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LARGE DEVIATIONS FOR THE EMPIRICAL FIELD OF A GIBBS MEASURE[1]

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Let S be a finite set and consider the space Ω of all configurations ω: $Z^d \to S$. For $j \in Z^d$, θ_j: $\Omega \to \Omega$ denotes the shift by j. Let V_n denote the cube $\{i \in Z^d\colon 0 \le i_k < n, 1 \le k \le d\}$. Let μ be a stationary Gibbs measure for a stationary summable interaction. Define ρ_{V_n} as the random probability measure on Ω given by

$$\rho_{V_n}(\omega) = n^{-d} \sum_{j \in V_n} \delta_{\theta_j \omega}.$$

Our principal result is that the sequence of measures $\mu \circ \rho_{V_n}^{-1}$, $n = 1, 2, \dots,$ satisfies the large deviation principle with normalization n^d and rate function the specific relative entropy $h(\cdot\,; \mu)$. Applying the contraction principle, we obtain a large deviation principle for the distribution of the empirical distributions; a detailed description of the resulting rate function is provided.

1. Introduction. Let S be a finite set and introduce the compact space $\Omega = S^{Z^d}$ of configurations ω: $Z^d \to S$. Let $\mathscr{M}(\Omega)$ denote the compact space of probability measures on Ω, and denote by $\mathscr{M}_s(\Omega)$ the subspace consisting of all elements of $\mathscr{M}(\Omega)$ that are invariant under all shifts θ_i: $\Omega \to \Omega$, $i \in Z^d$, where $(\theta_i \omega)(j) = \omega(j + i)$. A measure $\mu \in \mathscr{M}(\Omega)$ is called a Gibbs measure if its local conditional probabilities are specified by a stationary interaction potential. Precise definitions and conditions are given in Section 2.

Let us consider a stationary Gibbs measure $\mu \in \mathscr{M}_s(\Omega)$. Denote the cube $\{i = (i_1, \dots, i_d) \in Z^d\colon 0 \le i_k < n\}$ by V_n. Now define the empirical field ρ_{V_n} as the random element of $\mathscr{M}(\Omega)$ given by

$$\rho_{V_n}(\omega) = \frac{1}{n^d} \sum_{i \in V_n} \delta_{\theta_i \omega},$$

with δ_ω the unit point mass at ω. The distribution $\mu \circ \rho_{V_n}^{-1}$ of ρ_{V_n} under μ belongs to $\mathscr{M}(\mathscr{M}(\Omega))$. Our principal result is that the sequence

$$\mu \circ \rho_{V_n}^{-1}, \qquad n = 1, 2, \dots,$$

satisfies a large deviation principle, where the rate function is given by the specific relative entropy $h(\nu; \mu)$ of ν with respect to μ. More explicitly, we show in Section 3 that for open subsets A of $\mathscr{M}(\Omega)$,

$$\liminf_n \frac{1}{n^d} \log \mu\left[\rho_{V_n} \in A\right] \ge - \inf_{\nu \in A \cap \mathscr{M}_s(\Omega)} h(\nu; \mu),$$

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and in Section 4 it is shown that for closed subsets A of $\mathscr{M}(\Omega)$,

$$\limsup_n \frac{1}{n^d} \log \mu\left[\rho_{V_n} \in A\right] \le -\inf_{\nu \in A \cap \mathscr{M}_s(\Omega)} h(\nu; \mu).$$

By the contraction principle [13], page 5, we obtain, in particular, a large deviation principle for the empirical distributions

$$R_{V_n}(\omega) = \frac{1}{n^d} \sum_{i \in V_n} \delta_{\omega(i)}$$

on the state space S. In Section 5 we provide a detailed description of the resulting rate function.

For background on large deviations we recommend [13] and [12], but no prior knowledge of the subject will be assumed. The pioneering paper on large deviations for the empirical process of a Markov process is [3], and our indebtedness to this work will be clear.

Working independently of us, Olla [8] and Comets [1] obtained results similar to our Theorems 3.1 and 4.1; the method of proof of these authors is completely different from ours. For other work related to this paper, see [6], [4] and [7].

2. Preliminaries. The purpose of this section is to introduce notation, summarize some known results, which will be needed later, and derive some preliminary results. The reader may choose to skip this section and refer back to it as needed.

When working with a fixed space X, we write $Y^c := X - Y$ for any subset Y of X. If X is a compact metric space, $C(X)$ denotes the class of continuous functions from X to the reals, and $\mathscr{M}(X)$ is the class of probability measures on the Borel sets of X, topologized by weak convergence.

We will be concerned with the d-dimensional lattice Z^d, a fixed finite set S and the space Ω of all configurations ω: $Z^d \to S$ topologized by the compact product topology. For each $i \in Z^d$, the shift map θ_i: $\Omega \to \Omega$ is defined by $(\theta_i \omega)(k) = \omega(k + i)$, $k \in Z^d$. Let $\mathscr{M}_s(\Omega)$ denote the class of all $\mu \in \mathscr{M}(\Omega)$ that are *stationary* in the sense that $\mu = \mu \circ \theta_i$ for all $i \in Z^d$.

For any positive integer n, let V_n be the cube of all $t = (t_1, \dots, t_d) \in Z^d$ with all coordinates nonnegative and strictly less than n. For $s \in Z^d$ and $t \in Z^d$, the *distance* between s and t is the sum of the absolute values of differences between the components. For $V \subseteq Z^d$ and p a positive integer, $\partial_p V$ denotes the set of $t \in V^c$ whose distance from V is at most p. Let $N_p(0)$ be the set of all $t \in Z^d$ whose distance from 0 is at most p.

For $V \subseteq Z^d$, we denote by $\mathscr{F}_V$ the σ-field on Ω generated by the projections $\omega \to \omega(t)$, $t \in V$, and by ω_V the restriction of $\omega \in \Omega$ to V. Let $\mathscr{V}$ denote the class of finite subsets of Z^d. For $\mu \in \mathscr{M}(\Omega)$ and $V \in \mathscr{V}$, we denote by μ_V the marginal distribution of μ on S^V, by $\mu_V^*(\cdot|\eta)$ the regular *conditional distribution* on $\mathscr{F}_{Z^d}$ given $\mathscr{F}_{V^c}$ at the point $\eta \in \Omega$, and by $\mu_V(\cdot|\eta)$ its marginal distribution on S^V. Also, if p is a positive integer, $\mu_{V,p}^*(\cdot|\eta)$ or $\mu_{V,p}(\cdot|\eta)$ denotes the conditional distribution given $\mathscr{F}_{\partial_p V}$ viewed as a measure on Ω or S^V, respectively.

