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CONNECTING BROWNIAN PATHS[1]

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We study two processes obtained as follows: Take two independent d-dimensional Brownian motions started at points x, y, respectively. For the first process, let $d \geq 3$ and condition on $X_t = Y_t$ for some t (a set of probability 0). Run X out to the point of intersection and then run Y in reversed time from this point back to y. For the second process, let $d \geq 5$ and perform the same construction, this time conditioning on $X_s = Y_t$ for some s, t. The first process is shown to be Doob's conditioned (to go from x to y) Brownian motion Z, and the second has distribution absolutely continuous with respect to that of Z, the Radon–Nikodym density being a constant times the time Z takes to travel from x to y. Similar results (including extensions to the critical dimensions $d = 2$ and $d = 4$) are obtained by conditioning the motions to hit before they leave domains. We use the asymptotics of the probability of "near misses" and results on the weak convergence of h-transforms.

1. Introduction. If two Brownian paths in $\mathbb{R}^d$ come close to one another, either at the same or at different times, those paths yield a bond connecting the paths' initial points. We study these bonds. Since we wish such "near misses" to happen only once, we will do so in dimensions high enough for such events to be very unlikely.

Let $x \neq y$ be points in $\mathbb{R}^d$, $d \geq 3$, and let ${}_yZ_t^x$, $0 \leq t < \zeta(Z)$, be Doob's Brownian motion conditioned to go from x to y; that is, the Markov process which starts at x, has finite lifetime $\zeta(Z)$ and transition density

$$(1.1)\qquad {}_yp_t(z, w) = (|w - y|/|z - y|)^{d-2} p_t(z, w), \qquad z \neq y,$$

where p_t is the transition density of Brownian motion. Note that $\int {}_yp_t(x, w)\, dw < 1$, the excess being the probability that ${}_yZ^x$ exceeds its lifetime before time t. This process is, for all purposes we can imagine, as tractable as (unconditioned) Brownian motion. See Doob (1984) for much more information about conditioned Brownian motion and Durrett (1984) for an elementary account.

Let X_t^x, Y_t^y be independent Brownian motions started at x, y, respectively. Let $\varepsilon > 0$ and put

$$L_\varepsilon = \sup\{t > 0; |X_t - Y_t| \leq \varepsilon\}, \qquad \sup(\varnothing) = 0.$$

It is easily checked that $P(0 < L_\varepsilon) = (\varepsilon/|x - y|)^{d-2}$ if $|x - y| \geq \varepsilon$ and $L_\varepsilon < \infty$

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a.s. Let

$$W_t^\varepsilon = \begin{cases} X_t^x, & 0 \le t < L_\varepsilon, \\ Y_{2L_\varepsilon - t}^y, & L_\varepsilon < t < 2L_\varepsilon, \\ \Delta, & t \ge 2L_\varepsilon, \end{cases}$$

and make W^ε right continuous at L_ε. We also make ${}_yZ_t^x = \Delta$ for $t \ge \zeta(Z)$, so that Δ functions as a cemetery state.

1.2 **Theorem.** *As $\varepsilon \downarrow 0$, the law of W^ε conditioned on $\{L_\varepsilon > 0\}$ converges weakly to that of ${}_yZ^x$.*

The use here of the *last* time our processes are within ε of each other is for convenience only. The analogue of Theorem 1.2 with L_ε replaced by $\inf\{t; |X_t - Y_t| \le \varepsilon\}$, or by any other time our processes are within ε of each other, still holds and is easily derived from Theorem 1.2 and some of our lemmas on Brownian paths. Weak convergence is with respect to the usual Skorohod topology. This is discussed more fully in Sections 2 and 7. The proof of Theorem 1.2 is not difficult.

Now let $d \ge 5$ and put

$$M_\varepsilon = \sup\{s > 0; |X_s^x - Y_t^y| \le \varepsilon \text{ for some } t\},$$

$$N_\varepsilon = \sup\{t > 0; |X_s^x - Y_t^y| \le \varepsilon \text{ for some } s\}.$$

(Again the precise forms of M_ε, N_ε are not essential, only convenient.) Both M_ε and N_ε are finite a.s. Let

$$V_t^\varepsilon = \begin{cases} X_t^x, & 0 \le t < M_\varepsilon, \\ Y_{M_\varepsilon + N_\varepsilon - t}^y, & M_\varepsilon \le t < M_\varepsilon + N_\varepsilon, \\ \Delta, & t \ge M_\varepsilon + N_\varepsilon. \end{cases}$$

1.3 **Theorem.** *As $\varepsilon \downarrow 0$, the law of V^ε conditioned on $\{M_\varepsilon > 0\}$ converges weakly. The limiting law is absolutely continuous with respect to that of ${}_yZ^x$, with Radon–Nikodym density $\zeta({}_yZ^x)/E\zeta({}_yZ^x)$.*

Thus in Theorem 1.2 we condition massive particles to collide and in Theorem 1.3 we condition Wiener sausages to intersect.

The proof will in fact show that the joint law of $(V^\varepsilon, M^\varepsilon)$ converges weakly to that of a pair (V, M). It is also shown that conditional on V, the time M is uniformly distributed on $[0, \zeta(V)]$. In particular, it is not determined by V and occurs, in some sense, at a typical point along the path (rather than at, say, a place with rapid oscillation). In contrast, Martin Barlow has pointed out to us that the intersection of Brownian motion in $\mathbb{R}^2$ with space time Brownian motion occurs at atypical points; as a consequence of Makarov's theorem, harmonic measure in the domain below the space–time Brownian path is carried by a set with Hausdorff dimension strictly less than that of the whole path.

Neither of these two theorems extends to lower dimensions (in dimensions 2 and 4, respectively, the weak limits still exist, but are standard Brownian motions). However, we can go down one dimension by considering domains other than all of $\mathbb{R}^d$. (In even lower dimensions, the weak limits are not needed to make sense of the analogous results, but unfortunately these results are false.)

Let D be any domain in $\mathbb{R}^d$, $d \geq 2$, which has a Green function $G(x, y)$. For $x, y \in D$, $x \neq y$, let ${}_yZ^x$ now be Doob's Brownian motion conditioned to go from x to y before leaving D [in (1.1), $(|w - y|/z - y|)^{d-2}$ gets replaced by $G(w, y)/G(z, y)$ and p_t by the transition density of Brownian motion killed upon leaving D]. For $\zeta(X^x) = \inf\{s;\ X_s^x \notin D\}$, $\zeta(Y^y) = \inf\{t;\ Y_t^y \notin D\}$, we let $L_\varepsilon = \sup\{t < \min(\zeta(X^x), \zeta(Y^y);\ |X_t^x - Y_t^y| < \varepsilon\}$ and then define W^ε as before.

1.4 THEOREM. *As $\varepsilon \downarrow 0$, the law of W^ε conditioned on $\{L_\varepsilon > 0\}$ converges weakly to that of ${}_yZ^x$.*

Now let D be a domain in $\mathbb{R}^d$, $d \geq 4$, and suppose that

$$\int_D G(x, z) G(y, z)\, dz < \infty. \tag{1.5}$$

This condition always holds if $d \geq 5$, since it is true for $D = \mathbb{R}^d$ and the Green function for D is dominated by the Newtonian one. In contrast, the condition fails for $D = \mathbb{R}^4$, so is a real restriction here. It holds if D is bounded or is contained in the complement of any solid cone.

Define V^ε as before, now using $M_\varepsilon = \sup\{s < \zeta(X^x);\ |X_s^x - Y_t^y| < \varepsilon$ for some $t < \zeta(Y^y)\}$ and $N_\varepsilon = \sup\{t < \zeta(Y^y);\ |X_s^x - Y_t^y| < \varepsilon$ for some $s < \zeta(X^x)\}$.

1.6 THEOREM. *As $\varepsilon \downarrow 0$, the law of V^ε conditioned on $\{M_\varepsilon > 0\}$ converges weakly. The limiting law is absolutely continuous with respect to that of ${}_yZ^x$, with Radon–Nikodym density $\zeta({}_yZ^x)/E\zeta({}_yZ^x)$.*

Note that since $\int_D G(x, z)G(y, z)\, dz = G(x, y)E\zeta({}_yZ^x)$, we need (1.5) to state the theorem.

It is very likely that our Brownian motions can be replaced by random walks of step size $|x - y|/n$, and "coming within ε" by "intersection" and that limiting results will be the same as $n \to \infty$, but we do not attempt to prove this here (although our original argument was a heuristic nonstandard one and suggests the preceding result). A different question is that of whether there are random walk results similar to ours which do not involve taking limits. The exact analogues of the preceding theorems fail, but the following holds: Let X_m^x, Y_n^y be independent standard random walks on the standard d-dimensional lattice, $d \geq 3$. If $\Sigma(x_i - y_i)$ is even, let $L = \sup\{n, X_n^x = Y_n^y\}$. If it is odd, let $L = \sup\{n;\ X_n^x = Y_{n+1}^y\}$. Then conditioned on $\{L > 0\}$ (now a set of positive probability), the pasting together of X^x and Y^y is a random walk conditioned to go from x to y. In fact, if γ_d is the probability that $X_n^0 \neq Y_n^0$ for any $n \geq 1$, then it is easily seen that the probability of the previous pasting giving any fixed path of length m is $(2d)^{-m}\gamma_d$, and this easily implies our assertion. This argument clearly fails if we restrict our random walks to any proper subset of the lattice or if we replace last hitting time with first hitting time.

Our proofs of the preceding results are based on Doob's theory h-transforms. Since in Theorem 1.3 we have two time parameters, we will sometimes need h-transforms of two parameter processes; see Section 2 for the definition.

In Section 7 we prove some results about weak convergence of h-transforms. They are folklore in the one-parameter case, but we do not know a reference. In the two-parameter case they appear to be new. Section 2 will set notation, and the proofs of Theorems 1.2 and 1.3 are given in Section 3. Section 6 contains some discussion of the condition (1.5), and the estimates needed to make the arguments of Section 3 work are shown in Sections 4 and 5.

These estimates are close to many others in the literature. The sources Lawler (1982), (1985), Le Gall (1986a), (1986b), Aizenman (1985), Felder and Fröhlich (1985), Brydges and Spencer (1985) and especially Erdös and Taylor (1960a), (1960b) all contain related results from which we have profited; but whereas most of these papers deal with problems like the asymptotics in ε of $P(|X_s^x - Y_t^y| < \varepsilon$ for some $s, t)$, we are principally concerned with the dependence of these objects on x and y. Moreover, we work in domains and must deal with the possible presence of pathological boundaries. This is especially true in dimension 4 (see Section 6). In this context it should be pointed out that Le Gall (1986a) studied asymptotics like those in Section 4, but for stopping X_t^x and Y_t^y at some fixed t_0 rather than at $\zeta(X^x) \wedge \zeta(Y^y)$.

2. Notation. We now collect some of the notation used in the remainder of the paper.

1. $B(x, r)$ is the ball of radius r centered at x.
2. $B(K, r)$ is the r-neighborhood of K.
3. Λ_ε will be the lattice εZ^d.
4. b will usually be the number $1 + (\sqrt{d}/2)$.
5. $u_\varepsilon \sim v_\varepsilon$ will mean that $u_\varepsilon / v_\varepsilon \to 1$.
6. The letters X, Y, V, W, Z, L, M and N will keep the meanings given them in Section 1. In addition,

$$S_\varepsilon = \inf\{s; |X_s^x - Y_t^y| < \varepsilon \text{ for some } t\},$$

$$T_\varepsilon = \inf\{t; |X_t^x - Y_t^y| < \varepsilon\},$$

   with similar definitions when dealing with domains.
7. h_ε, g_ε and g will be special functions, to be defined in Section 3, but we will often use h or h_n for a generic excessive function.
8. We adopt the conventions that c denotes a generic constant whose value may change from line to line, and that all functions vanish at Δ.

