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CONTINUITY OF GAUSSIAN PROCESSES[1]

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We give sufficient conditions for local continuity of the isonormal process L at some point of its parameter set. Since a Gaussian process defined on a compact parameter space that is a.s. continuous at each point is sample continuous, our result can be applied to the problem of general sample continuity of Gaussian processes. It is shown that our sufficient conditions are strictly weaker than the classical sufficient conditions for sample continuity.

1. Introduction. Let $\{X(t), t \in \mathscr{C}\}$ be a Gaussian process with a continuous covariance function over a compact subset $\mathscr{C}$ of a metric space (S, d). Such a process is called *sample-continuous* if there is a version of the process with continuous sample functions. Equivalently, $\{X(t), t \in \mathscr{C}\}$ is sample-continuous if it is uniformly continuous for t restricted to a countable dense subset of $\mathscr{C}$. The process is said to be *sample-bounded* if it has a version with bounded sample functions.

Let t_0 be a point of $\mathscr{C}$. The process is said to be *continuous at* t_0 if there is a version of the process with sample functions that are continuous at t_0. Equivalently, the process is continuous at t_0 if $P(\lim_{t \to t_0, t \in \mathscr{C}^*} X(t) = X(t_0)) = 1$, $\mathscr{C}^*$ being a countable dense subset of $\mathscr{C}$.

Let H be a real, infinite-dimensional Hilbert space. A linear map L from H into real Gaussian variables with $ELx = 0$, $ELxLy = (x, y)$ for all $x, y \in H$ is called the *isonormal* Gaussian process on H. [As usual, $(\cdot, \cdot)$ denotes the inner product in H.]

A modern approach to the study of sample function continuity and boundedness of Gaussian processes reduces this problem to the study of those sets $C \subset H$ on which the isonormal L has continuous GC or bounded GB sample functions, called GC-sets and GB-sets, respectively [Dudley (1967), 1973), Feldman (1971) and Sudakov (1969, 1971)]. This approach relates GC and GB properties to certain measures of the size of a set C in H. Three such measures of the size of C have been extensively studied: *mixed volume* [Dudley (1967), Sudakov (1971) and Milman and Pisier (1987)], *majorizing measures* [Fernique (1975) and Talagrand (1987)] and *metric entropy*. In this paper we use the notion of metric entropy. The reader interested in mixed volumes or majorizing measures is referred to the sources mentioned previously.

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Let $\mathscr{C}$ be a subset of a metric space (S, d). Given $\varepsilon > 0$, let $N(\mathscr{C}, \varepsilon) \equiv N_{\mathscr{C}}(\varepsilon)$ be the minimal number of points $x_1, x_2, \dots, x_n$ from $\mathscr{C}$ such that for any $y \in \mathscr{C}$, there is an x_i such that $d(x_i, y) \le \varepsilon$. Then $H(\mathscr{C}, \varepsilon) = \ln N(\mathscr{C}, \varepsilon)$ is called the metric entropy of $\mathscr{C}$, and the *exponent of entropy* $r(\mathscr{C})$ is defined by

$$r(\mathscr{C}) = \limsup_{\varepsilon \to 0} \frac{\ln H(\mathscr{C}, \varepsilon)}{|\ln \varepsilon|}.$$

Dudley (1973) proved that $\mathscr{C} \subset H$ is always a GC-set if

$$\int_0^1 H(\mathscr{C}, x)^{1/2}\, dx < \infty. \tag{1.1}$$

This implies in particular that $\mathscr{C}$ is a GC-set if $r(\mathscr{C}) < 2$, and it is known that $\mathscr{C}$ cannot be a GB-set (and so not a GC-set) if $r(\mathscr{C}) > 2$ [Sudakov (1969)]. The case $r(\mathscr{C}) = 2$ includes an ambiguous range, however, where $H_{\mathscr{C}}(\varepsilon)$ cannot determine whether $\mathscr{C}$ is GB or GC [Dudley (1973)]. In particular, there are compact GC-sets for which the integral (1.1) diverges.

In this paper we find conditions under which the isonormal process L on a set $\mathscr{C} \subset H$ is a.s. continuous at some point $x_0 \in \mathscr{C}$. This is closely related to the question of whether or not $\mathscr{C}$ is GC. Clearly, if $\mathscr{C}$ is GC, then the isonormal process is a.s. continuous at each point of $\mathscr{C}$. Less evident is the converse statement: If the isonormal process is a.s. continuous at every point of $\mathscr{C}$, then $\mathscr{C}$ is a GC-set. This important property of Gaussian processes was noted for Gaussian processes on $[0,1]$ by Marcus and Shepp [(1971), page 436] who give credit for the idea to Dudley, and can be extended as follows.

THEOREM 1.1. *Let $\mathscr{C}$ be a compact subset of a metric space $\{S, d\}$ and $X(t)$ a Gaussian process on $\mathscr{C}$. If $X(t)$ is a.s. continuous at each point of $\mathscr{C}$, then it is sample-continuous.*

The proof of Theorem 1.1 will be given in the Appendix at the end of the paper.

In the following section we give sufficient conditions for local continuity of the isonormal process. These conditions turn out to be strictly weaker than those obtainable from (1.1). Consequently, our result (Theorem 2.1) can be used to establish the GC property in situations in which the integral (1.1) diverges.

2. Sufficient conditions for local continuity. We start with some additional definitions and preliminary results.

For a given set $\mathscr{C}$ in a Hilbert space H and a given point $x_0 \in \mathscr{C}$, set

$$\mathscr{C}(x_0; \delta) := \{x \in \mathscr{C} : \|x - x_0\| \le \delta\}, \qquad \delta > 0, \tag{2.1}$$

$$N(x_0; \delta, \varepsilon) := N(\mathscr{C}(X_0; \delta), \varepsilon), \qquad \delta > 0,\ \varepsilon > 0, \tag{2.2}$$

$$N(x_0; \delta_1, \delta_2, \varepsilon) := N(\mathscr{C}(x_0; \delta_2) \setminus \mathscr{C}(x_0; \delta_1), \varepsilon), \qquad \delta_2 > \delta_1 \ge 0,\ \varepsilon > 0. \tag{2.3}$$

A set $\mathscr{C}$ is said to satisfy Condition A if

$$\sup_{x, y \in \mathscr{C}, x \neq y} \frac{| \|x\| - \|y\| |}{\|x - y\|^2} = M < \infty. \tag{2.4}$$