An *interaction potential* is a collection of maps $U(V,\cdot)\colon S^V \to R$, $V \in \mathscr{V}$. It is *stationary if*

$$U(V, \omega_V) = U\big(V + t, (\theta_t^{-1}\omega)_{V+t}\big). \tag{2.1}$$

Henceforth U will denote a fixed stationary interaction potential, which also satisfies the condition

$$\sum_{0 \in V \in \mathscr{V}} \|U(V,\cdot)\| < \infty, \tag{2.2}$$

where the norm is the supremum norm. A measure $\mu \in \mathscr{M}(\Omega)$ is a *Gibbs measure* with respect to U if $\mu_V(\cdot|\eta)$ can be chosen as

$$\mu_V(\omega_V|\eta) = \big(Z_V(\eta)\big)^{-1}\exp(-E_V(\omega_V|\eta)), \tag{2.3}$$

where

$$E_V(\omega_V|\eta) := \sum_{A\colon A\cap V \neq \varnothing} U(A, \zeta_A), \tag{2.4}$$

with $\zeta(i) := \omega(i)$ for $i \in V$ and $\zeta(i) := \eta(i)$ for $i \in V^c$, and

$$Z_V(\eta) := \sum_{\xi \in S^V} \exp(-E_V(\xi|\eta)). \tag{2.5}$$

μ is called a *Markov random field* if (2.3) only depends on $\eta_{\partial_1 V}$, and this means that U is a nearest-neighbor potential, i.e., $U(A,\cdot) \equiv 0$ whenever A contains two points the distance between which exceeds 1; see [11]. We define $\mathscr{G}(U)$ as the class of all Gibbs measures with respect to the interaction potential U. Then

$$\mathscr{G}(U) \supseteq \mathscr{G}(U) \cap \mathscr{M}_s(\Omega) \equiv \varnothing; \tag{2.6}$$

see [11], Theorem 4.3.

Let us now introduce some thermodynamical quantities and recall the corresponding limit theorems; see [5] for proofs. For $\nu \in \mathscr{M}_s(\Omega)$ and $V \in \mathscr{V}$, we denote by

$$H_V(\nu) := -\sum_{\xi \in S^V} \nu_V(\xi)\log \nu_V(\xi)$$

the *entropy* of ν_V. The *specific entropy* of ν is defined as

$$h(\nu) := \lim_n \frac{1}{|V_n|} H_{V_n}(\nu), \tag{2.7}$$

where $|V_n|$ denotes the cardinality of V_n. More precisely, the d-dimensional Shannon–McMillan theorem holds:

$$\lim_n \frac{1}{|V_n|} \log \nu_{V_n}(\omega_{V_n}) \tag{2.8}$$

exists in $L^1(\nu)$ (see [5], Theorem 2.4), and even ν a.s. [see [5], (4.28) for a Gibbs measure ν and [10] for a general $\nu \in \mathscr{M}_s(\Omega)$]. The *specific energy* of ν is defined by

$$e_U(\nu) := \lim_n \frac{1}{|V_n|} \int E_{V_n}(\omega_{V_n}|\eta)\nu(d\eta), \tag{2.9}$$

where the limit does not depend on $n \in \Omega$. More precisely, for any sequence $\eta_n \in \Omega$, the limit

$$\lim_n \frac{1}{|V_n|} E_{V_n}(\omega_{V_n}|\eta_n) \tag{2.10}$$

exists ν a.s. and in $L^1(\nu)$ and coincides with the conditional expectation of

$$\sum_{0 \in V \in \mathscr{V}} \frac{U(V, \omega_V)}{|V|},$$

with respect to the σ-algebra of shift-invariant events. In particular,

$$e_U(\nu) = \int \sum_{0 \in V \in \mathscr{V}} \frac{U(V, \omega_V)}{|V|} \nu(d\omega), \tag{2.11}$$

and this shows, due to (2.2), that e_U is a continuous functional on $\mathscr{M}_s(\Omega)$. The proof of (2.10) uses the relation

$$\lim_n \frac{1}{|V_n|} \sup_{\omega, \eta, \eta'} \left| E_{V_n}(\omega_{V_n}|\eta) - E_{V_n}(\omega_{V_n}|\eta') \right| = 0, \tag{2.12}$$

and this will be needed in the following discussion; see (4.16) in [5].

Also the *pressure* defined by

$$p_U := \lim_n \frac{1}{|V_n|} \log Z_{V_n}(\eta) \tag{2.13}$$

exists and is independent of $\eta \in \Omega$.

Now take $\mu \in \mathscr{G}(U) \cap \mathscr{M}_s(\Omega)$ and $\nu \in \mathscr{M}_s(\Omega)$. Then

$$H_V(\nu; \mu) := \sum_{\xi \in S^V} \nu_V(\xi) \log \frac{\nu_V(\xi)}{\mu_V(\xi)} \geq 0 \tag{2.14}$$

defines the *relative entropy* of ν_V with respect to μ_V, and the *specific relative entropy* is given by

$$h(\nu; \mu) = \lim_n \frac{1}{|V_n|} H_{V_n}(\nu; \mu). \tag{2.15}$$

The limit in (2.15) exists, and it satisfies the following *variational principle*:

$$h(\nu; \mu) = e_U(\nu) - h(\nu) + p_U \geq 0, \tag{2.16}$$

and the last inequality becomes an equality if and only if $\nu \in \mathscr{G}(U)$; see [5], (4.25) and (4.27). Since e_U is continuous and h is upper semicontinuous, (2.16) shows that $h(\cdot\,; \mu)$ is a lower semicontinuous function on $\mathscr{M}_s(\Omega)$.

We shall need the fact that, for $\nu \in \mathscr{M}_s(\Omega)$,

$$\lim_n \frac{1}{|V_n|} \log \frac{\nu_{V_n}(\omega_{V_n})}{\mu_{V_n}(\omega_{V_n})} \tag{2.17}$$

exists in $L^1(\nu)$, and that this limit coincides with $h(\nu; \mu)$, ν a.s., if ν is ergodic.