9. We move x and y inside and outside expectations at will. Thus the following are taken to be tautologies:

$$ {}_yE^x[f(Z)] = E\big[f({}_yZ^x)\big], \qquad {}_yP^x(Z \in A) = P({}_yZ^x \in A), $$
$$ E^{x,y}[f(X,Y)] = E[f(X^x, Y^y)], $$
$$ P^{x,y}((X,Y) \in A) = P((X^x, Y^y) \in A). $$

Similarly, when dealing with general h-transforms, we write

$$ Ef({}_hX^x) = {}_hE^x[f(X)], $$

etc.

10. When dealing with domains, we use the same letters ($P^{x,y}, G, p_t, \dots$), when dealing with objects killed upon leaving D, as we did before, for the corresponding unkilled objects. This should cause no confusion because we are consistent within sections. To talk about the unkilled object in a domain section we just add a "0" (e.g., $P_0^{x,y}, G_0, p_t^0, \dots$).

We now describe the form of weak convergence used in the theorems in Section 1. We must allow jumps, since our processes usually have two: one from the "seam" between X and Y and another at ζ. Let $\overline{D}$ be the one-point compactification of D and let Δ be a point isolated from $\overline{D}$. Let Ω be the space of paths with values in $\overline{D} \cup \{\Delta\}$ which are right continuous with left limits and stay at Δ forever once they reach it. Endow Ω with the Skorohod topology [see Billingsley (1968) and Lindvall (1973)]. Weak convergence will always be that of probabilities on Ω (or $\Omega \times \Omega$; see Section 7).

To go along with Ω, we have other standard notation: For $\omega \in \Omega$ we write $\zeta(\omega) = \inf\{t;\ \omega(t) = \Delta\}$, $(\theta_t\omega)(s) = \omega(t+s)$ (respectively, the lifetime and shift operator). We use the σ-fields $\mathscr{F}_t$ generated by the evaluation maps $w \to w(s)$ for $s \le t$. Thus, since $X, Y, \dots$ will take values in Ω, we can write random variables depending on X_s for $s \le t$ in the form $\tau(X)$ for $\tau \in \mathscr{F}_t$. Similarly, the lifetime of X is $\zeta(X)$.

Recall that if h is excessive (i.e., superharmonic), we say that ${}_hX$ is an h-transform of X if for each positive $\tau \in \mathscr{F}_t$,

$$ E\big[\tau({}_hX), \zeta({}_hX) > t\big] = E\big[\tau(X)h(X_t)/h(X_0), \zeta(X) > t\big]. $$

It will be convenient to have a similar definition for two-parameter processes as well.

A *lower layer* is a set $\zeta \subset [0,\infty) \times [0,\infty)$ such that:

(a) If $(s,t) \in \zeta$ and $s' \le s$, $t' \le t$, then $(s', t') \in \zeta$.
(b) If $s_n \downarrow s$, $t_n \downarrow t$ and $(s_n, t_n) \notin \zeta$ for any n, then $(s,t) \notin \zeta$.

A *bipath* will be a function of the form

$$ \omega(s,t) = \begin{cases} (\omega^1(s), \omega^2(t)), & (s,t) \in \zeta, \\ (\Delta, \Delta), & (s,t) \notin \zeta, \end{cases} $$

where $\omega^1, \omega^2 \in \Omega$, ζ is a lower layer and $\zeta(\omega^1) = \inf\{s; (s,0) \notin \zeta\}$, $\zeta(\omega^2) =$

$\inf\{t; (0,t) \notin \zeta\}$ [so that ω^1 is the first component of $\omega(\cdot, 0)$ and ω^2 is the second component of $\omega(0, \cdot)$].

A *biprocess* is a random bipath U such that each $U_{s,t}$ is measurable. We write U_s^1 and U_t^2 for the first and second components of $U_{s,0}$ and $U_{0,t}$, respectively, and $\zeta(U)$ for its associated lower layer.

The simplest biprocess is called bi-Brownian motion. We let X and Y be independent Brownian motions and set $\zeta(U) = [0, \zeta(X)) \times [0, \zeta(Y))$, $U^1 = X$, $U^2 = Y$ [unless we are working in domains, clearly $\zeta(U) = [0,\infty) \times [0,\infty)$]. Thus $U_{s,t} = (X_s, Y_t)$ while both X and Y are alive.

Let $U_{s,t}$ be a bi-Brownian motion and let $h(x,y)$ be *biexcessive* [that is, $h(x,\cdot)$ and $h(\cdot, y)$ are each excessive]. A biprocess ${}_hU_{s,t}$ is called an *h-bitransform* of $U_{s,t}$ if for every $s, t \geq 0$ and positive $\sigma \in \mathscr{F}_s$, $\tau \in \mathscr{F}_t$, we have that

$$\begin{aligned} &E\left[\sigma({}_hU^1)\tau({}_hU^2), (s,t) \in \zeta({}_hU)\right] \\ &\quad = E\left[\sigma(U^1)\tau(U^2)h(U_{s,t})/h(U_{0,0}), (s,t) \in \zeta(U)\right]. \end{aligned}$$

Not much is known about these objects, but see, for example, Walsh (1981) and Cairoli (1968). All we will need will be the following fact: Let A be an open subset of $D \times D$ and let $h(x,y) = P((X_s^x, Y_t^y) \in A$ for some $s, t > 0)$. Kill bi-Brownian motion at the "last exit" from A; that is, set

$${}_hU_{s,t} = \begin{cases} (X_s, Y_t), & \text{if there are some } s' > s \text{ and } t' > t \text{ such that } (X_{s'}, Y_{t'}) \in A, \\ (\Delta, \Delta), & \text{otherwise.} \end{cases}$$

Now condition on $\Gamma = \{(X_s^x, Y_t^y) \in A$ for some $s, t\}$ [that is, on $\{{}_hU_{0,0} \neq (\Delta, \Delta)\}$].

2.1 Proposition. *When conditioned on* Γ, ${}_hU_{s,t}$ *is an h-bitransform of bi-Brownian motion.*

The proof is identical to the corresponding well known result in one parameter; see Doob [(1984), page 568].

We will be concerned with the preceding only when A is the event $|X_s - Y_t| < \varepsilon$ for some s, t, so that we are killing bi-Brownian motion at the last near miss. The set ζ for this process is just $\{(s,t): |X_{s'} - X_{t'}| < \varepsilon$ for some $s' \geq s$ and $t' \geq t\}$. This set is contained in $[0, \sup\{s: |X_s - Y_t| < \varepsilon$ for some $t\}) \times [0, \sup\{t: |X_s - Y_t| < \varepsilon$ for some $s\})$, but is not in general equal to this rectangle, since X_t can be within ε of the Y path several times, and the supremum of the t for which this happens can correspond to a smaller time for the Y process than for other t for which it happens. Part of what we prove is that the set ζ becomes more rectangular as $\varepsilon \to 0$.

3. Outlines of the proofs of Theorems 1.2 and 1.3. We start with a lemma on time reversal. Let $Z = {}_yZ^x$. Since Z has finite lifetime, we may define its reverse:

$$\hat{Z}_t = \begin{cases} y, & t = 0, \\ Z_{\zeta(Z)-t}, & 0 < t < \zeta(Z), \\ \Delta, & t \geq \zeta(Z). \end{cases}$$

In general, a caret ($\hat{}$) will be used to denote reversal of a process with finite lifetime.

3.1 LEMMA. *Let* $\sigma, \tau \in \mathscr{F}_t$ *be positive. Then*

$$_yE^x[\sigma(Z)\tau(\hat{Z}), \zeta(Z) > 2t] = E^{x,y}[\sigma(X)\tau(Y)G(X_t, Y_t)/G(x, y)].$$

PROOF.

$$\begin{aligned} &_yE^x[\sigma(Z)\tau(\hat{Z}), \zeta(Z) > 2t] \\ &= {}_yE^x[\sigma(Z)\,{}_yE^{Z_t}[\tau(\hat{Z}), \zeta(Z) > t], \zeta(Z) > t] \\ &= {}_yE^x[\sigma(Z)\,{}_{Z_t}E^y[\tau(Z), \zeta(Z) > t], \zeta(Z) > t] \\ &= \frac{1}{G(x, y)}E^x[\sigma(X)\,{}_{X_t}E^y[\tau(z), \zeta(Z) > t]G(X_t, y)] \\ &= \frac{1}{G(x, y)}E^x\left[\sigma(X)\frac{1}{G(y, X_t)}E^y[\tau(Y)G(Y_t, X_t)]G(X_t, y)\right] \\ &= E^{x,y}[\sigma(X)\tau(Y)G(X_t, Y_t)/G(x, y)] \end{aligned}$$

by symmetry of Brownian motion under time reversal. □

The following result may be found in the Appendix [Proposition (7.6)], generalized to domains.

3.2 LEMMA. *Let* h_n *and* h *be strictly positive and superharmonic on* $\mathbb{R}^d$ *with* $h_n \to h$ *a.e. and* $h_n(x) \to h(x) < \infty$. *Then* $_{h_n}P^x \to {}_hP^x$ *weakly.*

PROOF OF THEOREM 1.2. Let

$$\begin{aligned} h_\varepsilon(x, y) &= P(|X_t^x - Y_t^y| < \varepsilon \text{ for some } t) \\ &= P(|X_{2t}^{x-y}| < \varepsilon \text{ for some } t) \\ &= C_d\varepsilon^{d-2}G(x - y, 0) = C_d\varepsilon^{d-2}G(x, y), \quad \text{whenever } |x - y| \geq \varepsilon. \end{aligned}$$

Let $X^{\varepsilon,x}$ and $Y^{\varepsilon,y}$ be X^x and Y^y killed at L_ε. Then [see Doob (1984)] conditioned on $L_\varepsilon > 0$, the $\mathbb{R}^{2d}$-valued process $(X_t^{\varepsilon,x}, Y_t^{\varepsilon,y})$ is an h_ε-transform of (X_t^x, Y_t^y), so that by Lemma 3.2 its law converges weakly to that of a $G(\cdot, \cdot)$

transform. We have that $W^\varepsilon = \Phi(X^{\varepsilon,x}, Y^{\varepsilon,y})$, where

$$\Phi(\omega,\omega')(s) = \begin{cases} \omega(s), & s < \zeta(\omega), \\ \omega'([\zeta(\omega) + \zeta(\omega') - s]-), & \zeta(\omega) \le s < \zeta(\omega) + \zeta(\omega'), \\ \Delta, & s \ge \zeta(\omega) + \zeta(\omega') \end{cases}$$

[defined on $\Omega_0 = \{(\omega,\omega'); \zeta(\omega') < \infty\}$]. Since Φ is continuous on Ω_0, we will have that the law of W^ε converges weakly once we know that $G(\cdot,\cdot)$-transforms have finite lifetimes a.s. This is easily seen either by a direct calculation that $G(\cdot,\cdot)$ is a potential in $\mathbb{R}^{2d}$ or by the observation that the difference of the first and second components of a $G(\cdot,\cdot)$-transform is itself a transform by $G(0,\cdot)$ in $\mathbb{R}^d$. The identification of the limit law of W^ε as that of ${}_yZ^x$ follows immediately from Lemma 3.1.

We will follow a similar approach to Theorem 1.3. The following generalization of Lemma 3.1 has a similar proof and will be omitted.