The isonormal process L restricted to such a set $\mathscr{C}$ satisfies the following bound, which is derived in Samorodnitsky (1986):

$$\begin{aligned} &P\left\{ \sup_{x \in \mathscr{C}} Lx > \lambda_0 \sigma + \sum_{j=1}^{\infty} \varepsilon_j \lambda_j \right\} \\ &\quad \leq N_{\mathscr{C}}(\varepsilon_1) \frac{1}{\sqrt{2\pi}} \lambda_0^{-1} \exp\left\{ \frac{-\lambda_0^2}{2} \right\} \\ &\qquad + \frac{1}{\sqrt{2\pi}} \lambda_0^{-1} \exp\left\{ \frac{-\lambda_0^2}{2} \right\} \sum_{k=2}^{\infty} N_{\mathscr{C}}(\varepsilon_k) \exp\left\{ -\frac{1}{2} (\lambda_{k-1} - \rho_k^* \lambda_0)^2 \right\}, \end{aligned} \tag{2.5}$$

for any positive sequences $\{\varepsilon_j\}_{j=1}^{\infty}, \{\lambda_j\}_{j=0}^{\infty}$, $\varepsilon_j \to 0$ as $j \to \infty$, satisfying for all $k \geq 2$ the conditions

$$\lambda_{k-1} \geq 1, \tag{2.6}$$

$$\rho_k^* < \lambda_{k-1}/\lambda_0 < 1/\rho_k^*. \tag{2.7}$$

Here

$$\rho_k^* := \frac{2M\sigma + 1}{2\underline{\sigma}} \varepsilon_{k-1} \tag{2.8}$$

and

$$\sigma := \sup_{x \in \mathscr{C}} \|x\|, \qquad \underline{\sigma} := \inf_{x \in \mathscr{C}} \|x\| > 0.$$

The proof of Theorem 2.1 is based on the bound (2.5).

REMARK 2.1. The following property of metric entropy is used in the sequel. For any $\mathscr{C}_1 \subset \mathscr{C}_2$ and $\varepsilon > 0$,

$$N(\mathscr{C}_1, \varepsilon) \leq N(\mathscr{C}_2, \varepsilon/2). \tag{2.9}$$

This relation is semievident; details can be found in Samorodnitsky (1986).

THEOREM 2.1. *Suppose there is a constant $\theta > 0$ such that the function*

$$H(\delta, \varepsilon) := \ln N(x_0; \delta, \theta, \varepsilon), \qquad 0 < \delta \leq \theta, \ \varepsilon > 0, \tag{2.10}$$

satisfies the condition

$$\lim_{s \to 0} \int_0^s H(s, t)^{1/2} \, dt = 0. \tag{2.11}$$

Then the isonormal process L restricted to $\mathscr{C}$ is continuous at x_0.

Remark 2.2. An alternative sufficient condition for continuity of the isonormal process at x_0 that follows directly from (1.1) is

$$\int_0^1 H(\mathscr{C}(x_0;\delta),t)^{1/2}\,dt<\infty \quad \text{for all } \delta>0 \text{ small enough.} \tag{2.12}$$

It is easily seen that the condition given by Theorem 2.1 is strictly weaker than (2.12). [Suppose that (2.12) holds for all $\delta \le \theta$. Then by (2.9) for any $s \le \theta$, $H(s,t) \le H(\mathscr{C}(x_0;\theta), t/2)$, so (2.12) implies (2.11). The examples given in the end of this section represent situations in which Theorem 2.1 works while (2.12) fails.]

Proof of Theorem 2.1. By its definition $H(s,t)$ is nonincreasing in t for each s. Also (2.9) implies that for any $0<\delta_1<\delta_2\le\theta$, $\ln N(x_0;\delta_1,\delta_2,\varepsilon)\le H(\delta_1,\varepsilon/2)$. Let

$$H_0(s,t) := H\left(s,\frac{t}{12}\right), \tag{2.13}$$

$$H_1(s,t)^{1/2} := H_0(s,t)^{1/2} + |\ln s| + \ln|\ln t|, \tag{2.14}$$

$$H_2(s,t)^{1/2} := H_1(s,t)^{1/2} + \frac{1}{s}\left[\int_0^s H_1(s,u)^{1/2}\,du\right]^{1/2}. \tag{2.15}$$

These functions still satisfy (2.11).

Let $t_1>0$ be such that the function $\phi(t)=\ln|\ln t|$ is nonnegative and nonincreasing on the interval $(0,t_1]$. It follows then that for any $0\le t\le t_1$, $H_2(s,t)\ge H(s,t/12)$ and $H_2(s,t)$ is nonincreasing in t for each s. Since the function $H_1(s,t)$ satisfies (2.11), there is $s_1>0$ such that for any $0<s\le s_1$,

$$\int_0^s H_1(s,t)^{1/2}\,dt \le 1. \tag{2.16}$$

Finally, let $t_2>0$ be such that the inequality $|\ln t|/t\ge 81$ holds for any $0<t\le t_2$. Set

$$I(s) := \int_0^s H_2(s,t)^{1/2}\,dt \tag{2.17}$$

and

$$g(s) := \left[\sup_{u\le s}\left\{I(u)+I(u)^{1/2}\right\}\right]^{1/2}. \tag{2.18}$$

Then $g(s)\downarrow 0$ as $s\to 0$. We are going to show that for a separable version of the process

$$\limsup_{\delta\to 0}\left\{\frac{|Lx-Lx_0|}{g(\|x-x_0\|)} : x\in\mathscr{C}(x_0;\delta)\right\}\le 2 \quad \text{a.s.} \tag{2.19}$$

This implies, of course, that the isonormal process L restricted to $\mathscr{C}$ is continuous at x_0. As the supremum in (2.19) is taken over a decreasing family of sets, it is enough to prove that

$$\lim_{\delta \to 0} P\left[\sup\left\{\frac{|Lx - Lx_0|}{g(\|x - x_0\|)} : x \in \mathscr{C}(x_0; \delta)\right\} > 2\right] = 0. \tag{2.20}$$

Since we are talking in (2.20) about a limit as $\delta \to 0$, we may and will assume from now on that

$$0 < \delta \le \min(t_1, s_1, t_2). \tag{2.21}$$

Set $\delta_i := 3^{-i}\delta$, $i = 0, 1, 2, \dots$. Then $\delta_i \downarrow 0$ as $i \to \infty$. Consequently, the monotonicity of $g(s)$ implies that

$$\begin{aligned} &P\left[\sup\left\{\frac{|Lx - Lx_0|}{g(\|x - x_0\|)} : x \in \mathscr{C}(x_0; \delta)\right\} > 2\right] \\ &\le \sum_{i=0}^{\infty} P\left[\sup\left\{\frac{|Lx - Lx_0|}{g(\|x - x_0\|)} : x \in \mathscr{C}(x_0; \delta_i) \setminus \mathscr{C}(x_0; \delta_{i+1})\right\} > 2\right] \\ &\le \sum_{i=0}^{\infty} P\left[\sup\{|Lx - Lx_0| : x \in \mathscr{C}(x_0; \delta_i) \setminus \mathscr{C}(x_0; \delta_{i+1})\} > 2g(\delta_{i+1})\right]. \end{aligned} \tag{2.22}$$