To see this, observe that, by (2.12),

$$\lim_n \frac{1}{|V_n|} \log \frac{\mu_{V_n}(\omega_{V_n})}{\mu_{V_n}(\omega_{V_n}|\eta)} = 0,$$

for any $\eta \in \Omega$; see [5], (4.22). So the limit in (2.17) may be written as

$$\lim_n \frac{1}{|V_n|} \log \frac{\nu_{V_n}(\omega_{V_n})}{\mu_{V_n}(\omega_{V_n}|\eta)} = \lim_n \frac{1}{|V_n|} \log \nu_{V_n}(\omega_{V_n}) + p_U + \lim_n \frac{1}{|V_n|} E_{V_n}(\omega_{V_n}|\eta). \tag{2.18}$$

The assertion involving (2.17) is now justified by (2.8) and (2.10), and the convergence holds in fact ν a.e.

For a positive integer p, we introduce

$$\gamma_p = \sum_{A \ni 0,\ A \subsetneq N_p(0)} \|U(A, \cdot)\|. \tag{2.19}$$

LEMMA 2.1. *Let $\eta \in \Omega$ and $\eta' \in \Omega$ coincide on $\partial_p V$. Then*

$$\frac{\mu_V(\omega_V|\eta)}{\mu_V(\omega_V|\eta')} \le e^{4|V|\gamma_p}, \qquad \omega_V \in S^V. \tag{2.20}$$

PROOF. In (2.4) break the summation into two parts, $\Sigma = \Sigma' + \Sigma''$, where the first sum is over those A included in $V \cup \partial_p V$, whereas the second sum is over those A intersecting $(V \cup \partial_p V)^c$. Then

$$\left\|\Sigma''\right\| \le \sum_{i \in V} \sum_{A \ni i,\ A \cap (V \cup \partial_p V)^c \ne \varnothing} \|U(A, \cdot)\| \subseteq |V|\gamma_p.$$

So we obtain

$$|E_V(\omega_V|\eta) - E_V(\omega_V|\eta')| \le 2|V|\gamma_p$$

and

$$Z_V(\eta) \ge \sum_{\xi \in S^V} e^{-E_V(\xi|\eta') - 2|V|\gamma_p} = Z_V(\eta') e^{-2|V|\gamma_p}.$$

Hence

$$\frac{\mu_V(\omega|\eta)}{\mu_V(\omega_V|\eta')} = \frac{Z_V(\eta')}{Z_V(\eta)} e^{-|E_V(\omega_V|\eta) - E_V(\omega_V|\eta')|} \le e^{4|V|\gamma_p}. \qquad \square$$

We need a remark on the space $\mathscr{M}(\Omega)$. $\mathscr{M}(\Omega)$ can be viewed as a compact metric space, and we shall use the following explicit metric. Choose a sequence (ϕ_n) of continuous functions on Ω satisfying the following conditions: (i) the span of (ϕ_n) is dense in $C(\Omega)$; (ii) $\|\phi_n\| \le 1$, $n = 1, 2, \ldots$; (iii) for each n there exists $V \in \mathscr{V}$ so that ϕ_n is $\mathscr{F}_V$-measurable. Now define

$$\text{dist}(\nu, \mu) = \sum_{n=1}^{\infty} 2^{-n} \left| \int \phi_n \, d\nu - \int \phi_n \, d\mu \right|, \tag{2.21}$$

for $\nu, \mu \in \mathscr{M}(\Omega)$.

3. Lower bound. The following theorem will be proved in this section.

THEOREM 3.1. *Let $\mu \in \mathscr{G}(U) \cap \mathscr{M}_s(\Omega)$, where U is a stationary interaction potential satisfying* (2.4). *Then, for any open subset G of $\mathscr{M}(\Omega)$,*

$$\liminf_n \frac{1}{|V_n|} \log \mu\left(\rho_{V_n} \in G\right) \ge - \inf_{\nu \in G \cap \mathscr{M}_s(\Omega)} h(\nu; \mu). \tag{3.1}$$

The proof will use Lemma 3.2. A more general lemma (for $d = 1$) was given in [9], but it will be convenient to give a direct proof here.

LEMMA 3.2. *Let $\nu \in \mathscr{M}_s(\Omega)$. Then there exists a sequence (ν_n) of ergodic measures converging to ν such that*

$$\lim_n h(\nu_n) = h(\nu). \tag{3.2}$$

PROOF. For $n \ge 1$ denote by $\tilde{\nu}_n$ the measure that coincides with ν on each σ-field $\mathscr{F}_{V_n + n \cdot k}$, $k \in Z^d$, and makes these σ-fields independent. The measure

$$\nu_n := \frac{1}{|V_n|} \sum_{i \in V_n} \tilde{\nu}_n \circ \theta_i^{-1}$$

is stationary. Let $\theta_i \tilde{\nu}_n = \tilde{\nu}_n \circ \theta_i^{-1}$. To show that ν_n is ergodic, let B be an invariant event, that is, $\theta_i B = B$ for all $i \in Z^d$. It is easily seen that each $\theta_i \tilde{\nu}_n$ satisfies the Kolmogorov zero–one law, and so $\theta_i \tilde{\nu}_n(B) := b_i \in \{0, 1\}$. Also, for i and $k \in Z^d$, $\theta_{i+k} \tilde{\nu}_n(B) = \theta_i \tilde{\nu}_n(\theta_k^{-1} B) = \theta_k \tilde{\nu}(B)$, so that $b_{i+k} = b_k$, and so $b_i = b$ does not depend on i. By the definition of ν_n, $\nu_n(B) = b$.