Let $Z, \hat{Z}$ be as in Lemma 3.1. Set

$$Z_r^{s,t} = \begin{cases} Z_{s+r}, & s+r+t < \zeta(Z), \\ \Delta, & \text{otherwise.} \end{cases}$$

3.3 LEMMA. *Let $\sigma \in \mathscr{F}_s$, $\tau \in \mathscr{F}_t$, $\xi \in \mathscr{F}_\infty$ all be positive. Then*

$$ {}_yE^x[\sigma(Z)\tau(\hat{Z})\xi(Z^{s,t}), \zeta(Z) > s+t] = E^{x,y}\big[\sigma(X)\tau(Y)\,{}_{Y_t}E^{X_s}[\xi(Z)]G(X_s,Y_t)/G(x,y)\big].$$

Let

$$g_\varepsilon(x,y) = P^{x,y}(|X_s - Y_t| < \varepsilon \text{ for some } s,t).$$

As before, we must determine the asymptotic behavior of g_ε. Since this is trickier than before, we will merely record the result, postponing the proof to Section 5A. Let $g(x,y) = \int G(x,z)G(y,z)\,dz$, which, by scaling, is $c|x-z|^{-(d-4)}$ for a constant c not depending on x or y.

3.4 LEMMA. *$g_\varepsilon(x',y')/g_\varepsilon(x,y) \to g(x',y')/g(x,y)$ as $\varepsilon \downarrow 0$ for any $x' \ne y'$, $x \ne y$.*

We will show the following in Section 5A as well.

3.5 LEMMA. *For each $s > 0$ and $x \ne y$,*

$$P^{x,y}(S_\varepsilon < s < M_\varepsilon)/g_\varepsilon(x,y) \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

PROOF OF THEOREM 1.3. Let

$$U_{s,t}^\varepsilon = \begin{cases} (X_s^x, Y_t^y), & \text{if } \exists\, s' > s,\ t' > t \text{ s.t. } |X_{s'}^x - Y_{t'}^y| < \varepsilon, \\ (\Delta,\Delta), & \text{otherwise.} \end{cases}$$

Condition on $\{M_\varepsilon > 0\}$. By Proposition 2.1, U^ε is a g_ε-bitransform (see Section 2 for the definition). In Section 7 we prove a weak convergence result for such bitransforms, analogous to Lemma 3.2. To apply it, we must verify condition (7.7), but Lemma 3.5 does precisely this. The conclusion we obtain is exactly that the joint law of $(U^{1,\varepsilon}, U^{2,\varepsilon})$ on $\Omega \times \Omega$, conditioned on $\{M_\varepsilon > 0\}$, converges weakly to that of some pair (U^1, U^2), where

$$U_s^{1,\varepsilon} = \begin{cases} X_s^x, & s < M^\varepsilon, \\ \Delta, & s \geq M^\varepsilon, \end{cases}$$

$$U_t^{2,\varepsilon} = \begin{cases} Y_t^y, & t < N^\varepsilon, \\ \Delta, & t \geq N^\varepsilon. \end{cases}$$

Since $V^\varepsilon = \Phi(U^{1,\varepsilon}, U^{2,\varepsilon})$, we have as before that the law of V^ε converges weakly to that of $V = \Phi(U^1, U^2)$.

To identify the law of V, let I be uniform on $[0,1]$ and independent of ${}_yZ^x$. Let

$$Z_s^1 = \begin{cases} Z_s, & s < I\zeta(Z), \\ \Delta, & \text{otherwise}, \end{cases}$$

$$Z_t^2 = \begin{cases} \hat{Z}_t, & t < (1-I)\zeta(Z), \\ \Delta, & \text{otherwise}. \end{cases}$$

Since $V = \Phi(U^1, U^2)$ and $Z = \Phi(Z^1, Z^2)$, we need to show that the joint law of (U^1, U^2) is absolutely continuous with respect to that of (Z^1, Z^2), with Radon–Nikodym derivative $\zeta(Z)/{}_yE^x\zeta(Z)$. Thus let $\sigma \in \mathscr{F}_s$ and $\tau \in \mathscr{F}_t$ be positive. Recall that $g(x,y) = G(x,y){}_yE^x\zeta(Z)$. By Lemma 3.3,

$$\begin{aligned}
E&\big[\sigma(U^1)\tau(U^2), \zeta(U^1) > s \text{ and } \zeta(U^2) > t\big] \\
&= E^{x,y}\big[\sigma(X)\tau(Y)g(X_s, Y_t)/g(x,y)\big] \\
&= E^{x,y}\big[\sigma(X)\tau(Y)\big(G(X_s,Y_t){}_{Y_t}E^{X_s}\zeta(Z)\big)/\big(G(x,y){}_yE^x\zeta(Z)\big)\big] \\
&= {}_yE^x\big[\sigma(Z)\tau(\hat{Z})(\zeta(Z) - s - t)/{}_yE^x\zeta(Z), \zeta(Z) > s + t\big] \\
&= {}_yE^x\big[\sigma(Z)\tau(\hat{Z})\zeta(Z)/{}_yE^x\zeta(Z), I\zeta(Z) \in (s, \zeta(Z) - t)\big] \\
&= {}_yE^x\big[\sigma(Z^1)\tau(Z^2)\zeta(Z)/{}_yE^x\zeta(Z), \zeta(Z^1) > s \text{ and } \zeta(Z^2) > t\big],
\end{aligned}$$

as required. □

It is now clear how to modify the preceding argument to obtain

3.6 COROLLARY. *As* $\varepsilon \downarrow 0$, *the joint law of* $(V^\varepsilon, M_\varepsilon)$ *conditioned on* $\{M_\varepsilon > 0\}$ *converges weakly to that of a pair* (V, M). $M/\zeta(V)$ *is uniform on the interval* $[0,1]$ *and is independent of* V.

4. Asymptotics 1. This section covers the proof of Theorem 1.4. The arguments given in Section 3 apply equally well in this context, once we show the following [recall that X^x and Y^y are now Brownian motions, killed upon leaving

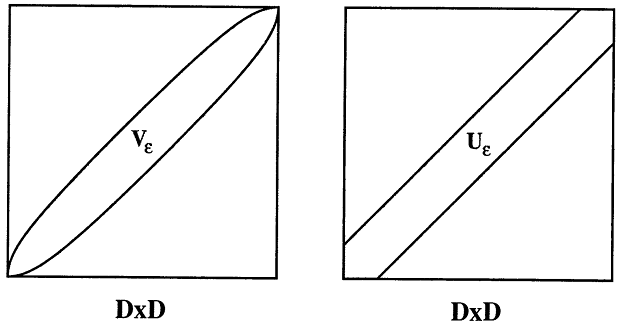

FIG. 1.

a domain $D \subset \mathbb{R}^d$, and that $h_\varepsilon(x, y) = P(|X_t^x - Y_t^y| < \varepsilon$ for some $t < \zeta(X^x) \wedge \zeta(Y^y))]$:

4.1 PROPOSITION. *Let $x, y \in D$, $x \neq y$.*

(a) *If $d \geq 3$, then $h_\varepsilon(x, y) \sim C_d G(x, y)\varepsilon^{d-2}$.*

(b) *If $d = 2$, then $h_\varepsilon(x, y) \sim C_2 G(x, y)/\log(1/\varepsilon)$ provided D has a Green function.*

These statements are proved in Sections 4A and 4B, respectively.

First we sketch the proof for $d \geq 3$. The analogue of (a) with h_ε replaced by $\tilde{h}_\varepsilon(x, y) = P(G(X_t^x, Y_t^y) = \varepsilon^{-(d-2)}$ for some $t < \zeta(X^x) \wedge \zeta(Y^y))$ is easily proved. Observe that h_ε is harmonic on $D \times D - U_\varepsilon$, where U_ε is all (z_1, z_2) satisfying $|z_2 - z_1| \leq \varepsilon$, with boundary values 1 on U_ε and 0 elsewhere, while $\tilde{h}_\varepsilon$ differs only in that U_ε is replaced by a different neighborhood V_ε of the diagonal in $D \times D$, which is asymptotically the same except at the boundary of $D \times D$ (see Figure 1). We must show that the parts near the boundary do not count.

Formally, let $T = T_\varepsilon$ be $\inf\{t > 0;\ |X_t^x - Y_t^y| < \varepsilon,\ t < \zeta(X^x) \wedge \zeta(Y^y)\}$ and let D_n be relatively compact domains $\overline{D}_n \subset D_{n+1}$, $D_n \uparrow D$. We prove

4.2 LEMMA. *Under the conditions of Proposition* 4.1,

$$\lim_{n \to \infty} \limsup_{\varepsilon \downarrow 0} P^{x,y}\big(T_\varepsilon < \infty,\ X_{T_\varepsilon} \notin D_n\big)/P^{x,y}\big(T_\varepsilon < \infty\big) = 0.$$

To prove this, we essentially find a density for $X_{T_\varepsilon}^x$.

4A. $D \subset \mathbb{R}^d$, $d \geq 3$.

WARNING. Recall that P, G, etc., refer to BM killed upon leaving D. If we need (unkilled) BM on all of $\mathbb{R}^d$, we write P_0, G_0, etc.

Before proving Lemma 4.2 in dimensions greater than 2, we use it.

PROOF OF PROPOSITION 4.1(a). Since $G_0 - G$ is continuous on $D \times D$, letting $\varepsilon \downarrow 0$ and then $n \to \infty$ we have, by Lemma 4.2 (writing $T = T_\varepsilon$),

$$\begin{aligned} E\left[G(X_T^x, Y_T^y), X_T^x \in D_n, T < \infty\right] &\sim E\left[G_0(X_T^x, Y_T^y), X_T^x \in D_n, T < \infty\right] \\ &= C_d \varepsilon^{-(d-2)} P^{x,y}(X_T \in D_n, T < \infty) \\ &\sim C_d \varepsilon^{-(d-2)} P^{x,y}(T < \infty) = C_d \varepsilon^{-(d-2)} h_\varepsilon(x, y). \end{aligned}$$

That is, given $\theta > 0$, we can find $\varepsilon_0(\theta) = \varepsilon_0 > 0$ and $n(\varepsilon)$ such that if $\varepsilon < \varepsilon_0$ and $n > n(\varepsilon)$, then the ratio of the first to the last terms in this expression is within θ of 1. Also,

$$\begin{aligned} G(x, y) - E\left[G(X_T^x, Y_T^y), X_T^x \in D_n, T < \infty\right] &= E\left[G(X_T^x, Y_T^y), X_T^x \notin D_n, T < \infty\right] \\ &\le C_d \varepsilon^{-(d-2)} P^{x,y}(T < \infty)\left[P^{x,y}(X_T \notin D_n, T < \infty)/P^{x,y}(T < \infty)\right] \\ &\le G(x, y) P^{x,y}(X_T \notin D_n, T < \infty)/P^{x,y}(T < \infty) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

by Lemma 4.2 again, showing the result. □

PROOF OF LEMMA 4.2, $d \ge 3$. Write

$$\Theta(x, y; z) = \int_0^\infty p_t(x, z) p_t(y, z)\, dt,$$

$$\Theta_0(x, y; z) = \int_0^\infty p_t^0(x, z) p_t^0(y, z)\, dt.$$

Since $p_t^0(x, z) p_t^0(y, z)$ is the transition density at (z, z) of Brownian motion in $\mathbb{R}^{2d}$ started at (x, y), $\Theta_0(\cdot, \cdot; z)$ is harmonic in (x, y) off any neighborhood of (z, z). Thus

$$\Theta_0(x, y; z) = E^{x,y}\left[\Theta_0(X_T, Y_T; z)\right]. \tag{4A.1}$$

Let $|\gamma| = 1$. Then

$$\int_{\mathbb{R}^d} \Theta_0(0, \gamma; z)\, dz = \int_0^\infty p_{2t}(0, \gamma)\, dt = \tfrac{1}{2} G_0(0, \gamma) < \infty,$$

so

$$\int_{|z|<k} \Theta_0(0, \gamma; z)\, dz \Big/ \int_{\mathbb{R}^d} \Theta_0(0, \gamma; z)\, dz \to 1 \quad \text{as } k \to \infty.$$

By scaling,

$$\int_{|z|<\delta} \Theta_0(0, \varepsilon\gamma; z)\, dz \Big/ \int_{\mathbb{R}^d} \Theta_0(0, \varepsilon\gamma; z)\, dz \to 1, \tag{4A.2}$$

as $\varepsilon \downarrow 0$ for each fixed $\delta > 0$.