We are going to estimate the probabilities in the last sum in (2.22). Denote

$$\mathscr{C}_i := \{x - x_0, x \in \mathscr{C}(x_0; \delta_i) \setminus \mathscr{C}(x_0; \delta_{i+1})\}.$$

A separable version of the isonormal process is still linear with probability 1 on countable sets. Therefore, letting $\mathscr{B}_i$ be a countable dense subset of $\mathscr{C}(x_0; \delta_i) \setminus \mathscr{C}(x_0; \delta_{i+1})$, we get

$$\begin{aligned} &P\left[\sup\{|Lx - Lx_0| : x \in \mathscr{C}(x_0; \delta_i) \setminus \mathscr{C}(x_0; \delta_{i+1})\} > 2g(\delta_{i+1})\right] \\ &\quad = P\left[\sup\{|Lx - Lx_0| : x \in \mathscr{B}_i\} > 2g(\delta_{i+1})\right] \\ &\quad = P\left[\sup\{|Lx| : x \in \mathscr{B}_i - x_0\} > 2g(\delta_{i+1})\right] \\ &\quad = P\left\{\sup_{x \in \mathscr{C}_i} |Lx| > 2g(\delta_{i+1})\right\}, \end{aligned} \tag{2.23}$$

since $\mathscr{B}_i - x_0 = \{x : x + x_0 \in \mathscr{B}_i\}$ is dense in $\mathscr{C}_i$.

We would like to apply the bound (2.5) to the last probability in (2.23). However, $\mathscr{C}_i$ may not satisfy Condition A; thus we first define

$$\hat{\mathscr{C}}_i := \left\{x \frac{\delta_i}{\|x\|}, x \in \mathscr{C}_i\right\}.$$

The same linearity argument used to establish (2.23) shows that with probability 1,

$$\sup_{x \in \hat{\mathscr{C}}_i} |Lx| = \sup_{x \in \mathscr{C}_i} \left\{|Lx| \frac{\delta_i}{\|x\|}\right\} \ge \sup_{x \in \mathscr{C}_i} |Lx|.$$

Thus,

$$P\left\{\sup_{x \in \hat{\mathscr{C}}_i} |Lx| > 2g(\delta_{i+1})\right\} \ge P\left\{\sup_{x \in \mathscr{C}_i} |Lx| > 2g(\delta_{i+1})\right\}. \tag{2.24}$$

Moreover, the points in $\hat{\mathscr{C}}_i$ have equal norms (δ_i), and so $\hat{\mathscr{C}}_i$ satisfies Condition A with $M = 0$. We apply the bound (2.5) to $\hat{\mathscr{C}}_i$, taking $\sigma = \underline{\sigma} = \delta_i$, $M = 0$. For any two sequences $\{\varepsilon_j^{(i)}\}_{j=1}^{\infty}$, $\{\lambda_j^{(i)}\}_{j=0}^{\infty}$ satisfying (2.6) and (2.7), we thus obtain

$$P\left\{\sup_{x\in\hat{\mathscr{C}}_i}|Lx| > \lambda_0^{(i)}\delta_i + \sum_{j=1}^{\infty}\varepsilon_j^{(i)}\lambda_j^{(i)}\right\}$$

$$\begin{aligned}(2.25)\qquad &\le \sqrt{\frac{2}{\pi}}\left(\lambda_0^{(i)}\right)^{-1}\exp\left\{-\frac{1}{2}\left(\lambda_0^{(i)}\right)^2\right\}N\left(\hat{\mathscr{C}}_i, \varepsilon_1^{(i)}\right)\\ &\quad+\sqrt{\frac{2}{\pi}}\left(\lambda_0^{(i)}\right)^{-1}\exp\left\{-\frac{1}{2}\left(\lambda_0^{(i)}\right)^2\right\}\sum_{j=2}^{\infty}N\left(\hat{\mathscr{C}}_i, \varepsilon_j^{(i)}\right)\\ &\quad\times\exp\left\{-\frac{1}{2}\left(\lambda_{j-1}^{(i)} - \rho_j^*\lambda_0^{(i)}\right)^2\right\}.\end{aligned}$$

Note that for any $x, y \in \mathscr{C}_i$,

$$\begin{aligned}\left\|x\frac{\delta_i}{\|x\|} - y\frac{\delta_i}{\|y\|}\right\| &= \delta_i\left\|\frac{x}{\|x\|} - \frac{y}{\|x\|} + \frac{y}{\|x\|} - \frac{y}{\|y\|}\right\|\\ &\le \delta_i\left(\frac{1}{\|x\|}\|x-y\| + \|y\|\left|\frac{1}{\|x\|} - \frac{1}{\|y\|}\right|\right)\\ &\le \frac{2\delta_i}{\|x\|}\|x-y\| \le \frac{2\delta_i}{\delta_{i+1}}\|x-y\|\\ &= 6\|x-y\|.\end{aligned}$$

Consequently, for any $\varepsilon > 0$,

$$(2.26)\qquad N\left(\hat{\mathscr{C}}_i, \varepsilon\right) \le N(\mathscr{C}_i, \varepsilon/6) = N\left(\mathscr{C}(x_0; \delta_i)\setminus\mathscr{C}(x_0; \delta_{i+1}), \varepsilon/6\right).$$