It is easy to check that the measures ν_n converge weakly to ν. Since the specific entropy $h(\cdot)$ is upper semicontinuous, it follows that

$$\limsup_n h(\nu_n) \le h(\nu).$$

For fixed $i \in V_n$, the cube $V_{N \cdot n}$ contains at least $(N-1)^d$ disjoint boxes of the form $V_n + i + n \cdot k$ with $k \in Z^d$, and on each such cube the entropy of $\theta_i \tilde{\nu}_n$ is given by $H_{V_n}(\nu)$ (here $\theta_i \tilde{\nu}_n := \tilde{\nu}_n \circ \theta_i^{-1}$). Thus

$$H_{V_n}(\theta_i \tilde{\nu}_n) \ge (N-1)^d H_{V_n}(\nu).$$

By Jensen's inequality,

$$H_{V_{N \cdot n}}(\nu_n) \ge \frac{1}{|V_n|} \sum_{i \in V_n} H_{V_{N \cdot n}}(\theta_i \tilde{\nu}_n) \ge (N-1)^d H_{V_n}(\nu).$$

Hence

$$\frac{1}{|V_{N\cdot n}|} H_{V_{N\cdot n}}(\nu_n) \geq \left(\frac{N-1}{N}\right)^d \frac{1}{|V_n|} H_{V_n}(\nu).$$

Letting $N \uparrow \infty$, we obtain

$$h(\nu_n) \geq \frac{1}{|V_n|} H_{V_n}(\nu)$$

and this implies

$$\liminf_n h(\nu_n) \geq h(\nu). \qquad \square$$

PROOF OF THEOREM 3.1. (1) Let $\nu \in \mathscr{M}(\Omega)$ and G an open neighborhood of ν. It will suffice to show

$$\liminf_n \frac{1}{|V_n|} \log \mu\left[\rho_{V_n} \in G\right] \geq -h(\nu; \mu). \tag{3.3}$$

(2) Assume now that ν is ergodic and G is a neighborhood of ν. In view of (2.21), there is a neighborhood

$$G_0 = \bigcup_{k=1}^{n} \left\{\nu' \middle| \left|\int f_k \, d\nu' - \int f_k \, d\nu\right| < \varepsilon\right\} \subseteq G,$$

of ν, where $f_1, \dots, f_n$ are $\mathscr{F}_{N_p(0)}$-measurable for some $p \geq 1$. Since ν a.s. ρ_{V_n} converges weakly to ν,

$$\lim_n \nu\left[\rho_{V_n} \in G_0\right] = 1.$$

Assume $h(\nu; \mu) < \infty$, as otherwise there is nothing to prove. Let $W_n := V_n \cup \partial_p V_n$. Then ν restricted to $\mathscr{F}_{W_n}$ is absolutely continuous with respect to μ restricted to $\mathscr{F}_{W_n}$; denote the corresponding Radon-Nikodym derivative by ϕ_{W_n}, so that

$$\lim_n \frac{1}{|V_n|} \log \phi_{W_n} = \lim_n \frac{1}{|W_n|} \log \phi_{W_n}$$
$$= h(\nu; \mu) \quad \text{in } L^1(\nu),$$

by (2.17). Now note that the set

$$\left\{\rho_{V_n} \in G_0\right\} = \bigcap_{k=1}^{n} \left\{\left|\frac{1}{|V_n|} \sum_{i \in V_n} f_k \circ \theta_i - \int f_k \, d\nu\right| < \varepsilon\right\}$$

belongs to $\mathscr{F}_{W_n}$. This allows us to write

$$\mu\left[\rho_{V_n} \in G\right] \geq \mu\left[\rho_{V_n} \in G, \frac{1}{|V_n|} \log \phi_{W_n} \leq h(\nu; \mu) + \varepsilon; \phi_{W_n} > 0\right]$$
$$\geq e^{-|V_n|(h(\nu; \mu) + \varepsilon)} \nu\left[\rho_{V_n} \in G_0, \frac{1}{|V_n|} \log \phi_{W_n} \leq h(\nu; \mu) + \varepsilon\right],$$

and, since the second factor in the last member approaches 1 by (2.17), we obtain (3.3).

(3) Now let $\nu \in \mathscr{M}_s(\Omega)$ and G a neighborhood of ν. Approximate ν by ergodic ν_n with $h(\nu_n) \to h(\nu)$ as ensured by Lemma 3.2. Then the left-hand side of (3.3) is greater or equal to $-h(\nu_n; \mu)$ for all large enough n. Remembering (2.16) and the fact that $e(\cdot)$ is continuous, we see that $h(\nu_n; \mu) \to h(\nu; \mu)$, and (3.3) follows. □

4. Upper bound. In this section we prove

THEOREM 4.1. *Let $\mu \in \mathscr{G}(U) \cap \mathscr{M}_s(\Omega)$. Then for any closed subset A of $\mathscr{M}(\Omega)$,*

$$\limsup_n \frac{1}{|V_n|} \log\left[\rho_{V_n} \in A\right] \le - \inf_{\nu \in A \cap \mathscr{M}_s(\Omega)} h(\nu; \mu)$$

($\inf \phi = \infty$).

Before we can prove the theorem, we need some definitions and several lemmas. In the remainder of this section $\mu \in \mathscr{G}(U) \cap \mathscr{M}_s(\Omega)$.

For $\nu \in M_s(\Omega)$, we introduce

$$\begin{aligned} H_{V,p}(\nu; \mu) &= \int H\left(\nu_{V,p}(\cdot|\eta); \mu_{V,p}(\cdot|\eta)\right)\nu(d\eta) \\ &= H_{V \cup \partial_p V}(\nu; \mu) - H_{\partial_p V}(\nu; \mu). \end{aligned} \tag{4.1}$$

Next let $C_{V,p}$ denote the class of all $\mathscr{F}_{V \cup \partial_p V}$-measurable functions ϕ on Ω that satisfy

$$\int e^{\phi}\, d\mu^*_{V,p}(\cdot|\eta) \le 1, \qquad \eta \in \Omega, \tag{4.2}$$

and also

$$\phi \le |V| \log \alpha^{-1}, \tag{4.3}$$

where $\alpha := \inf_{s \in S, \eta \in \Omega} \mu_{\{i\}}(s|\eta) > 0$.

LEMMA 4.2. $H_{V,p}(\nu; \mu) = \sup_{\phi \in C_{V,p}} \int \phi\, d\nu$.