Now fix n and let $\delta > 0$ be so small that $B(D_n, \delta) \subset D_{n+1}$. By (4A.1) we have

$$(4A.3)\qquad \int_{\mathbf{R}^d} \Theta_0(x, y; z)\, dz = \left[\int_{\mathbf{R}^d} \Theta_0(0, \varepsilon\gamma; z)\, dz\right] P_0^{x,y}(T < \infty),$$

$$\begin{aligned}\int_{D \setminus D_n} \Theta_0(x, y; z)\, dz &= E_0^{x,y}\left[\int_{D\setminus D_n} \Theta_0(X_T, Y_T; z)\, dz\right] \\ &\geq E_0^{x,y}\left[X_T \in D \setminus D_{n+1}, \int_{B(X_T, \delta)} \Theta_0(X_T, Y_T; z)\, dz\right] \\ &= \left[\int_{|z|<\delta} \Theta_0(0, \varepsilon\gamma; z)\, dz\right] P_0^{x,y}(X_T \in D \setminus D_{n+1}).\end{aligned}$$

Thus by (4A.2),

$$(4A.4)\qquad \lim_{n\to\infty} \limsup_{\varepsilon \downarrow 0} P_0^{x,y}(T < \infty, X_T \in D \setminus D_{n+1}) / P_0^{x,y}(T < \infty) = 0.$$

To complete the proof, we need to show that

$$\liminf_{\varepsilon\downarrow 0} P^{x,y}(T < \zeta)/P_0^{x,y}(T < \infty) > 0.$$

But

$$\begin{aligned}\tfrac{1}{2} G(x, y) &= \int_D \Theta(x, y; z)\, dz \\ &= E^{x,y}\left[\int_D \Theta(X_T, Y_T; z)\, dz, T < \zeta\right] \\ &\leq \left(\int_{\mathbf{R}^d} \Theta_0(0, \varepsilon\gamma; z)\, dz\right) P^{x,y}(T < \zeta),\end{aligned}$$

so by (4A.3),

$$P^{x,y}(T < \zeta)/P_0^{x,y}(T < \infty) \geq G(x, y)/G_0(x, y) > 0. \qquad \square$$

NOTE. In (4A.4) we are not claiming that

$$\lim_{n\to\infty} \limsup_{\varepsilon\downarrow 0} P_0(|X_t^x - Y_t^y| < \varepsilon \text{ for some } t \text{ at which } X_t^x \in D \setminus D_{n+1})$$
$$\div P_0^{x,y}(T < \infty) = 0.$$

In fact, if $|\partial D| > 0$, this is in general false.

4B. $D \subset \mathbf{R}^2$. First, note that the proof of Lemma 4.2 in the last section works for $d = 2$, provided D is bounded. [If $D \subset B(0, r)$, say, then replace p_t^0, G_0, Θ_0, etc., by the transition density, etc., for Brownian motion killed upon leaving $B(0,2r)$. The main fact then is that

$$\frac{\inf\left\{\int_{B(x,\delta)}\Theta_0(x, y; z)\,dz; |x-y| \le \varepsilon, |x| \le r\right\}}{\sup\left\{\int_{B(0,2r)}\Theta_0(x, y; z)\,dz; |x-y| \le \varepsilon, |x| \le r\right\}} \to 1,$$

as $\varepsilon \downarrow 0$ for each $\delta > 0$.]

This is the only honest ingredient in this section; we will bootstrap our way from it to the proof of Proposition 4.1(b). We first handle that part of D near ∞ [(4B.5)] and then, more generally, the part near ∂D.

Let $D \subset \mathbb{R}^2$ have a Green function G. Then

$$G(x, y) \ge C_2 \log^+(\delta/|x-y|) \quad \text{if } d(x, \partial D) > \delta,$$

so that for $T = T_\varepsilon$,

$$\begin{aligned} G(x, y) &= E\left[G(X_T^x, Y_T^y), T < \infty\right] \\ &\ge C_2 \log^+(\delta/\varepsilon) P^{x,y}\left(d(X_T, \partial D) > \delta, T < \infty\right). \end{aligned} \tag{4B.1}$$

Let D_n be relatively compact domains $\overline{D}_n \subset D_{n+1}$, $D_n \uparrow D$. In general, let σ_A be the first exit time from A and let G_A be the Green function of A. Then by the first remark of this section,

$$G_{D_n}(x, y) = \lim_{\varepsilon \downarrow 0} C_2 \log(1/\varepsilon) P^{x,y}\left(T < \sigma_{D_n}\right). \tag{4B.2}$$

Let $\delta_n = d(D_n, \partial D) > 0$. We have that

$$\begin{aligned} G(x, y) &\ge \lim_{\delta \downarrow 0} \limsup_{\varepsilon \downarrow 0} C_2 \log(1/\varepsilon) P^{x,y}\left(T < \infty, d(X_T, \partial D) > \delta\right) \\ &\ge \lim_{n \to \infty} \limsup_{\varepsilon \downarrow 0} C_2 \log(1/\varepsilon) P^{x,y}\left(T < \infty, T < \sigma_{D_n}\right) \\ &= G(x, y). \end{aligned} \tag{4B.3}$$

The first statement is by (4B.1), the second by the implication that $T < \sigma_{D_n} \Rightarrow d(X_T, \partial D) > \delta_n$ and the third by (4B.2) and the fact that $G = \lim G_{D_n}$.

Thus we have equality throughout and, hence, also

$$\lim_{n \to \infty} \limsup_{\varepsilon \downarrow 0} \log(1/\varepsilon) P^{x,y}\left(T < \infty, d(X_T, \partial D) > \delta_n, T \ge \sigma_{D_n}\right) = 0. \tag{4B.4}$$

Since D has a Green function, there are two disjoint closed balls B_1 and B_2 so that $H_k = D \cup B_k$ has a Green function too: $k = 1, 2$. Apply (4B.4) to H_1 and H_2 rather than to D to find $\delta < d(B_1, B_2)/2$ and $r > 0$ such that

$$\begin{aligned} \limsup_{\varepsilon \downarrow 0} \log(1/\varepsilon) P^{x,y}\big(T < \infty, T \ge \sigma_{B(0,r)}, d(X_T < B_1) > \delta \\ \text{or } d(X_T, B_2) > \delta\big) < \theta. \end{aligned}$$

Since $d(z, B_1) \vee d(z, B_2) > \delta$ for every $z \in \mathbb{R}^2$, we have in fact that

$$\limsup_{\varepsilon \downarrow 0} \log(1/\varepsilon) P^{x,y}\left(T < \infty, T > \sigma_{B(0,r)}\right) < \theta. \tag{4B.5}$$

Then apply our first remark once more, to find $\delta > 0$ such that

$$\text{(4B.6)} \quad \limsup_{\varepsilon \downarrow 0} \log(1/\varepsilon) P^{x,y}\big(T < \infty, T < \sigma_{B(0,r)}, \text{ but } d(X_T, \partial D) < \delta\big) < \theta.$$

Combining (4B.5) and (4B.6), we have that

$$\text{(4B.7)} \qquad \limsup_{\varepsilon \downarrow 0} \log(1/\varepsilon) P^{x,y}(T < \infty, X_T \in B(\partial D, \delta)) < 2\theta.$$

Together with (4B.3), this yields both Proposition 4.1(b) and Lemma 4.2 (for $d = 2$).

5. Asymptotics 2. To complete the proof of Theorem 1.3, we must establish Lemmas 3.4 and 3.5. We will do this in Section 5A and will show the analogous results for domains in Sections 5B and 5C. Since the arguments of Section 3 apply equally well to domains, this will also show Theorem 1.6.

We will restate the results to be shown and, for ease of reference, separate the hypotheses into three cases:

5.1 PROPOSITION. *For $x, x', y, y' \in D$, $x \neq y$, $x' \neq y'$, we have that $g_\varepsilon(x', y')/g_\varepsilon(x, y) \to g(x', y')/g(x, y)$ as $\varepsilon \downarrow 0$, provided*

(a) $D = \mathbb{R}^d$, $d \geq 5$, *or*
(b) *D is a domain in $\mathbb{R}^d$, $d \geq 5$, or*
(c) *D is a domain in $\mathbb{R}^4$ and $g(x, y) < \infty$.*

Recall that S_ε and M_ε are, respectively, the first and last times s such that $|X_s^x - Y_t^y| < \varepsilon$ for some t.

5.2 LEMMA. *Fix $x, y \in D$, $x \neq y$, $s > 0$. Then $P^{x,y}(S_\varepsilon < s < M_\varepsilon)/g_\varepsilon(x, y) \to 0$ as $\varepsilon \downarrow 0$, provided*

(a) $D = \mathbb{R}^d$, $d \geq 5$, *or*
(b) *D is a domain in $\mathbb{R}^d$, $d \geq 5$, or*
(c) *D is a domain in $\mathbb{R}^4$ and $g(x, y) < \infty$.*

Our approach is as before. The basic proof is that of (a). It requires minor modification in case (b), with a new lemma (Lemma 5B.1) needed, to the effect that the part of D near ∂D does not matter. In case (c) we must also worry about that part of D near ∞ [Lemma 5C.4(a)].

Though we have not stated them before [as we did in Proposition (4.1)], the decay rates of $g_\varepsilon(x, y)$ will emerge in the course of the proof. They are ε^{d-4}, ε^{d-4} and $1/\log(1/\varepsilon)$, respectively. With a little more work involving a scaling argument, we can show in cases (a) and (b) that $g_\varepsilon(x, y) \sim c(d)\varepsilon^{d-4} g(x, y)$, where the constants do not depend on the domain. Since we do not use this result, we omit the proof. We conjecture that in case (c), $g_\varepsilon(x, y) \sim cg(x, y)/\log(1/\varepsilon)$. We also have a completely different approach to these questions, which uses approximate Laplacians to compute the potential part of the superharmonic function $g_\varepsilon(\cdot, y)$, but it is significantly longer than the one given here.

5A. $D = \mathbb{R}^d$, $d \geq 5$.

5A.1 LEMMA. *Fix $x \neq y$.*

(a) *$\varepsilon^{-(d-4)} g_\varepsilon(x, y)$ is bounded in ε, above and away from zero.*

(b) *There is a constant c such that for each closed K,*

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^{-(d-4)} P^{x,y}\big(S_\varepsilon < \infty, X_{S_\varepsilon} \in K\big) \leq c \int_K G(x, z) G(y, z)\, dz.$$

(c) *For each $\rho > 0$,*

$$P\big(\text{diameter}\{z;\, z = X_s^x \text{ for some } s,\, |z - Y_t^y| < \varepsilon \text{ for some } t\} > \rho\big) = o(\varepsilon^{d-4}) \quad \text{as } \varepsilon \downarrow 0.$$

PROOF. (a) Consider first the upper bound on g_ε. Let Λ_ε be the lattice εZ^d and let $Q_\varepsilon = (-\varepsilon/2, \varepsilon/2)^d$. Set $b = 1 + \sqrt{d}/2$. We assume $2b\varepsilon < |x - y|$. Then

$$\begin{aligned} g_\varepsilon(x, y) &\leq \sum_{z \in \Lambda_\varepsilon} P(X^x \text{ and } Y^y \text{ hit } B(z, b\varepsilon)) \\ &\leq c\varepsilon^{d-2} + \sum_{\substack{z \in \Lambda_\varepsilon \\ |z-x|, |z-y| > 2b\varepsilon}} \big[b\varepsilon/|x - z|\big]^{d-2} \big[b\varepsilon/|y - z|\big]^{d-2} \\ &\leq c\varepsilon^{d-2} + c\varepsilon^{d-4} \sum_{\substack{z \in \Lambda_\varepsilon \\ |z-x|, |z-y| > 2b\varepsilon}} \int_{z+Q_\varepsilon} |x - z|^{-(d-2)} |y - z|^{-(d-2)}\, dz \\ &\leq c\varepsilon^{d-2} + c\varepsilon^{d-4} g(x, y), \end{aligned}$$

giving the upper bound.