Thus, (2.25) can be rewritten as

$$P\left\{\sup_{x\in\mathscr{C}_i}|Lx| > \lambda(i)\delta_i + \sum_{j=1}^{\infty}\varepsilon_j^{(i)}\lambda_j^{(i)}\right\}$$

$$\begin{aligned}(2.27)\qquad &\le (2/\pi)^{1/2}\lambda(i)^{-1}\exp\left\{-\lambda(i)^2/2\right\}N\left(x_0; \delta_{i+1}, \delta_i, \varepsilon_1^{(i)}/6\right)\\ &\quad+(2/\pi)^{1/2}\lambda(i)^{-1}\exp\left\{-\lambda(i)^2/2\right\}\sum_{j=2}^{\infty}N\left(x_0; \delta_{i+1}, \delta_i, \varepsilon_j^{(i)}/6\right)\\ &\quad\times\exp\left\{-\tfrac{1}{2}\left(\lambda_{j-1}^{(i)} - \rho_j^*\lambda(i)\right)^2\right\},\end{aligned}$$

where $\lambda(i)$ denotes $\lambda_0^{(i)}$. Now we specify the $\varepsilon_j^{(i)}$ and $\lambda_j^{(i)}$ sequences. Set

$$(2.28)\qquad \lambda(i) = \lambda_0^{(i)} := \frac{g(\delta_{i+1})}{\delta_i},$$

$$(2.29)\qquad \varepsilon_j^{(i)} := \frac{1}{3}\delta_i 2^{-(j-1)},\qquad j = 1, 2, \ldots,$$

$$(2.30)\qquad \lambda_j^{(i)} := \frac{1}{3}\lambda(i)\frac{H_2\left(\delta_{i+1}, 2^{-j}\delta_{i+1}\right)^{1/2}}{H_2\left(\delta_{i+1}, 2^{-1}\delta_{i+1}\right)^{1/2}},\qquad j = 1, 2, \ldots.$$

Note that $g(s) \geq s^{1/2}|\ln s|^{1/2}$; consequently, by (2.21), $\lambda_j^{(i)} \geq 1$ for all $i \geq 0$ and $j \geq 0$. Furthermore, note that $\rho_j^* = 2^{-(j-1)}/3$ does not depend on i, and that the condition $\rho_j^* < \lambda_{j-1}^{(i)}/\lambda(i)$ trivially holds for all $i \geq 0$, $j \geq 2$. The second part of condition (2.7), namely $\lambda_{j-1}^{(i)}/\lambda(i) < 1/\rho_j^*$, becomes, after a substitution,

$$\frac{3H_2(\delta_{i+1}, 2^{-1}\delta_{i+1})^{1/2}}{H_2(\delta_{i+1}, 2^{-(j-1)}\delta_{i+1})^{1/2}} > \frac{2^{-(j-1)}}{3}. \tag{2.31}$$

Note that for any $0 < d < 1$ and for any v, specifically $v = t/2$ or $v = t$, we have, by (2.16),

$$\begin{aligned}\frac{H_2(t, dt)^{1/2}}{H_2(t, v)^{1/2}} &= \frac{H_1(t, dt)^{1/2} + 1/t\left[\int_0^t H_1(t, u)^{1/2}\,du\right]^{1/2}}{H_1(t, v)^{1/2} + 1/t\left[\int_0^t H_1(t, u)^{1/2}\,du\right]^{1/2}} \\ &\leq \frac{(1/dt)\int_0^{dt} H_1(t, u)^{1/2}\,du + 1/t\left[\int_0^t H_1(t, u)^{1/2}\,du\right]^{1/2}}{1/t\left[\int_0^t H_1(t, u)^{1/2}\,du\right]^{1/2}} \\ &\leq \frac{1}{d}\left[\int_0^t H_1(t, u)^{1/2}\,du\right]^{1/2} + 1 \leq \frac{2}{d},\end{aligned} \tag{2.32}$$

whenever $0 < t \leq s_1$. Then (2.31) follows from (2.32) with $t = \delta_{i+1}$, $d = 2^{-(j-1)}$. Furthermore,

$$\begin{aligned}\lambda(i)\delta_i + \sum_{j=1}^{\infty} \varepsilon_j^{(i)}\lambda_j^{(i)} &= \lambda(i)\delta_i\Bigg[1 + \frac{2}{9H_2(\delta_{i+1}, 2^{-1}\delta_{i+1})^{1/2}} \\ &\qquad \times \sum_{j=1}^{\infty} 2^{-j}H_2(\delta_{i+1}, 2^{-j}\delta_{i+1})^{1/2}\Bigg] \\ &\leq \lambda(i)\delta_i\left[1 + \frac{4}{9}\frac{\int_0^{\delta_{i+1}/2} H_2(\delta_{i+1}, u)^{1/2}\,du}{\delta_{i+1}H_2(\delta_{i+1}, 2^{-1}\delta_{i+1})^{1/2}}\right].\end{aligned} \tag{2.33}$$

Here the monotonicity of $H_2(s, t)$ in t has been used. Similar to (2.32) we have for $0 < d < 1$,

$$\begin{aligned}\frac{\int_0^{dt} H_2(t, u)^{1/2}\,du}{dt H_2(t, dt)^{1/2}} &= \frac{\int_0^{dt}\left\{H_1(t, u)^{1/2} + 1/t\left[\int_0^t H_1(t, s)^{1/2}\,ds\right]^{1/2}\right\}du}{dt\left\{H_1(t, dt)^{1/2} + 1/t\left[\int_0^t H_1(t, s)^{1/2}\,ds\right]^{1/2}\right\}} \\ &\leq \frac{\int_0^{dt} H_1(t, u)^{1/2}\,du + d\left[\int_0^t H_1(t, s)^{1/2}\,ds\right]^{1/2}}{d\left[\int_0^t H_1(t, s)^{1/2}\,ds\right]^{1/2}} \\ &\leq 1 + \frac{1}{d}\left[\int_0^t H_1(t, s)^{1/2}\,ds\right]^{1/2} \leq \frac{2}{d},\end{aligned} \tag{2.34}$$

as long as $0 < t \leq s_1$. Consequently, taking $t = \delta_{i+1}$ and $d = \frac{1}{2}$,

$$\lambda(i)\delta_i + \sum_{j=1}^{\infty} \varepsilon_j^{(i)}\lambda_j^{(i)} \leq \lambda(i)\delta_i(1 + 8/9) < 2\lambda(i)\delta_i. \tag{2.35}$$

Consequently, (2.27) implies that

$$P\Big\{\sup_{x\in\mathscr{C}_i}|Lx| > 2g(\delta_{i+1})\Big\}$$

$$\begin{aligned}
&\le (2/\pi)^{1/2}\lambda(i)^{-1}\exp\{-\lambda(i)^2/2\}N(x_0;\,\delta_{i+1},\delta_i,\varepsilon_1^{(i)}/6)\\
&\quad+(2/\pi)^{1/2}\lambda(i)^{-1}\exp\{-\lambda(i)^2/2\}\\
&\quad\times\sum_{j=2}^{\infty}N(x_0;\,\delta_{i+1},\delta_i,\varepsilon_j^{(i)}/6)\exp\Big\{-\tfrac{1}{2}\big(\lambda_{j-1}^{(i)}-\rho_j^*\lambda(i)\big)^2\Big\}.
\end{aligned}\tag{2.36}$$

