(X_{\tau_n}) = E_x\left(\int_{[T_A,\infty]} \mathbb{1}_{\{s\geq\tau_n\}}\, e^{-p\cdot s}g(X_s)\,ds|\mathscr{F}_{\tau_n}\right).$$

For each $n \in \mathbb{N} \cup \{\infty\}$, writing τ_∞ for τ, set

$$M_n = \int_{[T_A,\infty]} \mathbb{1}_{\{s\geq\tau_n\}}\, e^{-p\cdot s}g(X_s)\,ds.$$

The sequence $(M_n;\ n \in \mathbb{N})$ converges a.s. towards M_∞ and is uniformly bounded. Thus, it follows [see Dellacherie and Meyer (1975), V-Théorème 45] that

$$\begin{aligned}\lim_{n\to\infty} e^{-p\cdot\tau_n}H_A^p f(X_{\tau_n}) &= \lim_{n\to\infty} E_x(M_n/\mathscr{F}_{\tau_n})\\ &= E_x\left(\int_{[T_A,\infty]} \mathbb{1}_{\{s\geq\tau\}}\, e^{-p\cdot s}g(X_s)\,ds\,\middle|\, \bigvee_{n\in\mathbb{N}} \mathscr{F}_{\tau_n}\right).\end{aligned}$$

Since $\mathbb{P}_x(\tau \in T_A) = 1$, this last expression can also be written as

$$\begin{aligned}&E_x\left(\int_{[\tau,\infty]} e^{-p\cdot s}g(X_s)\,ds\,\middle|\, \bigvee_{n\in\mathbb{N}} \mathscr{F}_{\tau_n}\right)\\ &\quad = E_x\left(e^{-p\cdot\tau}U_p g(X_\tau)\middle|\, \bigvee_{n\in\mathbb{N}} \mathscr{F}_{\tau_n}\right).\end{aligned}$$

Since the function $f = U_p g$ is continuous and the process X is quasi-left continuous, we have

$$\lim_{n\to\infty} e^{-p\cdot\tau_n}f(X_{\tau_n}) = e^{-p\cdot\tau}f(X_\tau) \quad \text{a.s.}$$

This implies that $e^{-p\cdot\tau}f(X_\tau)$ is measurable with respect to the σ-field $\bigvee_{n\in\mathbb{N}}\mathscr{F}_{\tau_n}$. Finally, we get $\mathbb{P}_x$-a.s.,

$$\lim_{n\to\infty} e^{-p\cdot\tau_n}H_A^p f(X_{\tau_n}) = e^{-p\cdot\tau}f(X_\tau),$$

and this completes the proof. □

To conclude this paper, let us consider the problem stated in Section 1.1, under the hypothesis that X is continuous.

Given $p \gg 0$, an open subset $A \subset E$, and a function f which is the p-potential of a continuous function g: $f = U_p g$, find a function φ such that

φ is p-harmonic on A,
$\varphi = f$ on the interior of A^c,
for almost all $x \in \partial A$, there exists a sequence $(x_n;\ n \in \mathbb{N})$ in A such that $(x_n, \varphi(x_n);\ n \in \mathbb{N})$ converges towards $(x, f(x))$.

In the classical Dirichlet problem [see Dynkin (1965)] the boundary condition is only

$$\varphi = f, \quad \text{on } \partial A.$$

As pointed out in Helms (1967), Dynkin (1981) or Vanderbei (1983), the problem involving a fourth-order differential operator needs to be some well-posed supplementary conditions on the derivatives of φ at ∂A.

In this paper we have replaced these conditions by a stronger one, which says that

$$\varphi = U_p g, \quad \text{on } A^c \text{ with } g \text{ continuous on } E.$$

By Proposition 3.3, the function φ defined as $\varphi = H^p_{A^c} f$ is p-harmonic on A. Moreover, by Proposition 3.2, $\varphi = f$ on the set of the regular points of A^c. The set A being open, this is a fortiori true for the set of the interior points of A^c. As in the proof of Proposition 3.4, the stopping line T_A is predictable and is announced by a sequence $(L_n;\ n \in \mathbb{N})$ of stopping lines. Therefore, for any stopping point τ which belongs a.s. to T_A, there exists a sequence $(\tau_n;\ n \in \mathbb{N})$ of stopping points which announces τ in the sense of Proposition 3.5 [for each n, the stopping point τ_n can be defined as the intersection of the stopping line L_n and of the optional increasing path which goes through τ; see Walsh (1981)]. Denote by B the subset of ∂A made of the points x such that there exists no sequence., $(x_n;\ n \in \mathbb{N})$ in A such that $(x_n, \varphi(x_n);\ n \in \mathbb{N})$ converges towards $(x, f(x))$. Then, by Proposition 3.5, we get that for any stopping point τ which belongs $\mathbb{P}_y$-a.s. to T_{A^c}: $\mathbb{P}_y(X_\tau \in B) = 0$, for any fixed $y \in E$. This is the meaning we give to the expression "for almost all $x \in \partial A$."

Moreover, if we are able to show that the function $\varphi = H^p_{A^c} f$ is the p-potential of some continuous bounded function, then, by Proposition 3.4, we have

$$\left(\mathscr{L}^1 - p_1 I\right) \cdot \left(\mathscr{L}^2 - p_2 I\right)\varphi = 0, \quad \text{on } A.$$

Since φ is continuous on E, we also have from Proposition 3.4 that

$$\varphi = f, \quad \text{on } \partial A,$$

and φ is the unique p-potential verifying the above conditions.