For the lower bound we assume, without loss of generality, that $x = (0, \ldots, 0)$, $y = (1, 0, \ldots, 0)$ and ε is of the form 2^{-n} (so that we have $x, y \in \Lambda_\varepsilon$). Then for each $\beta \in (0, \frac{1}{2})$, it follows as before that

$$\begin{aligned} g_\varepsilon(x, y) &\geq P\bigg(\bigcup_{z \in \Lambda_\varepsilon \setminus \{x, y\}} \{X^x \text{ and } Y^y \text{ hit } B(z, \beta\varepsilon)\}\bigg) \\ &\geq \sum_{z \in \Lambda_\varepsilon \setminus \{x, y\}} P(X^x \text{ and } Y^y \text{ hit } B(z, \beta\varepsilon)) \\ &\quad - \frac{1}{2} \sum_{\substack{z, z' \in \Lambda_\varepsilon \setminus \{x, y\} \\ z \neq z'}} P(X^x \text{ and } Y^y \text{ hit both } B(z, \beta\varepsilon) \text{ and } B(z', \beta\varepsilon)). \end{aligned}$$

Now $P(X^x$ and Y^y hit both $B(z, \beta\varepsilon)$ and $B(z', \beta\varepsilon))$ is majorized by $P_{zz} + P_{zz'} + P_{z'z} + P_{z'z'}$, where, for example, $P_{zz'}$ is the probability that X^x hits first $B(z, \beta\varepsilon)$ and then $B(z', \beta\varepsilon)$ and that Y^y hits first $B(z', \beta\varepsilon)$ and then $B(z, \beta\varepsilon)$. Each of these four probabilities is easily bounded, since the strong Markov property can be used. This, together with the preceding inequality, yields

$$
\begin{aligned}
g_\varepsilon(x, y) \geq{}& (\beta\varepsilon)^{2(d-2)} \sum_{z \in \Lambda_\varepsilon \setminus \{x, y\}} |x - z|^{-(d-2)} |y - z|^{-(d-2)} \\
&- \sum_{\substack{z, z' \in \Lambda_\varepsilon \setminus \{x, y\} \\ z \neq z'}} (\beta\varepsilon)^{4(d-2)} |x - z|^{-(d-2)} |z - z'|^{-2(d-2)} \\
&\qquad \times \left[(y - z)^{-(d-2)} + |y - z'|^{-(d-2)} \right] \\
\geq{}& c\beta^{2(d-2)} \varepsilon^{d-4} \int_{\mathbf{R}^d} |x - z|^{-(d-2)} |y - z|^{-(d-2)}\, dz \\
&- c\beta^{4(d-2)} \varepsilon^{2d-8} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d \setminus B(z, \varepsilon/2)} |x - z|^{-(d-2)} |z - z'|^{-2(d-2)} \\
&\qquad \times \left[|y - z|^{-(d-2)} + |y - z'|^{-(d-2)} \right] dz'\, dz.
\end{aligned}
$$

The second integral is $O(\varepsilon^{-(d-4)})$, so that $g_\varepsilon(x, y) \geq \varepsilon^{d-4}(c\beta^{2(d-2)} - c\beta^{4(d-2)})$, where the two (different) constants c do not depend on β. Thus we may choose β so small that the second factor is strictly positive, giving the lower bound.

(b) Let $F(K, \varepsilon) = F = \{z \in \Lambda_\varepsilon;\ z + \overline{Q}_\varepsilon \cap K \neq \varnothing\}$. Then as before,

$$
\begin{aligned}
&P^{x, y}\big(S_\varepsilon < \infty, X_{S_\varepsilon} \in K\big) \\
&\quad \leq \sum_{z \in F} P\big(X^x \text{ and } Y^y \text{ hit } B(z, b\varepsilon)\big) \\
&\quad \leq c\varepsilon^{d-2} + c\varepsilon^{d-4} \sum_{\substack{z \in F \\ |z - x|, |z - y| > 2b\varepsilon}} \int_{z + Q_\varepsilon} |x - w|^{-(d-2)} |y - w|^{-(d-2)}\, dw \\
&\quad \leq c\varepsilon^{d-2} + c\varepsilon^{d-4} \int_{B(K, b\varepsilon)} |x - z|^{-(d-2)} |y - z|^{-(d-2)}\, dz,
\end{aligned}
$$

from which the result follows.

(c) For ε small, the probability in question is bounded by

$$
\sum_{\substack{z, w \in \Lambda_\varepsilon \\ |z - w| \geq \rho/2}} P\big(X^x \text{ and } Y^y \text{ hit both } B(z, b\varepsilon) \text{ and } B(w, b\varepsilon)\big) \tag{5A.2}
$$

and as in part (a), this is easily seen to be $o(\varepsilon^{d-4})$. □

Proof of Proposition 5.1(a). We will approximate $g_\varepsilon(x, y)$ by $\sum_{z \in \Lambda_\delta} P(|X_s^x - Y_t^y| < \varepsilon$ for some s, t at which $X_s^x \in z + Q_{\delta - \rho})$ for suitable δ, ρ. Recall that $b = 1 + \sqrt{d}/2$ and set μ to be the uniform distribution on $\partial B(0, b)$. For $\varepsilon > 0$ let $\eta(\varepsilon) = P(|X_s^\mu - Y_t^\mu| < \varepsilon$ for some s, t at which $X_s^\mu \in Q_1)$.

The argument for the upper bound in Lemma 5A.1(a) shows that

$$
\eta(\varepsilon) = O(\varepsilon^{d-4}) \quad \text{as } \varepsilon \downarrow 0. \tag{5A.3}
$$

Let $\theta > 0$ and choose $\lambda > 0$ so small that

$$
\int_{B(x, 2\lambda) \cup B(y, 2\lambda)} |x - z|^{-(d-2)} |y - z|^{-(d-2)}\, dz \leq \theta. \tag{5A.4}
$$

Choose δ so that if $|z-x|>\lambda$ and $\delta'\le\delta$, then the hitting density of X^x on $\partial B(z,b\delta')$ (with respect to normalized surface area) lies between $(1-\theta)(b\delta'/|x-z|)^{d-2}$ and $(1+\theta)(b\delta'/|x-z|)^{d-2}$. The same is of course true for Y^y if $|z-y|>\lambda$ as well, so that in this case,

$$\begin{aligned}(1-\theta)^2(b\delta'/|x-z|)^{d-2}(b\delta'/|y-z|)^{d-2}\eta(\varepsilon/\delta')\\ \le P(|X_s^x-Y_t^y|<\varepsilon \text{ for some } s,t \text{ at which } X_s^x\in z+Q_{\delta'})\\ \le(1+\theta)^2(b\delta'/|x-z|)^{d-2}(b\delta'/|y-z|)^{d-2}\eta(\varepsilon/\delta'),\end{aligned}\tag{5A.5}$$

for ε small. By choosing δ possibly even smaller, we can also guarantee that if $|z-x|,|z-y|>\lambda$, then

$$\begin{aligned}(1-\theta)\delta^d|x-z|^{-(d-2)}|y-z|^{-(d-2)}\\ \le\int_{z+Q_\delta}|x-w|^{-(d-2)}|y-w|^{-(d-2)}\,dw\\ \le(1+\theta)\delta^d|x-z|^{-(d-2)}|y-z|^{-(d-2)}.\end{aligned}\tag{5A.6}$$

For $\rho\in(0,\delta/2)$, let $K=\bigcup_{z\in\Lambda_\delta}z+(Q_\delta\setminus Q_{\delta-\rho})$. Choose ρ so small that

$$\int_K|x-w|^{-(d-2)}|y-w|^{-(d-2)}\,dw\le\theta.\tag{5A.7}$$

Write $A_z^\varepsilon=\{|X_s^x-Y_t^y|<\varepsilon \text{ for some } s,t \text{ at which } X_s\in z+Q_{\delta-\rho}\}$. Then

$$\begin{aligned}\sum_{\substack{z\in\Lambda_\delta\\|z-x|,|z-y|>\lambda}}P^{x,y}(A_z^\varepsilon)-\sum_{\substack{z,w\in\Lambda_\delta\\z\ne w\\|z-x|,|z-y|,\\|w-x|,|w-y|>\lambda}}P^{x,y}(A_z^\varepsilon\cap A_w^\varepsilon)\\ \le g_\varepsilon(x,y)\\ \le\sum_{\substack{z\in\Lambda_\delta\\|z-x|,|z-y|>\lambda}}P^{x,y}(A_z^\varepsilon)+P^{x,y}(S_\varepsilon<\infty,X_{S_\varepsilon}\in K)\\ +P^{x,y}(S_\varepsilon<\infty,X_{S_\varepsilon}\in B(x,2\lambda)\cup B(y,2\lambda)).\end{aligned}\tag{5A.8}$$

Looking first at the upper bound, we use Lemma 5A.1(b) [applied with (5A.4) and (5A.7)], then (5A.5) (with $\delta'=\delta-\rho$), (5A.6) and (5A.3) to see that

$$\begin{aligned}g_\varepsilon(x,y)&\le\frac{(1+\theta)^2}{1-\theta}\frac{(b(\delta-\rho))^{2(d-2)}}{\delta^d}\eta\left(\frac{\varepsilon}{\delta-\rho}\right)\\ &\quad\times\sum_{\substack{z\in\Lambda_\delta\\|z-x|,|z-y|>\lambda}}\int_{z+Q_\delta}|x-w|^{-(d-2)}|y-w|^{-(d-2)}\,dw+c\theta\varepsilon^{d-4}\\ &\le\frac{(1+\theta)^2}{1-\theta}\frac{(b(\delta-\rho))^{2(d-2)}}{\delta^d}\eta\left(\frac{\varepsilon}{\delta-\rho}\right)g(x,y)+c\theta\varepsilon^{d-4}\\ &\le\frac{(b(\delta-\rho))^{2(d-2)}}{\delta^d}\eta\left(\frac{\varepsilon}{\delta-\rho}\right)g(x,y)+c\theta\varepsilon^{d-4}.\end{aligned}$$

Observe that the second sum in the left side of (5A.8) is bounded by a sum such as (5A.2), hence is $o(\varepsilon^{d-4})$. Thus we obtain a lower bound

$$g_\varepsilon(x, y) \geq \big(b(\delta - \rho)\big)^{2(d-2)}\delta^{-d}\eta(\varepsilon/(\delta - \rho))g(x, y) - c\theta\varepsilon^{d-4},$$

now using (5A.5), (5A.6), (5A.4), (5A.7) and (5A.3). If $x' \neq y'$, we may choose δ and ρ to give the same conclusions for x' and y' as well. Thus, for small enough ε,

$$\left|\frac{g_\varepsilon(x, y)}{g(x, y)} - \frac{g_\varepsilon(x', y')}{g(x', y')}\right| \leq c\theta\varepsilon^{d-4}.$$

The conclusion of Proposition 5.1 now follows, using the lower bound in Lemma 5A.1(a) and letting ε and then $\theta \to 0$. □

PROOF OF LEMMA 5.2(a). Fix $x \neq y$ and let $\alpha > 0$. For $\delta > 0$, let σ_δ be the first time X^x leaves $B(x, \delta)$. Then Lemma 5A.1(b) lets us find δ so small that

$$P^{x,y}(S_\varepsilon \leq \sigma_\delta) \leq \alpha\varepsilon^{d-4} \quad \text{as } \varepsilon \downarrow 0. \tag{5A.9}$$