For every fixed $i = 0,1,2,\dots,$ let a_i and b_i be the first and the second terms, respectively, in the right-hand-side of (2.36). Then (2.22)–(2.24) and (2.36) imply that

$$P\left[\sup\left\{\frac{|Lx-Lx_0|}{g(\|x-x_0\|)}: x\in\mathscr{C}(x_0;\,\delta)\right\}>2\right]\le\sum_{i=0}^{\infty}a_i+\sum_{i=0}^{\infty}b_i.\tag{2.37}$$

We are going to estimate each of these sums separately. We have

$$\begin{aligned}
\sum_{i=0}^{\infty}a_i &\le \sum_{i=0}^{\infty}\exp\{-g(\delta_{i+1})^2/2\delta_i^2+\ln N(x_0;\,\delta_{i+1},\delta_i,\varepsilon_1^{(i)}/6)\}\\
&\le \sum_{i=0}^{\infty}\exp\{-g(\delta_{i+1})^2/2\delta_i^2+H(\delta_{i+1},\varepsilon_1^{(i)}/12)\}\\
&\le \sum_{i=0}^{\infty}\exp\{-g(\delta_{i+1})^2/2\delta_i^2+H_2(\delta_{i+1},\varepsilon_1^{(i)})\}\\
&\le \sum_{i=0}^{\infty}\exp\{-I(\delta_{i+1})/2\delta_i^2+H_2(\delta_{i+1},\delta_i/3)\}\\
&\le \sum_{i=1}^{\infty}\exp\{-H_2(\delta_i,\delta_i)^{1/2}/6\delta_{i-1}+H_2(\delta_i,\delta_i/3)\}\\
&= \sum_{i=1}^{\infty}\exp\Big\{-\big(H_2(\delta_i,\delta_i)^{1/2}/6\delta_{i-1}\big)\\
&\qquad\times\Big[1-6\delta_{i-1}\big(H_2(\delta_i,3^{-1}\delta_i)/H_2(\delta_i,\delta_i)^{1/2}\big)\Big]\Big\}.
\end{aligned}$$

Note that (2.32) implies that

$$\frac{H_2(\delta_i,3^{-1}\delta_i)^{1/2}}{H_2(\delta_i,\delta_i)^{1/2}}\le 6,$$

so

$$\delta_{i-1}H_2(\delta_i,3^{-1}\delta_i)^{1/2}\le 18\delta_iH_2(\delta_i,\delta_i)^{1/2}$$

and the last expression converges uniformly in i to zero as $\delta\to 0$. Consequently, for δ small,

$$\sum_{i=0}^{\infty}a_i\le\sum_{i=1}^{\infty}\exp\left\{-\frac{1}{12\delta_{i-1}}H_2(\delta_i,\delta_i)^{1/2}\right\}\le\sum_{i=1}^{\infty}\exp\{-|\ln\delta_i|\}=\sum_{i=1}^{\infty}\delta_i=\frac{\delta}{2}\to 0$$

as $\delta \to 0$. Next we consider the sum $\sum_{i=0}^{\infty} b_i$. Note that for all $j \ge 2$,

$$\begin{aligned} \lambda_{j-1}^{(i)} - \rho_j^* \lambda(i) &= \lambda(i) \left[\frac{H_2(\delta_{i+1}, 2^{-(j-1)}\delta_{i+1})^{1/2}}{3H_2(\delta_{i+1}, 2^{-1}\delta_{i+1})^{1/2}} - \frac{2^{-(j-1)}}{3} \right] \\ &\ge \lambda(i) \frac{H_2(\delta_{i+1}, 2^{-(j-1)}\delta_{i+1})^{1/2}}{6H_2(\delta_{i+1}, 2^{-1}\delta_{i+1})^{1/2}}. \end{aligned} \tag{2.38}$$

Using (2.38), we get for small δ,

$$\begin{aligned} &\sum_{j=2}^{\infty} N\left(x_0; \delta_{i+1}, \delta_i, \frac{\varepsilon_j^{(i)}}{6}\right) \exp\left\{ -\frac{1}{2}\left(\lambda_{j-1}^{(i)} - \rho_j^* \lambda(i)\right)^2 \right\} \\ &\le \sum_{j=2}^{\infty} \exp\left\{ H_2(\delta_{i+1}, 3^{-1}\delta_i 2^{-(j-1)}) - \frac{g(\delta_{i+1})^2}{72\delta_i^2} \frac{H_2(\delta_{i+1}, 2^{-(j-1)}\delta_{i+1})}{H_2(\delta_{i+1}, 2^{-1}\delta_{i+1})} \right\} \\ &\le \sum_{j=2}^{\infty} \exp\left\{ H_2(\delta_{i+1}, 2^{-(j-1)}\delta_{i+1}) - \frac{g(\delta_{i+1})^2}{72\delta_i^2} \frac{H_2(\delta_{i+1}, 2^{-(j-1)}\delta_{i+1})}{H_2(\delta_{i+1}, 2^{-1}\delta_{i+1})} \right\} \\ &\le \sum_{j=2}^{\infty} \exp\left\{ -\frac{1}{72\delta_i} \frac{H_2(\delta_{i+1}, 2^{-(j-1)}\delta_{i+1})}{H_2(\delta_{i+1}, \delta_{i+1}/2)} \frac{I(\delta_{i+1})}{\delta_i} \right. \\ &\qquad\qquad \left. \times \left[1 - 72 \frac{H_2(\delta_{i+1}, \delta_{i+1}/2)\delta_i^2}{I(\delta_{i+1})} \right] \right\} \\ &\le \sum_{j=2}^{\infty} \exp\left\{ -\frac{1}{216\delta_i} \frac{H_2(\delta_{i+1}, 2^{-(j-1)}\delta_{i+1})}{H_2(\delta_{i+1}, \delta_{i+1}/2)} H_2(\delta_{i+1}, \delta_{i+1})^{1/2} \right. \\ &\qquad\qquad \left. \times \left[1 - 216 \frac{H_2(\delta_{i+1}, \delta_{i+1}/2)}{H_2(\delta_{i+1}, \delta_{i+1})^{1/2}} \delta_i \right] \right\} \\ &\le \sum_{j=2}^{\infty} \exp\left\{ -\frac{1}{216\delta_i} \frac{H_2(\delta_{i+1}, 2^{-(j-1)}\delta_{i+1})^{1/2}}{H_2(\delta_{i+1}, \delta_{i+1}/2)^{1/2}} H_2(\delta_{i+1}, \delta_{i+1})^{1/2} \right. \\ &\qquad\qquad \left. \times \left[1 - 216 \frac{H_2(\delta_{i+1}, \delta_{i+1}/2)^{1/2}}{H_2(\delta_{i+1}, \delta_{i+1})^{1/2}} \delta_i H_2\left(\delta_{i+1}, \frac{\delta_{i+1}}{2}\right)^{1/2} \right] \right\} \\ &\le \sum_{j=2}^{\infty} \exp\left\{ -\left(\frac{1}{216}\right)\left(\frac{1}{4}\right) H_2(\delta_{i+1}, 2^{-(j-1)}\delta_{i+1})^{1/2} \left(\frac{1}{\delta_i}\right) \right. \\ &\qquad\qquad \left. \times \left[1 - 216 \cdot 4\delta_i H_2(\delta_{i+1}, \delta_{i+1})^{1/2} 4 \right] \right\}. \end{aligned}$$