Acknowledgment. The author is indebted to an anonymous referee for his helpful comments.

REFERENCES

BAKRY, D. (1980). Limites quadrantales des martingales. *Processus aléatoires à deux indices. Lecture Notes in Math.* **863** 40–49. Springer, Berlin.

BAKRY, D. (1981). Théorèmes de section et de projection pour processus à deux indices. *Z. Wahrsch. verw. Gebiete* **55** 51–71.

BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory.* Academic, New York.

BRENNAN, M. D. (1979). Planar semimartingales. *J. Multivariate Anal.* **9** 465–486.

CAIROLI, R. (1966). Produits de semi-groupes de transition et produits de processus. *Publ. Inst. Statist. Univ. Paris* **15** 311–384.

CAIROLI, R. (1966). Produits de semi-groupes de transition et produits de processus. *Publ. Inst. Statist. Univ. Paris* **15** 311–384.

CAIROLI, R. (1968). Une représentation intégrale pour fonctions séparément excessives. *Ann. Inst. Fourier (Grenoble)* **18**(1) 317–338.

CAIROLI, R. and WALSH, J. B. (1975). Stochastic integrals in the plane. *Acta Math.* **134** 111–183.

DELLACHERIE, C. and MEYER, P.-A. (1975). *Probabilités et Potentiel.* Hermann, Paris.

DOPPEL, K. and JACOB, N. (1983). A non-hypoelliptic Dirichlet problem from stochastics. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **8** 375–390.

DYNKIN, E. B. (1965). *Markov Processes.* Springer, Berlin.

DYNKIN, E. B. (1981). Harmonic functions associated with several Markov processes. *Adv. in Appl. Math.* **2** 260–283.

HAS'MINSKII, R. Z. (1960). Probabilistic representation of solutions of some differential equations. *Proc. Sixth All-Union Conf. Theory Probab. Math. Statist. Vilnius, 1960* 177–182. (In Russian.)

HELMS, L. L. (1967). Biharmonic functions and Brownian motion. *J. Appl. Probab.* **4** 130–136.

JACOB, N. (1985). On the differential equation $\Delta_1 \Delta_2 u = 0$. Preprint.

KOREZLIOGLU, H., LEFORT, P. and MAZZIOTTO, G.(1981). Une propriété markovienne et diffusions associées. *Processus aléatoires à deux indices. Lecture Notes in Math.* **863** 245–274. Springer, Berlin.

MAZZIOTTO, G. (1985). Two parameter optimal stopping and bi-Markov processes. *Z. Wahrsch. verw. Gebiete* **69** 99–135.

MAZZIOTTO, G. and MERZBACH, E. (1985). Regularity and decomposition of two-parameter supermartingales. *J. Multivariate Anal.* **17** 38–55.

MAZZIOTTO, G. and SZPIRGLAS, J. (1981). Sur les discontinuités d'un processus cad-lag à deux indices. *Processus aléatoires à deux indices. Lecture Notes in Math.* **863** 84–90. Springer, Berlin.

MERZBACH, E. (1979). Processus stochastiques à indices partiellement ordonnés. Rapport interne 55, Ecole Polytechnique, Paris.

MERZBACH, E. (1980). Stopping for two-dimensional stochastic processes. *Stochastic Process. Appl.* **10** 49–63.

MEYER, P.-A. (1967). *Processus de Markov. Lecture Notes in Math.* **26**. Springer, Berlin.

MEYER, P.-A. (1981). Théorie élémentaire des processus à deux indices. *Processus aléatoires à deux indices. Lecture Notes in Math.* **863** 1–39. Springer, Berlin.

MICHEL, D. (1979). Produit de deux diffusions. *C. R. Acad. Sci. Paris Ser. A-B* **289** 143–146.

MILLET, A. and SUCHESTON, L. (1981). On regularity of multiparameter amarts and martingales. *Z. Wahrsch. verw. Gebiete* **56** 21–45.

NEVEU, J. (1964). *Bases Mathématiques des Probabilités.* Masson, Paris.

NUALART, D. and SANZ, M. (1979). A Markov property for two-parameter Gaussian processes. *Stochastica* **3** 1–16.

VANDERBEI, R. J. (1983). Towards a stochastic calculus for several Markov processes. *Adv. in Appl. Math.* **4** 125–144.

VANDERBEI, R. J. (1984). Probabilistic solution of the Dirichlet problem for biharmonic functions in discrete space. *Ann. Probab.* **12** 311–324.

WALSH, J. B. (1968). Probability and a Dirichlet problem for multiply superharmonic functions. *Ann. Inst. Fourier (Grenoble)* **18**(2) 221–279.

WALSH, J. B. (1981). Optional increasing paths. *Processus aléatoires à deux indices. Lecture Notes in Math.* **863** 172–201. Springer, Berlin.

WONG, E. and ZAKAI, M. (1974). Martingales and stochastic integrals for processes with a multidimensional parameter. *Z. Wahrsch. verw. Gebiete* **29** 109–122.

PAA / RDS / RCM
CENTRE NATIONAL D'ETUDES DES TÉLÉCOMMUNICATIONS
38-40, RUE DU GENERAL LECLERC
92131-ISSY LES MOULINEAUX
FRANCE