PROOF. For $\phi \in C_{V,p}$, $\eta \in \Omega$, let

$$d\lambda = \left(\int e^{\phi}\, d\mu^*_{V,p}(\cdot|\eta)\right)^{-1} e^{\phi}\, d\mu^*_{V,p}(\ |\eta).$$

Condition (4.2) and the fact that relative entropy is nonnegative give

$$\begin{aligned} H\left(\nu_{V,p}(\cdot|\eta); \mu_{V,p}(\cdot|\eta)\right) &= H\left(\nu_{V,p}(\cdot|\eta); \lambda_V\right) \\ &\quad + \int \phi\, d\nu^*_{V,p}(\cdot|\eta) - \log \int e^{\phi}\, d\mu^*_{V,p}(\cdot|\eta) \\ &\ge \int \phi\, d\nu^*_{V,p}(\cdot|\eta) \end{aligned}$$

and hence

$$H_{V,p}(\nu;\mu) \geq \int \phi \, d\nu. \tag{4.4}$$

Now define

$$\psi(\omega) = \frac{\nu_{V,p}(\omega_V|\omega)}{\mu_{V,p}(\omega_V|\omega)} \leq \frac{1}{\alpha^{|V|}}.$$

For $c \in (0, \alpha^{-|V|})$, define

$$\psi_c = (\psi \vee c)\left(\int \psi \vee c \, d\mu^*_{V,p}(\cdot|\eta)\right)^{-1} \tag{4.5}$$

and note that $\phi_c := \log \psi_c$ belongs to $C_{V,p}$. By monotone convergence,

$$\begin{aligned}\lim_{c \downarrow 0} \int \phi_c \, d\nu &= \lim_{c \downarrow 0} \int \log(\Psi \vee c) \, d\nu - \lim_{c \downarrow 0} \int \log\left[\int \psi \vee c \, d\mu^*_{V,p}(\cdot|\eta)\right] d\nu \\ &= \int \log \Psi \, d\nu = H_{V,p}(\nu;\mu).\end{aligned}$$

□

LEMMA 4.3. *Let p_n, $n = 1, 2, \ldots,$ satisfy $\lim_n |\partial_{p_n} V_n| \cdot |V_n|^{-1} = 0$. Then*

$$h(\nu;\mu) = \lim_n \frac{1}{|V_n|} H_{V_n,p_n}(\nu;\mu). \tag{4.6}$$

PROOF. Note

$$H(\nu_{V,p}(\cdot|\eta); \mu_{V,p}(\cdot|\eta)) = -H_V(\nu_{V,p}(\cdot|\eta)) - \int \log \mu_{V,p}(\cdot|\eta) \, d\nu_{V,p}(\cdot|\eta)$$

and by (2.12)

$$-\lim_n \frac{1}{|V_n|} \int \log \mu_{V,p}(\cdot|\eta)\nu(d\eta) = e_U(\nu) + p_U$$

and so, remembering (2.16), it suffices to show

$$\lim_n \frac{1}{|V_n|} \int H_{V_n}(\nu_{V,p}(\cdot|\eta))\nu(d\eta) = h(\gamma).$$

But this follows from

$$\int H_V(\nu_{V,p}(\cdot|\eta)) \, d\nu = H_{V \cup \partial_p V}(\nu) - H_{\partial_p V}(\nu)$$

and the assumption on (p_n), since $H_{\partial_p V}(\nu) \leq |\partial_p V| \log |S|$. □

In order to obtain our upper bound, we follow [2] and introduce a stationary modification of the empirical field ρ_V. For each nonnegative integer n define π_n: $\Omega \to \Omega$ by

$$\pi_n(\omega)(i + n \cdot t) = \omega(i), \qquad i \in V_n, \ t \in Z^d.$$

Now let

$$\rho^s_{V_n} = \rho_{V_n} \circ \pi_n.$$

Clearly, $\rho^s_{V_n} \in \mathscr{M}_s(\Omega)$.

LEMMA 4.4. *Let ϕ: $\Omega \to R$ be measurable with respect to $\mathscr{F}_{N_m(0)}$. Then*

$$\left|\int \phi \, d\rho^s_{V_n} - \int \phi \, d\rho_{V_n}\right| \le \frac{|V_n| - |V_{n-2m}|}{|V_n|} 2\|\phi\|, \tag{4.7}$$

where $\|\phi\|$ denotes the supremum norm of ϕ.

PROOF. Note that

$$\int \phi \, d\rho^s_{V_n} = \frac{1}{|V_n|} \sum_{i \in V_n} \phi \circ \theta_i \circ \pi_n$$

and

$$\int \phi \, d\rho_{V_n} = \frac{1}{|V_n|} \sum_{i \in V_n} \phi \circ \theta_i$$

and that the summands on the right-hand side agree except for those $i \in V_n$ that satisfy $i + N_m(0) \not\subseteq V_n$. □

LEMMA 4.5. *For $k, p \ge 1$, $n > k + p$, $\phi \in C_{V_k, p}$ and any measurable subset A of $\mathscr{M}_s(\Omega)$,*

$$\begin{aligned} \frac{1}{|V_n|} \log \mu\left[\rho^s_{V_n} \in A\right] \le &- \inf_{\nu \in A} \frac{1}{|V_k|} \int \phi \, d\nu + \frac{|V_n - V_{n-2k-2p}|}{|V_n|} \frac{2\|\phi\|}{|V_k|} \\ &+ \left[1 - \left(\frac{k}{k+p} - \frac{k}{n}\right)^d\right] \log \alpha^{-1} + 4 \frac{|V_{n+k}|}{|V_k|} \gamma_p. \end{aligned} \tag{4.8}$$

PROOF. For n large, decompose V_n into a family of blocks $B_j = V_k + j$ and separating corridors, arranged so that each block B_j is separated from each of its neighbors by a corridor of width p. Let $W_{n,k,p}$ be the set of points in V_n belonging to one of the blocks B_j. Let

$$\psi_i = \sum_{t \in R_i} \phi \circ \theta_t, \qquad i \in V_k,$$

where R_i is the set of $t \in V_n$ such that $t = i + j$ for some j corresponding to a block B_j in our decomposition. Using (4.3), we obtain

$$\frac{1}{|V_k|} \sum_{i \in V_n} \phi \circ \theta_i \le |V_n - W_{n,k,p}| \log \alpha^{-1} + \frac{1}{|V_k|} \sum_{i \in V_k} \psi_i. \tag{4.9}$$

By Jensen's inequality,

$$\exp\left(\frac{1}{|V_k|} \sum_{i \in V_k} \psi_i\right) \le \frac{1}{|V_k|} \sum_{i \in V_k} \exp(\psi_i). \tag{4.10}$$