But $\delta H_2(\delta,\delta)^{1/2} \to 0$ as $\delta \to 0$. Consequently for small δ, uniformly over i,

$$\begin{aligned}\sum_{j=2}^{\infty} & N\big(x_0;\, \delta_{i+1}, \delta_i, \varepsilon_j^{(i)}/6\big)\exp\Big\{-\tfrac{1}{2}\big(\lambda_{j-1}^{(i)} - \rho_j^* \lambda(i)\big)^2\Big\} \\ &\le \sum_{j=2}^{\infty} \exp\Big\{-(1/1728)(1/\delta_i) H_2\big(\delta_{i+1}, 2^{-(j-1)}\delta_{i+1}\big)^{1/2}\Big\} \\ &\le \sum_{j=2}^{\infty} \exp\big\{-2\ln|\ln 2^{-(j-1)}\delta_{i+1}|\big\} \\ &\le \sum_{j=2}^{\infty} \big((j-1)\ln 2\big)^{-2} := c < \infty,\end{aligned}$$

independently of i. Consequently for small δ,

$$\sum_{i=0}^{\infty} b_i \le c \sum_{i=0}^{\infty} a_i.$$

Thus $\sum_{i=0}^{\infty} b_i$ is finite for small δ and $\sum_{i=0}^{\infty} b_i \to 0$ as $\delta \to 0$. Consequently, (2.20) is proven and then (2.19) follows. □

The following two examples are taken from Dudley (1973).

EXAMPLE 2.1. Let $\{a_k\}$ be a sequence of positive numbers with $1 \ge a_k \downarrow 0$. For $k = 1, 2, \dots$, let $\mathscr{C}_k$ be a cube of dimension k^2 and side $2a_k/k^2$ centered at 0. Let the cubes $\mathscr{C}_k$ lie in orthogonal subspaces. Let $\mathscr{C} = \bigcup_{k=1}^{\infty} \mathscr{C}_k$.

Consider the origin $x_0 = 0$. Letting $a_k \downarrow 0$ slowly, we can make $\varepsilon^2 H(C_0(\delta), \varepsilon) \to 0$ as slowly as desired; thus the integral (2.12) can diverge for all $\delta > 0$. Nevertheless, the isonormal process is continuous a.s. at $x_0 = 0$. We show that Theorem 2.1 works here.

For every fixed k and ε,

$$N(\mathscr{C}_k, \varepsilon) \le \max\left\{\left(\frac{2a_k}{k\varepsilon}\right)^{k^2}, 1\right\}$$

and for fixed $0 < \delta_1 < \delta_2$, $\varepsilon > 0$,

$$N(0;\, \delta_1, \delta_2, \varepsilon) \le N\big(\mathscr{C}_0(\delta_1)^c, \varepsilon/2\big) \le \sum_{k=1}^{n(\delta_1)} \left(\frac{4a_k}{k\varepsilon}\right)^{k^2} + 1, \tag{2.39}$$

where

$$n(s) := \max\{k\colon a_k/k \ge s\}.$$

Define also

$$m(s) := \max\{k\colon a_k/k \ge s^{1/4}\}.$$

Both $n(s)$ and $m(s)$ increase to infinity as $s \downarrow 0$. We have

$$H(s,t) \le \ln\left[\sum_{k=1}^{n(s)} \left(\frac{4a_k}{kt}\right)^{k^2} + 1\right]. \tag{2.40}$$

We will show that

$$\lim_{s\to 0}\int_0^s\left\{\ln\left[\sum_{k=1}^{n(s)}\left(\frac{4a_k}{kt}\right)^{k^2}+1\right]\right\}^{1/2}dt=0. \tag{2.41}$$

The obvious relations $\ln(x+y)\le \ln 2x+\ln 2y$, $x, y\ge 1$ and $(x+y)^{1/2}\le x^{1/2}+y^{1/2}$ yield that

$$\begin{aligned}\int_0^s&\left\{\ln\left[\sum_{k=1}^{n(s)}\left(\frac{4a_k}{kt}\right)^{k^2}+1\right]\right\}^{1/2}dt\\ &\le\int_0^s\left\{\ln\left[\sum_{k=1}^{m(s)}\left(\frac{4a_k}{kt}\right)^{k^2}\right]\right\}^{1/2}dt\\ &\quad+\int_0^s\left\{\ln\left[\sum_{k=m(s)+1}^{n(s)}\left(\frac{4a_k}{kt}\right)^{k^2}\right]\right\}^{1/2}dt+s(4\ln 2)^{1/2}.\end{aligned} \tag{2.42}$$

Denote by I_1 and I_2 the first and second integrals in the right-hand side of (2.42). We are going to show that

$$\lim_{s\to 0} I_1=\lim_{s\to 0} I_2=0.$$

We have, for small s,

$$\begin{aligned}I_1&\le\int_0^s\left\{\sum_{k=1}^{m(s)}\left[k^2\ln\left(\frac{4a_k}{kt}\right)+\ln 2\right]\right\}^{1/2}dt\\ &\le s(m(s)\ln 2)^{1/2}+\int_0^s\sum_{k=1}^{m(s)}k\left[\ln\left(\frac{4a_k}{kt}\right)\right]^{1/2}dt\\ &\le s(m(s)\ln 2)^{1/2}+\sum_{k=1}^{m(s)}k\int_0^s\left[\ln\left(\frac{4a_1}{t}\right)\right]^{1/2}dt\\ &\le s(m(s)\ln 2)^{1/2}+\frac{m(s)(m(s)+1)}{2}2s\left[\ln\left(\frac{4a_1}{s}\right)\right]^{1/2}\end{aligned}$$

and this goes to zero as $s\to 0$ because $m(s)\le a_1 s^{-1/4}$. Further, for every $m(s)<k\le n(s)$,