Let β be a bound on the density of σ_δ. For any $\varepsilon > 0$, it will also be a bound on the density of $\sigma_\delta + S_\varepsilon \circ \theta_{\sigma_\delta}$, conditioned on $S_\varepsilon \circ \theta_{\sigma_\delta} < \infty$. Thus, if we choose $\gamma < \alpha/\beta$, we obtain

$$\begin{aligned} P^{x,y}\big(\sigma_\delta < S_\varepsilon, S_\varepsilon \in (s - \gamma, s)\big) &< \alpha P^{x,y}(S_\varepsilon < \infty) \\ &\leq c\alpha\varepsilon^{d-4} \quad \text{as } \varepsilon \downarrow 0. \end{aligned} \tag{5A.10}$$

By the strong Markov property of X at S_ε we can find ρ so small that

$$P^{x,y}\big(S_\varepsilon < \infty, |X_{S_\varepsilon + t} - X_{S_\varepsilon}| < \rho \text{ for some } t \geq \gamma\big) \leq \alpha\varepsilon^{d-4} \quad \text{as } \varepsilon \downarrow 0. \tag{5A.11}$$

Finally, Lemma 5A.1(c) shows that for ε sufficiently small,

$$P^{x,y}\big(S_\varepsilon < \infty, |X_{S_\varepsilon} - X_{M_\varepsilon}| > \rho\big) \leq \alpha\varepsilon^{d-4}. \tag{5A.12}$$

Putting (5A.9)–(5A.12) together shows that

$$P^{x,y}(S_\varepsilon < s < M_\varepsilon) < \alpha(3 + c)\varepsilon^{d-4}$$

as required. □

5B. $D \subset \mathbb{R}^d$, $d \geq 5$. Observe that Lemma 5.2(b) follows from Lemma 5.2(a) [note that Lemma 5A.1(a) easily generalizes], so that only Proposition 5.1(b) need concern us here. Recall once more that $P_0^{x,y}$ refers to Brownian motion on all of $\mathbb{R}^d$, the notation $P^{x,y}$ now being reserved for Brownian motion killed upon leaving D.

As in Section 4 we must handle the part of D near ∂D. Let D_n be relatively compact domains, $\overline{D}_n \subset D_{n+1}$, $D_n \uparrow D$. If ∂D has Lebesgue measure 0, then the following is a consequence of Lemma 5A.1(b).

5B.1 LEMMA.

$$\lim_{n\to\infty} \limsup_{\varepsilon\downarrow 0} \varepsilon^{-(d-4)} P^{x,y}\big(S_\varepsilon < \infty, X_{S_\varepsilon} \notin D_n\big) = 0.$$

PROOF. Lemma 5A.1(b) shows that

$$\lim_{r\to\infty} \limsup_{\varepsilon\downarrow 0} \varepsilon^{-(d-4)} P_0^{x,y}\big(S_\varepsilon < \infty, |X_{S_\varepsilon}| \geq r\big) = 0.$$

Thus it will suffice to show, for $r > 0$ fixed, that

$$\text{(5B.2)} \quad \lim_{n\to\infty} \limsup_{\varepsilon\downarrow 0} \varepsilon^{-(d-4)} P_0^{x,y}\big(S_\varepsilon < \infty, X_{S_\varepsilon} \in B(0,r) \cap D \setminus D_n\big) = 0.$$

Without loss of generality, we assume $B(x,3) \cup B(y,3) \subset D_n$ for each n. Let $\mu(dz, dw)$ be the probability on $B(x,1) \times B(y,1)$ under which z is uniform on $B(x,1)$ and $w = z + y - x$. Write

$$P_0^\mu = \int P_0^{z,w} \mu(dz, dw).$$

Let σ_ρ be the first time X leaves $B(x,\rho)$ and let τ_ρ be the first time Y leaves $B(y,\rho)$. In addition to

$$S_\varepsilon = \inf\{s > 0; |X_s - Y_t| < \varepsilon \text{ for some } t > 0\},$$

we will consider

$$\Sigma_\varepsilon = \inf\{s > \sigma_2; |X_s - Y_t| < \varepsilon \text{ for some } t > \tau_2\}.$$

We will compare the measures

$$\begin{aligned}
a_\varepsilon(dz) &= P_0^{x,y}\big(S_\varepsilon < \infty, X_{S_\varepsilon} \in dz\big),\\
b_\varepsilon(dz) &= P_0^{\mu}\big(S_\varepsilon < \infty, X_{S_\varepsilon} \in dz\big),\\
\alpha_\varepsilon(dz) &= P_0^{x,y}\big(\Sigma_\varepsilon < \infty, X_{\Sigma_\varepsilon} \in dz\big),\\
\beta_\varepsilon(dz) &= P_0^{\mu}\big(\Sigma_\varepsilon < \infty, X_{\Sigma_\varepsilon} \in dz\big).
\end{aligned}$$

Let A be the event $\{|X_s - Y_t| < \varepsilon$ for some $X_s \in B(x,2) \cup B(y,2)$ and also for some $X_s \notin B(x,3) \cup B(y,3)\}$. By Lemma 5A.1(c), $P_0^\mu(A) = o(\varepsilon^{d-4})$. If B is any set which is disjoint from $B(x,3) \cup B(y,3)$, then clearly (for $\varepsilon < 1$)

$$\begin{aligned}
a_\varepsilon(B) &\leq \alpha_\varepsilon(B) \leq a_\varepsilon(B) + P_0^{x,y}(A),\\
b_\varepsilon(B) &\leq \beta_\varepsilon(B) \leq b_\varepsilon(B) + P_0^{\mu}(A).
\end{aligned}$$

The law of $(X_{\sigma_3}, Y_{\tau_3})$ under $P_0^{x,y}$ has a bounded density with respect to its law under P_0^μ. If C is such a bound and B is a set as before, we therefore get an inequality,

$$\alpha_\varepsilon(B) \leq C\beta_\varepsilon(B).$$

Finally, notice that the law of X_{S_ε} under $P_0^{x+z,y+z}$ is just a translate of its law under $P_0^{x,y}$. Thus b_ε is a convolution of a_ε with the uniform distribution on $B(x,1)$. Since $a_\varepsilon(\mathbb{R}^d) = O(\varepsilon^{d-4})$, we see that b_ε has a density which is bounded by $c\varepsilon^{d-4}$. Putting all this together, we see that for B as before,

$$a_\varepsilon(B) \le \alpha_\varepsilon(B) \le C\beta_\varepsilon(B) \le C(b_\varepsilon(B) + P_0^\mu(A))$$
$$\le C|B|\varepsilon^{d-4} + o(\varepsilon^{d-4}), \qquad |B| \text{ being the Lebesgue measure of } B.$$

In particular, since $B(0,r) \cap D \setminus D_n \downarrow \varnothing$, we have $|B(0,r) \cap D \setminus D_n| \downarrow 0$ and (5B.2) follows. □

The proof of Proposition 5.1(b) now proceeds as in Section 5A with only minor modifications: We choose n so large that $P^{x,y}(S_\varepsilon < \infty, X_{S_\varepsilon} \notin D_n) \le \theta\varepsilon^{d-4}$ and then approximate $g_\varepsilon(x, y)$ by

$$\sum_{\substack{z \in \Lambda_\delta \cap D_{n+1} \\ |z-x|, |z-y| > \lambda}} P\big(|X_s^x - Y_t^y| < \varepsilon \text{ for some } s, t \text{ at which } X_s^x \in z + Q_{\delta-\rho}\big)$$

for λ, δ and ρ sufficiently small. As long as we restrict z to be in D_{n+1}, all the approximations of Section 5A carry over (now of course using the Green function of D rather than the Newtonian one). For example, in the left-hand side of (5A.5) we need to replace $\eta(\varepsilon/\delta')$ by $\eta(\varepsilon/\delta', r/\delta')$, where

$$\eta(\varepsilon, r) = P_0\big(|X_s^\mu - Y_t^\mu| < \varepsilon \text{ for some } s, t \text{ such that } X_s^\mu \in Q_1, \text{ and } X^\mu \text{ and } Y^\mu \text{ stay inside } B(0,r) \text{ before times } s \text{ and } t, \text{ respectively}\big)$$

and the particular choice of r is $d(D_{n+1}, \partial D)$. Then, to approximate $\eta(\varepsilon/\delta', r/\delta')$ by $\eta(\varepsilon/\delta')$, we need to choose δ so small that $r/\delta \ge r_0$, where

$$\eta(\varepsilon, r_0) \ge (1-\theta)\eta(\varepsilon) \quad \text{as } \varepsilon \downarrow 0.$$

5C. $D \subset \mathbb{R}^d$, $d = 4$. In contrast to Sections 5A and 5B, we must first spend some time deriving the order of magnitude of g_ε. Once that is settled, we will prove a version of Lemmas 5A.1 and 5B.1 suitable for dimension 4, after which the argument settles into the pattern set in the preceding two sections.

Let $B = B(0,1)$. Lemma 5C.1 uses an argument of Erdös and Taylor (1960b) for the lower bound and one of Lawler (1982) for the upper.

5C.1 LEMMA. *There are constants c, c' such that if ε is small and if $|x|, |y| \le 100$ and $|x - y| \ge 1/100$, then*

$$c \le (\log 1/\varepsilon) P_0\big(|X_s^x - Y_t^y| < \varepsilon \text{ for some } s \text{ at which } X_s^x \in B\big) \le c'.$$

PROOF. *Lower bound.* Set $\delta(\varepsilon) = \varepsilon(\log 1/\varepsilon)^{1/4}$ and let $\beta < \frac{1}{2}$ remain to be chosen later. Then

$$P_0\big(|X_s^x - Y_t^y| < \varepsilon \text{ for some } s \text{ at which } X_s^x \in B\big)$$
$$\ge \sum_{z \in \Lambda_{\delta(\varepsilon)} \cap B(0,1/2)} P_0(X^x \text{ and } Y^y \text{ hit } B(z, \beta\varepsilon))$$
$$- \sum_{\substack{z, w \in \Lambda_{\delta(\varepsilon)} \cap B(0,1/2) \\ z \ne w}} P_0(X^x \text{ and } Y^y \text{ hit both } B(z,\beta\varepsilon) \text{ and } B(w, \beta\varepsilon)).$$

Arguing as in Lemma 5A.1, the first sum is bounded below by $c(\beta\varepsilon/\delta(\varepsilon))^4 =$

$c\beta^4/\log(1/\varepsilon)$ and the second is bounded above by

$$c(\beta\varepsilon/\delta(\varepsilon))^8 \int_B \int_{B \setminus B(0,\delta(\varepsilon)/2)} G_0(x,z) G_0(z,z')^2 [G_0(y,z) + G_0(y,z')] \, dz' \, dz$$

$$\leq c(\beta\varepsilon/\delta(\varepsilon))^8 \log(1/\delta(\varepsilon)) \leq c\beta^8/\log(1/\varepsilon).$$

The two constants c may differ, but they do not depend on β, so that choosing β small enough yields the desired lower bound.