Now

$$\int \exp(\psi_i)\,d\mu = \int \exp\Bigl(\sum_{t\in R_i} \phi\circ\theta_t\Bigr)\,d\mu = \int \prod_{t\in R_i} \exp(\phi\circ\theta_t)\,d\mu$$

and this is to be estimated by successive conditioning. For each $t \in R_i$, the factors $\exp(\phi\circ\theta_s)$ with $s \in R_i - \{t\}$ are $\mathscr{F}_{(V_k+t)^c}$-measurable. On the other hand, we obtain from Lemma 2.1

$$E_\mu\bigl[e^{\phi\circ\theta_t}|\mathscr{F}_{(V_k+t)^c}\bigr] = \int e^{\phi}\mu^*_{V_k}(d\omega|\,\cdot)\circ\theta_t$$

$$\le e^{4|V_k|\gamma_p}\int e^{\phi}\mu^*_{V_{k,p}}(d\omega|\,\cdot)\circ\theta_t \le e^{4|V_k|\gamma_p}.$$

The cardinality of R_i is at most $((n+k)/k)^d$ and so

$$\int e^{\psi_i}\,d\mu \le \exp\bigl(4(n+k)^d\gamma_p\bigr). \tag{4.11}$$

The inequalities (4.9)–(4.11) imply

$$\int \exp\biggl(\frac{|V_n|}{|V_k|}\int\phi\,d\rho_{V_n}\biggr)\,d\mu \le \exp\bigl(|V_n - W_{n,k,p}|\log\alpha^{-1} + 4|V_{n+k}|\gamma_p\bigr). \tag{4.12}$$

The Chebyshev inequality and Lemma 4.4 with $m = k+p$ imply

$$\begin{aligned}\mu\bigl[\rho^s_{V_n}\in A\bigr] &\le \exp\biggl(\frac{-|V_n|}{|V_k|}\inf_{\nu\in A}\int\phi\,d\nu\biggr)\int\exp\biggl(\frac{|V_n|}{|V_k|}\int\phi\,d\rho^s_{V_n}\biggr)\,d\mu\\ &\le \exp\biggl(-\frac{|V_n|}{|V_k|}\inf_{\nu\in A}\int\phi\,d\nu\biggr)\\ &\quad+\frac{|V_n - V_{n-2k-2p}|}{|V_k|}2\|\phi\|\int\exp\biggl(\frac{|V_n|}{|V_k|}\int\phi\,d\rho_{V_n}\biggr)\,d\mu.\end{aligned} \tag{4.13}$$

The last factor is estimated by (4.12), and noting that

$$|V_n - W_{n,k,p}| \le n^d\biggl(1 - \biggl(\frac{k}{k+p} - \frac{k}{n}\biggr)^d\biggr)$$

leads to (4.8). □

Proof of Theorem 4.1. (1) Let us first show that for any closed subset A of $\mathscr{M}_s(\Omega)$,

$$J^s(A) := \limsup_n \frac{1}{|V_k|}\log\mu\bigl[\rho^s_{V_n}\in A\bigr] \le -\inf_{\nu\in A}h(\nu;\mu). \tag{4.14}$$

Letting n approach infinity in (4.8) gives us, for $\phi \in C_{k,p}$,

$$J^s(A) \le -\inf_{\nu\in A}\frac{1}{|V_k|}\int\phi\,d\nu + 4\gamma_p + \biggl(1 - \biggl(\frac{k}{k+p}\biggr)^d\biggr)\log\alpha^{-1}.$$

Now choose $p = p(k)$ so that p tends to ∞ with k but $p(k)/k$ tends to 0. Write C_k for $C_{k,p(k)}$. For $\varepsilon > 0$ there exists k_ε such that $|\gamma_{p(k)}| < \varepsilon$ for $k \geq k_\varepsilon$. For $k \geq k_\varepsilon$, $\phi \in C_k$,

$$J^s(A) - \varepsilon \leq -\inf_{\nu \in A} \frac{1}{|V_k|} \int \phi \, d\nu.$$

Thus

$$J^s(A) - \varepsilon \leq -\sup_{k \geq k_\varepsilon} \sup_{\phi \in C_k} \inf_{\nu \in A} \frac{1}{|V_k|} \int \phi \, d\nu$$

and, in fact,

$$J^s(A) - \varepsilon \leq - \sup_{\substack{A_1, \ldots, A_q \\ A \subseteq \cup A_j}} \inf_{1 \leq j \leq q} \sup_{k \geq k_\varepsilon} \sup_{\phi \in C_k} \inf_{\nu \in A_j} \frac{1}{|V_k|} \int \phi \, d\nu, \tag{4.15}$$

since $J^s(A) = \max J^s(A_j)$ for any finite covering $A_1, \ldots, A_q$ of A.

For any $\nu \in A$ there exists $k \geq k_\varepsilon$ and $\phi \in C_k$ so that

$$\frac{1}{|V_k|} \int \phi \, d\nu \geq \inf_{\nu \in A} h(\nu; \mu) - \varepsilon/2,$$

by Lemmas 4.2 and 4.3. Since $\int \phi \, d\nu$ is a continuous functional on $\mathscr{M}(\Omega)$, we can conclude that

$$\frac{1}{|V_k|} \int \phi \, d\tilde{\nu} \geq \inf_{\nu \in A} h(\nu; \mu) - \varepsilon,$$

for all $\tilde{\nu}$ in some open neighborhood G_ν of ν. Now we use the assumption that A is closed, hence compact. There exists a finite covering of A by open sets $G_1, \ldots, G_q$ such that

$$\inf_{1 \leq j \leq q} \sup_{k \geq k_\varepsilon} \sup_{\phi \in C_k} \inf_{\nu \in G_j} \frac{1}{|V_k|} \int \phi \, d\nu \geq \inf_{\nu \in A} h(\nu; \mu) - \varepsilon.$$

This shows that the right-hand side of (4.15) is bounded above by $-\inf_{\nu \in A} h(\nu; \mu) + \varepsilon$. Since ε was arbitrary this implies (4.14).