$$\begin{aligned}\left(\frac{4a_k}{kt}\right)^{k^2}&=\left[\left(\frac{4a_k}{kt}\right)^{kt/4a_k}\right]^{4ka_k/t}\\ &\le\left(e^{1/e}\right)^{4ka_k/t}\le\left(e^{1/e}\right)^{4a_k^2/st}\le\exp\left\{\frac{4a^2_{m(s)+1}}{est}\right\},\end{aligned} \tag{2.43}$$

where the first inequality in (2.43) follows from the fact that

$$\max_{x>0}x^{1/x}=e^{1/e}$$

and the second inequality follows from the fact that $a_k/k \geq s$. We conclude that

$$(2.44)\quad \begin{aligned} I_2 &\leq \int_0^s \left\{ \ln\left[n(s)\exp\left(\frac{4a_{m(s)+1}^2}{est} \right) \right] \right\}^{1/2} dt \\ &\leq s[\ln n(s)]^{1/2} + 2e^{-1/2}a_{m(s)+1}s^{-1/2}\int_0^s t^{-1/2}\,dt \\ &= s[\ln n(s)]^{1/2} + 4e^{-1/2}a_{m(s)+1}. \end{aligned}$$

The first term in (2.44) converges to zero because $n(s) \leq s^{-1}a_{n(s)}$, while the second term converges to zero because $m(s)\uparrow\infty$ as $s\downarrow 0$. Therefore, the conditions of Theorem 2.1 hold.

REMARK 2.3. Note that the GC property of $\mathscr{C}$ follows. We have just proved that the isonormal process is continuous at $x_0 = 0$. Certainly, for any $x_0 \in \mathscr{C}$ other than the origin, for any sufficiently small $\delta > 0$ the metric entropy of $\mathscr{C}(x_0, \delta)$ is bounded by a logarithmic function of ε, and so the integral (2.12) is finite. In this example, consequently, we have proved that $\mathscr{C}$ is a GC class by proving that the isonormal process is continuous at the (only) "difficult" point $x_0 = 0$.

EXAMPLE 2.2. Again, let $\{a_k\}$ be a sequence of positive numbers with $a_k \downarrow 0$. Let

$$\mathscr{C} = \left\{ \phi_n a_n (\ln n)^{-1/2}, n \geq 2 \right\} \cup \{0\},$$

ϕ_n orthonormal. Consider $x_0 = 0$. If $a_k \downarrow 0$ slowly ($a_k = (\ln\ln k)^{-1/2}$ is slow enough), then the integral (2.12) diverges for all $\delta > 0$. Let us show that Theorem 2.1 can be applied to this example as well. Set

$$M(s) := \min\left\{ n\colon a_n^2/\ln n \leq s^2 \right\}.$$

Then, for any $0 < \delta_1 < \delta_2$, $\varepsilon > 0$,

$$(2.45)\quad N(0; \delta_1, \delta_2, \varepsilon) \leq M(\delta_1) - M(\delta_2) + 1.$$

This implies that

$$(2.46)\quad N_0(\delta_1, \delta_2, \varepsilon) \leq 2M(\delta_1),$$

so that

$$(2.47)\quad H(s,t) \leq \ln 2 + \ln M(s).$$

Then

$$\int_0^s H(s,t)^{1/2}\,dt \leq s[\ln 2 + \ln M(s)]^{1/2}$$

and we have to show that

$$\lim_{s\to 0} s^2 \ln M(s) = 0.$$

But this is clear, since

$$\frac{a_{M(s)-1}^2}{\ln(M(s)-1)} > s^2,$$

so

$$M(s) < 2\exp\{s^{-2}a_{M(s)-1}\},$$

and also $M(s)\uparrow\infty$ as $s\downarrow 0$.

REMARK 2.4. Again, we have proved the GC property of $\mathscr{C}$, since at any point $x_0 \in \mathscr{C}$ other than the origin the isonormal process cannot have a discontinuity.

APPENDIX

Theorem 1.1 is proven here. The idea of the proof is the same as that of Marcus and Shepp (1971) in the case of Gaussian processes on $[0,1]$.

Let $\mathscr{C}^*$ be a countable dense subset of $\mathscr{C}$. For any $I \subset \mathscr{C}$ and $\varepsilon > 0$, define

$$A_\varepsilon(I) = \left\{\omega \,\middle|\, \begin{array}{l}\text{there is an } r > 0 \text{ such that for every } \delta > 0 \text{ there are}\\ t_1 \in \mathscr{C}^* \cap I \text{ and } t_2 \in \mathscr{C}^* \text{ such that } d(t_1, t_2) < \delta \text{ and}\\ |X(t_1) - X(t_2)| \geq \varepsilon + r\end{array}\right\}.$$

LEMMA A.1.

$$P(A_\varepsilon(I)) = 0 \text{ or } 1.$$

PROOF. Let $s_1, s_2, \dots$ be an enumeration of the points of $\mathscr{C}^*$. Then there exists a sequence of orthonormal Gaussian variables $Y_1, Y_2, \dots$ and real numbers $\{a_{ij}\}$, $i \leq j$, such that for each i,

$$X(s_i) = \sum_{j=1}^{i} a_{ij}Y_j. \tag{A.1}$$

Let $a_{ij} = 0$ for $j > i$ and let for $t_1, t_2 \in \mathscr{C}^*$, $i(1)$ and $i(2)$ denote the places of t_1 and t_2 correspondingly in the fixed enumeration of $\mathscr{C}^*$. Then

$$\sigma^2(t_1, t_2) = E[X(t_1) - X(t_2)]^2 = \sum_{j=1}^{\infty} (a_{i(1)j} - a_{i(2)j})^2. \tag{A.2}$$

Note that the covariance function of the process $R(s,t)$ is continuous on $\mathscr{C}\times\mathscr{C}$, since $X(t)$ is a.s. continuous at each point of $\mathscr{C}$. Then the function $\sigma^2(s,t)$ is also continuous on $\mathscr{C}\times\mathscr{C}$, and, because of compactness, it is uniformly continuous. Consequently, for every $\theta > 0$ there is an $\eta = \eta(\theta) > 0$ such that $d(t_1, t_2) \leq \eta$ implies that $\sigma^2(t_1, t_2) \leq \theta^2$. If also $t_1, t_2 \in \mathscr{C}^*$, then (A.2) implies that $|a_{i(1)j} - a_{i(2)j}| \leq \theta$ for every j. Let us rewrite the event $A_\varepsilon(I)$ as

$$A_\varepsilon(I) = \left\{\omega \,\middle|\, \begin{array}{l}\text{there is an } r > 0 \text{ such that for every } \delta > 0 \text{ there are}\\ t_1 \in \mathscr{C}^* \cap I \text{ and } t_2 \in \mathscr{C}^* \text{ such that } d(t_1, t_2) < \delta \text{ and}\\ |\sum_{j=1}^{\infty} Y_j(\omega)(a_{i(1)j} - a_{i(2)j})| \geq \varepsilon + r\end{array}\right\}.$$

We claim that $A_\varepsilon(I)$ is a tail event for the sequence $Y_1, Y_2, \dots$. It is sufficient to show that if ω_1 and ω_2 are such that for some finite m, $Y_n(\omega_1) = Y_n(\omega_2)$ for all $n > m$, then $\omega_1 \in A_\varepsilon(I)$ implies $\omega_2 \in A_\varepsilon(I)$.