Upper bound. Let $|\gamma| = 1$ and consider

$$\Gamma_r = \int_0^\infty 1_{B(0,r)}(Y_t^\gamma) G(0, Y_t^\gamma) \, dt.$$

Then it is easily seen that

$$E_0 \Gamma_r = \int_{B(0,r)} G_0(0,z) G_0(\gamma,z) \, dz = 2\pi^2 \log r + O(1),$$

$$E_0 \Gamma_r^2 = 2 \int_{B(0,r)} \int_{B(0,r)} G_0(0,z) G_0(0,w) G_0(\gamma,z) G_0(z,w) \, dw \, dz$$

$$= (2\pi^2 \log r)^2 + O(\log r) \quad \text{as } r \to \infty.$$

Thus $\mathrm{Var}(\Gamma_r) = O(\log r)$. Let

$$\Phi(\varepsilon) = \bigcup \{ B(z, 2\varepsilon/3); \, Y^y \text{ hits } B(z,\varepsilon/3) \cap B, \, z \in \Lambda_{\varepsilon/3} \}.$$

Let σ be the first time X^x hits $\Phi(\varepsilon)$ and, if it does so in $B(z, 2\varepsilon/3)$, let τ be the first time Y^y hits $B(z, \varepsilon/3)$. Then

$$\begin{aligned}
&\int_B G_0(x,z) G_0(y,z) \, dz \\
&\quad = E_0 \left[\int_0^\infty 1_B(Y_t^y) G_0(x, Y_t^y) \, dt \right] \\
&\quad = E_0 \left[\sigma < \infty, \int_0^\infty 1_B(Y_t^y) G_0(X_\sigma^x, Y_t^y) \, dt \right] \\
&\quad \geq (\pi^2/2)(\log 1/\varepsilon) \Big[P_0^{x,y}(\sigma < \infty) \\
&\qquad\qquad - P_0\Big(\sigma < \infty, \int_\tau^\infty 1_B(Y_t^y) G_0(X_\sigma^x, Y_t^y) \, dt \leq (\pi^2/2) \log 1/\varepsilon \Big) \Big].
\end{aligned}$$

Thus

$$\begin{aligned}
&P_0(|X_s^x - Y_t^y| < \varepsilon \text{ for some } s, t \text{ at which } Y_t^y \in B) \\
&\quad \leq P_0^{x,y}(\sigma < \infty) \\
&\quad \leq c[\log 1/\varepsilon]^{-1} + P_0\Big(\sigma < \infty, \int_\tau^\infty 1_B(Y_t^y) G_0(X_\sigma^x, Y_t^y) \, dt \leq (\pi^2/2) \log 1/\varepsilon \Big).
\end{aligned}$$

The latter probability is majorized by

$$O(\varepsilon^2) + \sum_{\substack{z \in \Lambda_{\varepsilon/3} \cap B(0,2) \\ |x-z|, |y-z| > 2\varepsilon}} \left[P_0(X^x \text{ hits } B(z, 2\varepsilon/3), Y^y \text{ hits } B(z, \varepsilon/3)) \right] \tag{5C.2}$$
$$\times \sup_{\substack{x' \in \partial B(z, 2\varepsilon/3) \\ y' \in \partial B(z, \varepsilon/3)}} P_0\left(\int_0^\infty 1_B(Y_t^{y'}) G_0(x', Y_t^{y'})\, dt \le c \log 1/\varepsilon \right) \Bigg].$$

As usual, we must worry about the boundary. Since the first factor in the preceding sum is $O(\varepsilon^4)$, we may peel off a strip about the boundary of thickness $2\varepsilon^{1/2}$ and get that (5C.2) is

$$O(\varepsilon^{1/2}) + O\left(P_0\left(\int_0^\infty 1_{B(x', \varepsilon^{1/2})}(Y_t^{y'}) G_0(x', Y_t^{y'})\, dt \le c \log 1/\varepsilon \right) \right),$$

where x' and y' are fixed, $|x' - y'| = \varepsilon$. By scaling, this is

$$O(\varepsilon^{1/2}) + O\left(P_0\left(\int_0^\infty 1_{B(0, \varepsilon^{-1/2})}(Y_t^\gamma) G_0(0, Y_t^\gamma)\, dt \le c \log(\varepsilon^{-1/2}) \right) \right)$$

and by Chebyshev's inequality and our computations involving Γ_r, this is $O(1/\log(1/\varepsilon))$ as required. □

For Γ open write G_Γ for the Green function of Γ and τ_Γ for the time of first exit from Γ. We take the convention that $G_\Gamma(\cdot, \cdot)$ vanishes off $\Gamma \times \Gamma$. Let $B' = B(0,2)$, $B = B(0,4)$ and set $\eta = \inf\{G_B(x, z); x, z \in B'\} > 0$.

5C.3 Lemma. *Given $\theta > 0$, there is an $\alpha = \alpha(\theta) > 0$ such that if Γ is an open subset of B, μ is a probability measure on $\overline{B}' \cap \Gamma$ and $G_\Gamma\mu(z) \le \eta/2$ on a subset of B' of Lebesgue measure at least θ, then*

$$P_0^\mu(\tau_\Gamma < \tau_B) \ge \alpha.$$

Proof. Let A be a Borel subset of B. Then

$$\begin{aligned}
\int_A G_B\mu(z)\, dz &= \int \left(\int_A G(x, z)\, dz \right) \mu(dx) \\
&= P_0^\mu\left(\int_0^{\tau_B} 1_A(X_t)\, dt \right) \\
&= P_0^\mu\left(\int_0^{\tau_\Gamma} 1_A(X_t)\, dt \right) + P_0^\mu\left(\int_{\tau_\Gamma}^{\tau_B} 1_A(X_t)\, dt \right) \\
&= \int_A G_\Gamma\mu(z)\, dz + \int_A G_B\nu(z)\, dz,
\end{aligned}$$

where

$$\nu(dx) = P_0^\mu(\tau_\Gamma < \tau_B, X_{\tau_\Gamma} \in dx).$$

Thus, $G_B\mu(z) = G_\Gamma\mu(z) + G_B\nu(z)$ for dz-a.a. $z \in B$ (recall our convention that

$G_\Gamma\mu = 0$ off Γ). In particular,

$$G_B\nu(z) \geq \eta - G_\Gamma\mu(z)$$

for dz-a.a. $z \in B'$. By our hypothesis on μ, we therefore have that $G_B\nu \geq \eta/2$ on a set of Lebesgue measure at least θ. Set

$$c = \sup_{x \in B} \int_B G_B(x, z)\,dz < \infty.$$

Then

$$c|\nu| \geq \int_B G_B\nu(z)\,dz \geq \eta\theta/2.$$

Taking α to be $\eta\theta/2c$, we get $|\nu| \geq \alpha$ as required. □

Now let D be a domain in $\mathbb{R}^4$ with $\int_D G(x,z)G(y,z)\,dz < \infty$ for some (and hence every) $x \neq y$. Aside from a new argument in part (a), Lemma 5C.4 amounts to reproving Lemmas 5A.1 and 5B.1. Recall that we take relatively compact domains D_n with $\overline{D}_n \subset D_{n+1}$ and $D_n \uparrow D$.

5C.4 LEMMA. (a) $\lim_{r\to\infty}\limsup_{\varepsilon\downarrow 0}(\log 1/\varepsilon)P(|X_s^x - Y_t^y| < \varepsilon$ *for some* s, t *at which* $|X_s^x| > r) = 0$.

(b) *For each* $x \neq y$, $(\log 1/\varepsilon)g_\varepsilon(x, y)$ *is bounded in* ε, *above and away from zero.*

(c) *There is a constant* c *such that for each compact* $K \subset D$,

$$\limsup_{\varepsilon\downarrow 0} (\log 1/\varepsilon)P^{x,y}\big(S_\varepsilon < \infty, X_{S_\varepsilon} \in K\big) \leq c\int_K G(x,z)G(y,z)\,dz.$$

(d) *For each* $\rho > 0$,

$$P\big(\textit{diameter}\,\{z;\, z = X_s^x \textit{ for some } s, |z - Y_t^y| < \varepsilon \textit{ for some } t\} > \rho\big)$$
$$= o(1/(\log 1/\varepsilon)) \textit{ as } \varepsilon \downarrow 0.$$

(e) $\lim_{n\to\infty}\limsup_{\varepsilon\downarrow 0}(\log 1/\varepsilon)P^{x,y}(S_\varepsilon < \infty, X_{S_\varepsilon} \notin D_n) = 0$.

PROOF. (a)

$$P(|X_s^x - Y_t^y| < \varepsilon \text{ for some } s, t \text{ at which } |X_s^x| > r)$$
$$\leq \sum_{\substack{z\in\Lambda_1\\ |z|>r/2}} P(|X_s^x - Y_t^y| < \varepsilon \text{ for some } s, t \text{ at which } X_s^x \in B(z,1))$$
$$\leq c(\log 1/\varepsilon)^{-1} \sum_{\substack{z\in\Lambda_1\\ |z|>r/2}} P(X^x \text{ and } Y^y \text{ hit } B(z,2)),$$

the latter by Lemma 5C.1 and the strong Markov properties of X^x and Y^y, say, at the first times they hit $B(z,1)$ and $B(z,2)$, respectively. Thus it will suffice to show that

$$\sum_{z \in \Lambda_1} P(X^x \text{ and } Y^y \text{ hit } B(z,2)) < \infty.$$

Let η be the constant of Lemma 5C.3. If μ and ν are measures on $\overline{B(z,2)} \cap D$ and

$$\int_{(z+Q_1)} G_{B(z,4)\cap D}\mu(z') G_{B(z,4)\cap D}\nu(z')\, dz' \le \eta^2/8$$

(with the convention of Lemma 5C.3 that $G_\Gamma \mu = 0$ off Γ), then at least one of $G_{B(z,4)\cap D}\mu$ or $G_{B(z,4)\cap D}\nu$ is less than $\eta/2$ on a set of $z' \in z + Q_1$ of Lebesgue measure bigger than $\theta = \frac{1}{4}$. Thus by Lemma 5C.3,

$$P_0^\mu\,(X \text{ leaves } D \text{ before it leaves } B(z,4)) \ge \alpha$$

or

$$P_0^\nu\,(Y \text{ leaves } D \text{ before it leaves } B(z,4)) \ge \alpha.$$

If we condition X^x and Y^y to hit $B(z,2)$ before leaving D and take μ and ν to be their respective hitting distributions, then we see that for each $z \in \Lambda_1$, at least one of the following holds:

$$\int_{(z+Q_1)\cap D} G(x,z')G(y,z')\, dz' \ge (\eta^2/8) P(X^x \text{ and } Y^y \text{ hit } B(z,2));$$

$$P(X^x \text{ leaves } D \text{ in } B(z,4)) \ge \alpha P(X^x \text{ hits } B(z,2));$$

$$P(Y^y \text{ leaves } D \text{ in } B(z,4)) \ge \alpha P(Y^y \text{ hits } B(z,2)).$$

Let I, II and III, respectively, be the sets of $z \in \Lambda_1$ such that these conditions hold. Then

$$\begin{aligned}\sum_{z\in\Lambda_1} & P(X^x \text{ and } Y^y \text{ hit } B(z,2)) \\ &\le \sum_{z\in \mathrm{I}} P(X^x \text{ and } Y^y \text{ hit } B(z,2)) + \sum_{z \in \mathrm{II}} P(X^x \text{ hits } B(z,2)) \\ &\quad + \sum_{z\in\mathrm{III}} P(Y^y \text{ hits } B(z,2)) \\ &\le (8/\eta^2)\int_D G(x,z')G(y,z')\,dz' + 2c'/\alpha < \infty,\end{aligned}$$

where $c' = \#\{z \in \Lambda_1;\ B(z,4) \text{ intersects } B(0,4)\}$.

(b) The upper bound in (b) has essentially just been shown. The lower bound follows from the argument of Lemma 5C.1 with trivial modifications.

(c) For $0 < \delta < d(K, \partial D)/2$, let $\kappa = \{z \in \Lambda_\delta;\ z + \overline{Q}_\delta \cap K \ne \varnothing\}$. Then

$$\begin{aligned}P^{x,y}(S_\varepsilon < \infty, X_{S_\varepsilon} \in K) &\le \sum_{z\in\kappa} P(|X_s^x - Y_t^y| < \varepsilon \text{ for some } s \\ &\qquad\qquad \text{at which } X_s^x \in B(z,\delta)) \\ &\le c(\log \delta/\varepsilon)^{-1} \sum_{z\in\kappa} P(X^x \text{ and } Y^y \text{ hit } B(z,2\delta))\end{aligned}$$

by Lemma 5C.1 and the strong Markov property [as in part (a)]. The conclusion of (c) now follows as in Lemma 5A.1, letting $\varepsilon \to 0$ and then $\delta \to 0$.