(2) Metrize $\mathscr{M}(\Omega)$ as in (2.21). Then by Lemma 4.4 the distance between ρ_{V_n} and $\rho^s_{V_n}$ converges to 0 uniformly in ω (recall that these are random measures) as $n \to \infty$. Hence if $\varepsilon > 0$ and A_ε is a closed ε-neighborhood of A, there exists $n_0(\varepsilon)$ such that $n \geq n_0(\varepsilon)$ implies

$$[\rho_{V_n} \in A] \subseteq [\rho^s_{V_n} \in A_\varepsilon] = [\rho^s_{V_n} \in A_\varepsilon \cap \mathscr{M}_s(\Omega)].$$

Thus (4.14) implies

$$J(A) := \limsup_n \frac{1}{|V_n|} \log \mu[\rho_{V_n} \in A] \leq - \inf_{\nu \in A_\varepsilon \cap \mathscr{M}_s(\Omega)} h(\nu; \mu).$$

Now let $\varepsilon \downarrow 0$ and use the fact that $h(\nu; \mu)$ is a lower semicontinuous function of ν to obtain Theorem 4.1. □

5. Large deviations for the empirical distribution. Let $\mu \in \mathscr{M}(\Omega) \cap \mathscr{G}(U)$ be a stationary Gibbs measure with interaction potential U. In the language of [13] we have shown that the sequence of measures $\mu \circ \rho_{V_n}^{-1}$ on $\mathscr{M}(\Omega)$ satisfies the large deviation principle with rate function

$$I(\nu) = \begin{cases} h(\nu; \mu), & \nu \in \mathscr{M}_s(\Omega), \\ \infty, & \text{otherwise.} \end{cases}$$

One important consequence is Varadhan's abstraction of the Laplace method: For any $\phi \in C(\Omega)$,

$$p(\phi; \mu) := \lim_n \frac{1}{|V_n|} \log \int \exp\Big(\sum_{i \in V_n} \phi \circ \theta_i \Big)\, d\mu \tag{5.1}$$

exists and satisfies

$$\begin{aligned} p(\phi; \mu) &= \sup_{\nu \in \mathscr{M}(\Omega)} \left[\int \phi\, d\nu - I(\nu) \right] \\ &= \sup_{\nu \in \mathscr{M}_s(\Omega)} \left[\int \phi\, d\nu - h(\nu; \mu) \right], \end{aligned} \tag{5.2}$$

as shown in [13], Theorem 2.2. By convex duality (5.2) implies

$$h(\nu; \mu) = \sup_{\phi \in C(\Omega)} \left[\int \phi\, d\nu - p(\phi; \mu) \right], \tag{5.3}$$

for any $\nu \in \mathscr{M}_s(\Omega)$. This follows, e.g., from [12], Theorem 7.15, if we use the natural embedding of $\mathscr{M}(\Omega)$ into the topological vector space of signed measures on Ω.

Let us now consider large deviations for the empirical distribution

$$R_{V_n}(\omega) := \frac{1}{|V_n|} \sum_{i \in V_n} \delta_{\omega(i)} \in \mathscr{M}(S).$$

Let π: $\mathscr{M}(\Omega) \to \mathscr{M}(S)$ associate with each $\nu \in \mathscr{M}(\Omega)$ its marginal distribution $\nu_0 = \pi(\nu)$ at the origin $0 \in Z^d$. Then

$$R_{V_n} = \pi(\rho_{V_n})$$

and the contraction principle of [13], 2, Remark 1, shows that the sequence $\mu \circ R_{V_n}^{-1}$, $n = 1, 2, \dots$, satisfies the large deviation principle with rate function

$$I_0(Q) = \inf_{\nu \in \mathscr{M}(\Omega):\, \pi(\nu) = Q} I(\nu) = \inf_{\nu \in \mathscr{M}_s(\Omega):\, \pi(\nu) = Q} h(\nu; \mu). \tag{5.4}$$

More explicitly,

$$\liminf_n \frac{1}{|V_n|} \log \mu\big[R_{V_n} \in A \big] \ge - \inf_{Q \in A} I_0(Q), \tag{5.5}$$

for any open set $A \subseteq \mathscr{M}(S)$, and

$$\limsup_n \frac{1}{|V_n|} \log \mu\big[R_{V_n} \in A \big] \le - \inf_{Q \in A} I_0(Q), \tag{5.6}$$

for any closed set $A \subseteq \mathscr{M}(S)$. Since $h(\cdot\,;\mu)$ is lower semicontinuous the infimum in (5.4) is actually attained. In Theorem (5.5) the class of minimizing measures $\nu \in \mathscr{M}_s(\Omega)$ is shown to coincide with $G(U^f)$, where U^f is a generalized interaction potential obtained from U and some function f on S that depends on Q.

Let us first note that, again by Laplace's method [13], the limit

$$p_0(f;\mu) := \lim_n \frac{1}{|V_n|}\log\int\exp\Big(\sum_{i\in V_n} f(\omega(i))\Big)\,d\mu$$

exists for any real-valued function f on S and satisfies the relations

$$p_0(f;\mu) = \sup_{Q\in\mathscr{M}_s}\Big[\int f\,dQ - I_0(Q)\Big] \tag{5.7}$$

and

$$I_0(Q) = \sup_{f:\,S\to R}\Big[\int f\,dQ - p_0(f;\mu)\Big]. \tag{5.8}$$

For our purposes, we will have to consider functions f on S that may assume the value $-\infty$.

Let f: $S \to [-\infty, \infty)$ be a function not identically equal to $-\infty$. Define U^f by $U^f(V,\cdot) = U(V,\cdot)$ for $|V| > 1$ and

$$U^f(\{i\},\omega) = U(\{i\},\omega) - f(\omega(i)), \qquad i \in Z^d.$$

The corresponding class of Gibbs measures $\mathscr{G}(U^f)$ will consist of probability measure μ^f such that for $V \in \mathscr{V}$ and $\nu \in \Omega$, a conditional distribution on S^V given $\mathscr{F}_{V^c}$ is given by

$$\mu^f(\omega_V|\eta) = \big(Z_V^f(\eta)\big)^{-1}\exp\Big[\sum_{i\in V} f(\omega(i))\Big]\mu_V(\omega_V|\eta), \tag{5.9}$$

where

$$Z_V^f(\eta) = \sum_{\xi\in S^V}\exp\Big[\sum_{i\in V} f(\xi(i))\Big]\mu_V(\xi|\eta). \tag{5.10}$$