Suppose the contrary, i.e., $\omega_1 \in A_\varepsilon(I)$ and $\omega_2 \notin A_\varepsilon(I)$. Then for some positive $r(\omega_1)$, for all sufficiently small $\delta > 0$ there are $t_1 \in \mathscr{C}^* \cap I$ and $t_2 \in \mathscr{C}^*$ such that $d(t_1, t_2) < \delta$, while

$$\left| \sum_{j=1}^{\infty} Y_j(\omega_1)(a_{i(1)j} - a_{i(2)j}) \right| \geq \varepsilon + r(\omega_1),$$

$$\left| \sum_{j=1}^{\infty} Y_j(\omega_2)(a_{i(1)j} - a_{i(2)j}) \right| < \varepsilon + \tfrac{1}{2}r(\omega_1).$$

[Here, as before, $i(1)$ and $i(2)$ denote the places of t_1 and t_2, respectively, in the fixed enumeration of $\mathscr{C}^*$.] We conclude (recall that all sums have finite numbers of terms) that

$$\begin{aligned}
\tfrac{1}{2}r(\omega_1) &< \left| \sum_{j=1}^{\infty} Y_j(\omega_1)(a_{i(1)j} - a_{i(2)j}) \right| - \left| \sum_{j=1}^{\infty} Y_j(\omega_2)(a_{i(1)j} - a_{i(2)j}) \right| \\
&\leq \left| \sum_{j=1}^{\infty} Y_j(\omega_1)(a_{i(1)j} - a_{i(2)j}) - \sum_{j=1}^{\infty} Y_j(\omega_2)(a_{i(1)j} - a_{i(2)j}) \right| \qquad \text{(A.3)} \\
&= \left| \sum_{j=1}^{m} \big(Y_j(\omega_1) - Y_j(\omega_2)\big)(a_{i(1)j} - a_{i(2)j}) \right|.
\end{aligned}$$

The inequality (A.3), however, cannot hold for all positive δ, since its left-hand side is positive, while the right-hand side goes to zero as $\delta \to 0$. This contradiction shows that $A_\varepsilon(I)$ is a tail event. Consequently, Kolmogorov's zero–one law implies that $P(A_\varepsilon(I)) = 0$ or 1. □

We return now to the proof of Theorem 1.1. For any $I \subset \mathscr{C}$ and $\varepsilon > 0$ define

$$B_\varepsilon(I) = \left\{ \omega \middle| \begin{array}{l} \text{for any } \delta > 0 \text{ there are } t_1 \in \mathscr{C}^* \cap I \text{ and } t_2 \in \mathscr{C}^* \\ \text{such that } d(t_1, t_2) < \delta \text{ and } |X(t_1) - X(t_2)| \geq \varepsilon \end{array} \right\}.$$

Then $B_\varepsilon(I) \subset A_{\varepsilon/2}(I)$. If $X(t)$ is not sample-continuous, then $P(B_\varepsilon(\mathscr{C})) > 0$ for some $\varepsilon > 0$. Then $P(A_{\varepsilon^*}(\mathscr{C})) > 0$ for some $\varepsilon^* > 0$. This implies that $P(A_{\varepsilon^*}(\mathscr{C})) = 1$.

Let $d_0 = \sup_{t_1, t_2 \in \mathscr{C}} d(t_1, t_2) < \infty$. The compactness of $\mathscr{C}$ implies that we can cover it by a finite number of compact sets $\mathscr{C}_1^{(1)}, \mathscr{C}_2^{(1)}, \dots, \mathscr{C}_{k(1)}^{(1)}$ of diameter at most $d_0/2$ each. Since

$$A_{\varepsilon^*}(\mathscr{C}) = \bigcup_{i=1}^{k(1)} A_{\varepsilon^*}(\mathscr{C}_i^{(1)}), \qquad \text{(A.4)}$$

we conclude, that for some $i(1)$, $1 \leq i(1) \leq k(1)$, $P(A_{\varepsilon^*}(\mathscr{C}_{i(1)}^{(1)})) > 0$. Thus $P(A_{\varepsilon^*}(\mathscr{C}_{i(1)}^{(1)})) = 1$. We divide now $\mathscr{C}_{i(1)}^{(1)}$ into compact subsets of diameter at most $d_0/4$, and so on. We obtain a sequence of nested compact nonempty sets

$$\mathscr{C} = \mathscr{C}_1^{(0)} > \mathscr{C}_{i(1)}^{(1)} > \mathscr{C}_{i(2)}^{(2)} > \cdots$$

with the following two properties: For each $k = 1, 2, \dots$ $P(A_{\varepsilon^*}(\mathscr{C}_{i(k)}^{(k)})) = 1$ and the diameter of $\mathscr{C}_{i(k)}^{(k)}$ is at most $d_0/2^k$. This sequence has to converge to a point $t_\infty \in \mathscr{C}$. Then, by the definition of $A_{\varepsilon^*}(I)$,

$$P\Big(\limsup_{t\to t_\infty,\, t\in\mathscr{C}^*} X(t) - \liminf_{t\to t_\infty,\, t\in\mathscr{C}^*} X(t) \ge \varepsilon^*\Big) \ge P\Big(\bigcap_{k=0}^{\infty} A_{\varepsilon^*}\big(\mathscr{C}_{i(k)}^{(k)}\big)\Big) = 1.$$

This contradicts the assumption that $X(t)$ is a.s. continuous at t_∞. This contradiction shows that $X(t)$ is sample-continuous.

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Note added in proof. M. Talagrand has pointed out that the proof of Theorem 2.1 can be simplified by using Borell's inequality [C. Borell, The Brunn–Minkowski inequality in Gauss space, *Invent. Math.* **30** (1975) 207–216] instead of our inequality (2.5).

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