Again $\mathscr{M}_s(\Omega) \cap \mathscr{G}(U^f) \neq \varnothing$; see [11], Theorem 4.3. Observe that for $\mu^f \in G(U^f)$ the marginal distribution $\mu_0^f = \pi(\mu^f)$ is concentrated on the set

$$S_f = \{s \in S\colon f(s) > -\infty\},$$

since

$$\mu_0^f(S_f^c) = \sum_{s\in S_f^c}\int\mu_{\{0\}}^f(s|\eta)\mu^f(d\eta) = 0.$$

Lemma 5.1. *Let f: $S \to [-\infty,\infty)$, $f \not\equiv -\infty$. The limit*

$$p_0(f;\mu) := \lim_n \frac{1}{|V_n|}\int\log\int\exp\Big(\sum_{i\in V_n} f(\omega(i))\Big)\mu(d\omega) \tag{5.11}$$

exists and is finite and satisfies the equation

$$p_0(f;\mu) = \int f\,d\mu_0^f - h(\mu^f;\mu), \tag{5.12}$$

for any $\mu^f \in \mathscr{G}(U^f) \cap \mathscr{M}_s$. More generally, for $\nu \in \mathscr{M}_s(\Omega)$ such that $Q := \pi(\nu)$ is concentrated on S_f and $\mu^f \in \mathscr{G}(U^f) \cap \mathscr{M}_s$,

$$h(\nu;\mu) = h(\nu;\mu^f) + \int f\,dQ - p_0(f;\mu). \tag{5.13}$$

PROOF. Let us compute $h(\nu;\mu^f)$ and $h(\nu;\mu)$ by (2.16) and compare the resulting expressions. One finds

$$e_{U^f}(\nu) - e_U(\nu) = \lim_n \frac{-1}{|V_n|} \int \sum_{i\in V_n} f(\omega_i)\nu(d\omega) = \int f\,dQ.$$

Also, for any η,

$$\begin{aligned} p_{U^f} &= \lim_n \frac{1}{|V_n|} \log \sum_{\xi \in S^{V_n}} \exp\left(-E_{V_n}(\xi|\eta) + \sum_{i\in V_n} f(\xi_i)\right) \\ &= \lim_n \frac{1}{|V_n|} \log \int \exp\left(\sum_{i\in V_n} f(\xi_i)\right) d\mu + \lim_n \frac{\log Z_{V_n}(\eta)}{V_n} \\ &= p_0(f;\mu) + p_U, \end{aligned}$$

where the limit defining $p_0(f;\mu)$ must exist and be finite because the same is true of all the other limits that occur. This proves (5.13), and the choice $\nu = \nu^f$ gives (5.12). □

THEOREM 5.2. *Suppose $I_0(Q) < \infty$. Then there exists a function f: $S \to [-\infty, \infty)$ with the following properties:*

(i) *A measure $\nu \in \mathscr{M}_s(\Omega)$ with marginal $\pi(\nu) = Q$ satisfies*

$$h(\nu;\mu) = I_0(Q), \tag{5.14}$$

if and only if $\nu \in \mathscr{G}(U^f)$.

(ii) *Both $\int f\,dQ$ and $p_0(f;\mu)$ are finite and*

$$I_0(Q) = \int f\,dQ - p_0(f;\mu). \tag{5.15}$$

PROOF. (1) By (5.8) we can choose a sequence (f_k) of functions on $C(S)$ such that

$$I_0(Q) = \lim_k \left[\int f_k\,dQ - p_0(f_k;\mu)\right]. \tag{5.16}$$

The expression in brackets is unchanged if f_k is replaced by $f_k + c$ for any constant c. We may therefore assume

$$\max_{i\in S} f_k(i) = 0. \tag{5.17}$$

Going over to a subsequence if necessary, we may assume

$$f(s) := \lim_k f_k(s) \in [-\infty, 0]$$

exists for each $s \in S$, and $f \not\equiv -\infty$. This will be the function whose existence is asserted by the theorem.

(2) Remembering (5.17), we obtain

$$
\begin{aligned}
Z_V^{f_k}(\eta) &= \sum_{\xi \in S^V} \exp\Big(\sum_{i \in V} f_k(\xi(i)) \Big) \mu_V(\xi|\eta) \\
&\geq \inf_{\xi \in S^V} \mu_V(\xi|\eta) \geq \alpha^{|V|},
\end{aligned} \tag{5.18}
$$

where $\alpha := \inf_{s \in S, \eta \in \Omega} \mu_i(s|\eta) > 0$. From (5.18) it follows that

$$p_0(f_k; \mu) = \lim_n \frac{1}{|V_n|} \log Z_V^{f_k}(\eta) \geq \alpha.$$

This implies, due to (5.16),

$$\lim_k \int f_k \, dQ > -\infty, \tag{5.19}$$

and so $Q(S_f) = 1$.

(3) Using (2), we can pass from (5.16) to

$$I_0(Q) = \int f \, dQ - \lim_k p_0(f_k; \mu). \tag{5.20}$$

We must still show that the last term in (5.20) agrees with $p_0(f; \mu)$ as given in (5.15). For $\varepsilon > 0$ and k sufficiently large, $f_k > f - \varepsilon$, and from this one obtains immediately that

$$\lim_k p_0(f_k; \mu) \geq p_0(f; \mu)$$

and hence (5.20) implies

$$I_0(Q) \leq \int f \, dQ - p_0(f; \mu). \tag{5.21}$$

On the other hand, if $\nu \in \mathscr{M}_s$ satisfies $\pi(\nu) = Q$ and $h(\nu; \mu) = I_0(Q)$, then by (5.13) for $\mu^f \in \mathscr{G}(U^f) \cap \mathscr{M}_s$

$$I_0(Q) = h(\nu; \mu^f) + \int f \, dQ - p_0(f; \mu) \geq \int f \, dQ - p_0(f; \mu). \tag{5.22}$$

Now from (5.21) and (5.22) follows the desired equality (5.15), and also $h(\nu; \mu^f) = 0$, and this implies $\nu \in \mathscr{G}(U^f)$ by [11], Theorem 7.1.

(4) Suppose $\mu^f \in \mathscr{G}(U^f) \cap \mathscr{M}_s$ has marginal $\pi(\mu^f) = Q$. Then $h(\mu^f; \mu) = I_0(Q)$ follows from (5.12) and (5.15). □

Remark. Let μ be a Markov random field, and let Q be a measure on S with $I_0(Q) \leq \infty$. In this case we have shown that there exists a new Markov random field of the form $\mu^f \in \mathscr{G}(U^f)$ such that μ^f has marginal distribution Q and satisfies

$$I_0(Q) = h(\mu^f; \mu).$$

This is a spatial analog to a result of [2], Theorem 2.1; see also [13], (13.7).

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