(d) For ε small, the probability in question is bounded by

$$\sum_{\substack{z, w \in \Lambda_{\rho'} \\ |z-w| > 2\rho'}} P\big(|X_s^x - Y_t^y| < \varepsilon \text{ for some } s, t \text{ at which } X_s^x \in B(z, \rho') \text{ and for some } s, t \text{ at which } X_s^x \in B(w, \rho')\big),$$

where $\rho' = \rho/5$ and we handle this as before.

(e) The argument is the same as that in Lemma 5B.1, except that instead of appealing to Lemma 5A.1(b), we use Lemma 5C.4(a). □

At this point we have obtained four dimensional versions of all the lemmas used in the proofs of Propositions 5.1(a), (b) and Lemma 5.2. It should be clear that (c) now follows as well, with only minor modifications of the old arguments.

6. Remarks. In dimension 4, there are several forms of our condition

$$\int G(x, z) G(y, z)\, dz < \infty. \tag{6.1}$$

It is clearly equivalent to having

$$ {}_yE^x \zeta(z) < \infty \tag{6.2}$$

(and we used this in Section 3). Moreover it is not hard to see that it is equivalent to

$$E^{x,y}\left[\int_0^{\zeta(X)} \int_0^{\zeta(Y)} 1_{B(0,\varepsilon)}(X_s - Y_t)\, ds\, dt\right] < \infty \tag{6.3}$$

for some $\varepsilon > 0$ [in one direction, use the argument of Lemma 5C.1(a)]. We have another proof of the dimension 4 result that involves looking at X only at the times

$$\sigma_1 = \inf\{s > 0; |X_s| \in \{2^k; k \geq 0\}\},$$

$$\sigma_{n+1} = \inf\{s > \sigma_n; |X_s| \in \{2^k; k \geq 0\} \setminus |X_{\sigma_n}|\}$$

(and similarly for Y). This argument shows that for $m_k = \#\{n; |X_{\sigma_n}| = 2^k\}$, the preceding conditions are equivalent to

$$\sum_{k=1}^{\infty} E^x[m_k^2] < \infty. \tag{6.4}$$

An interesting condition that is not equivalent is that $\zeta(X^x) < \infty$ a.s. Since $P_0(|X_s^x - Y_t^y| < \varepsilon$ for some $s, t) = 1$ [see, for example, Le Gall (1986b)] and (6.1) implies that $g_\varepsilon(x, y) \to 0$ as $\varepsilon \downarrow 0$, this condition is implied by any of the preceding [and moreover is equivalent to $g_\varepsilon(x, y) \to 0$]. The converse is not true; for example, consider a domain obtained by taking $\mathbb{R}^4$ and deleting concentric half-spheres whose radii increase so rapidly that X and Y expect to spend a long time within $1/n$ of each other during a transition between the nth radii.

In fact, in terms of the m_k, the condition that $\zeta(X^x) < \infty$ a.s. amounts to having only $E^x m_k \to 0$ as $k \to \infty$. In this case, the decay rate of $g_\varepsilon(x, y)$ need not be $1/\log(1/\varepsilon)$, as in general $\liminf(\log 1/\varepsilon) g_\varepsilon(x, y) \geq c\int G(x, z)G(y, z)\,dz$. Thus another condition equivalent to (6.1) is that

$$g_\varepsilon(x, y) = O(1/\log(1/\varepsilon)) \quad \text{for each } x, y. \tag{6.5}$$

Moreover, if (6.1) does not hold and we condition on $S_\varepsilon < \infty$, then $|X_{S_\varepsilon}|$ gets increasingly large as $\varepsilon \downarrow 0$. That is, the conditioning forces the paths to leave compact sets before dying. Since we worked hard in Section 5 to control exactly this kind of behavior, it is reassuring that this is where the result breaks down in general.

APPENDIX

Weak convergence of *h*-transforms. We turn to the weak convergence results needed in the foregoing. Let $D \subset \mathbb{R}^d$ be a domain with a Green function $G(x, y)$. Recall that Ω is the Skorohod space based on $\overline{D} \cup \{\Delta\}$, where $\overline{D} = D \cup \{\partial\}$ is the one-point compactification of D and Δ is an additional isolated point. Weak convergence is that of probabilities on Ω or $\Omega \times \Omega$.

In Lemma 7.1 there are two obstacles: $\zeta \to 0$ [hypothesized away by (7.2)] and oscillatory discontinuities at ζ. The latter is eliminated by our choice of compactification [we have not tried to verify the result for the Martin compactification, but it certainly fails for the Euclidean one (viz. Littlewood's crocodile)].

7.1 LEMMA. *Let (h_n) be strictly positive and superharmonic on D and let $x \in D$. Then ${}_{h_n}P^x$ are tight provided $h_n(x) < \infty\ \forall\, n$ and*

$$\inf\{h_n(y)/h_n(x);\ y \in B(x, \delta)\} \to 1 \tag{7.2}$$

uniformly in n as $\delta \downarrow 0$.

PROOF. Let ρ be a metric on $\overline{D} \cup \{\Delta\}$, compatible with its topology and satisfying $\rho(y, z) \geq |y - z|$ for $y, z \in D$. Write $|\Delta - y| = \infty$ for $y \in D$. Fix ε and R. For $\delta > 0$ let $K = \{y; \rho(y, \partial) > \varepsilon\}$ and define

$$\tau = \inf\{t; |X_s - X_t| > \varepsilon \text{ for some } s \in (t - \delta, t)\} \wedge R,$$

$$\tau' = \inf\{t > \tau;\ X_t \in K\} \wedge R.$$

If $\tau > \delta$, we may divide $[0, \tau)$ into equal intervals of length greater than or equal to $\delta/2$ on which the Euclidean increments of X are less than or equal to ε. By enlarging the last such interval, and only requiring the ρ-increments to be less than or equal to 2ε, the same is true on $[0, \tau')$ (now assuming only that $\tau' > \delta$). Thus we will have tightness [see Billingsley (1968) and Lindvall (1973)] provided the following hold (again, for x, ε, R fixed):

(7.3) $$_{h_n}P^x(\tau' < \zeta \wedge R) \to 0 \quad \text{uniformly in } n \text{ as } \delta \downarrow 0,$$

(7.4) $$_{h_n}P^x(\zeta \le \delta) \to 0 \quad \text{uniformly in } n \text{ as } \delta \downarrow 0.$$

To show (7.3), let h'_n be the réduite of h_n on K. Without loss of generality, we assume $x \in K$. We have that

$$\begin{aligned} _{h_n}P^x(\tau' < \zeta \wedge R) &= E^x[h_n(X_{\tau'})/h_n(x), \tau' < \zeta \wedge R] \\ &= E^x[h'_n(X_{\tau'})/h'_n(x), \tau' < \zeta \wedge R] \\ &= E^x[h'_n(X_\tau)/h'_n(x), \tau < \zeta \wedge R]. \end{aligned}$$

Since $P^x(\tau < \zeta \wedge R) \to 0$, we need only show that the $h'_n(X_\tau)/h'_n(x)$ are uniformly integrable (in n, δ).

First let $d \ge 3$. We build up to this by noting that if U is uniform on the unit sphere in $\mathbb{R}^d$, the following are uniformly integrable:

$$|U - z|^{-(d-2)}, \qquad z \in B(0,2)$$

[approximate $\partial B(0,1)$ by a disk, to get

$$E\left[|U - z|^{-(d-2)}, |U - z|^{-(d-2)} \ge n\right] \le c\int_{B^{d-1}(0,1)} |y|^{-(d-2)} 1_{\{|y|^{-(d-2)} \ge cn\}}\, dy,$$

which approaches 0 as $n \to \infty$],

$$|x + rU - z|^{-(d-2)}/|x - z|^{-(d-2)}, \qquad z \in \mathbb{R}^d \text{ and } r > 0,$$

(translation, scaling and then boundedness as $|z| \to \infty$),

$$|X_\tau^x - z|^{-(d-2)}/|x - z|^{-(d-2)}, \qquad z \in \mathbb{R}^d \text{ and } \varepsilon > 0, \text{ under } P_0^x$$

(condition on $|X_\tau^x - x|$, as X_τ^x is spherically symmetric about x under P_0^x),

$$G(X_\tau^x, z)/G(x, z), \qquad z \in K \text{ and } \delta > 0, \text{ under } P^x$$

[as $G(x, z) \ge c|x - z|^{-(d-2)}$ for $z \in K$ and $G(y, z) \le |y - z|^{-(d-2)}$ for $y, z \in D$] and

$$h'_n(X_\tau^x)/h'_n(x), \qquad n > 0 \text{ and } \varepsilon > 0, \text{ under } P^x$$

[as $h'_n(y)/h'_n(x) = \int_K G(y, z)/G(x, z)\nu(dz)$ for some probability ν], showing (7.3). The same argument works for $d = 2$, replacing $|y - z|^{-(d-2)}$ by $k + \log^+(1/|y - z|)$, where k is so large that this dominates $G(y, z)$ for $y \in D$, $z \in K$.

To show (7.4), let $\sigma_r = \inf\{t; |X_t - x| \ge r\}$. It is enough to show that

(7.5) $$_{h_n}P^x(\sigma_r < \zeta) \to 1 \quad \text{uniformly in } n \text{ as } r \downarrow 0,$$

and this follows easily from (7.2) since

$$_{h_n}P^x(\sigma_r < \zeta) = \frac{1}{h_n(x)} E^x\left[h_n(X_{\sigma_r})\right]. \qquad \square$$

7.6 PROPOSITION. *Let h_n and h be strictly positive and superharmonic on D, with $h_n \to h$ a.e., and $h_n(x) \to h(x) < \infty$. Then ${}_{h_n}P^x \to {}_hP^x$ weakly.*

PROOF. Let $\hat{h}_n$ be the superharmonic regularization of $\inf\{h_m; m \geq n\}$. Then $\lim \hat{h}_n = \lim h_n = h$ a.e., so since the $\hat{h}_n$ are increasing, in fact, $\hat{h}_n \to h$ everywhere. We have that $\hat{h}_n$ is lower semicontinuous. Thus for every $\varepsilon > 0$ and for every n_0, we may find $\delta > 0$ such that

$$h_n(y) \geq \hat{h}_n(y) \geq h_{n_0}(x) - \varepsilon \quad \text{for each } y \in B(x, \delta) \text{ and } n \geq n_0.$$

Since $h_n(x) \to h(x)$ and $\hat{h}_n(x) \to h(x)$, (7.2) follows immediately, showing tightness.

Thus we need only show convergence of the finite dimensional distributions. Let $\xi \in \mathscr{F}_t$. Then

$${}_{h_n}E^x[\xi, \zeta > t] = E^x[\xi h_n(X_t)/h_n(x)].$$

Without loss of generality, $\xi \leq 1_K(X_t)$ for some compact $K \subset D$, so that it suffices to show the uniform integrability of $1_K(X_t)h_n(X_t)/h_n(x)$ in n (t, K fixed). This follows as in Lemma 7.1. □

Recall from Section 2 that the paths of (two-parameter) h-bitransforms ${}_hU_{s,t}$ have the form

$$\omega_{s,t} = \begin{cases} (\omega_s^1, \omega_t^2), & (s,t) \in \zeta, \\ (\Delta, \Delta), & (s,t) \notin \zeta, \end{cases}$$

where $\zeta \subset \mathbb{R}_t^2$ is a lower layer. We are interested in limits with ζ a rectangle $[0, \zeta^1) \times [0, \zeta^2)$, but we need to approach such objects by h-transforms with nonrectangular ζ. Our approach will be to look at weak convergence (over $\Omega \times \Omega$) of the law of the pair $({}_hU^1, {}_hU^2)$, where ${}_hU_s^1$ is the first component of ${}_hU_{s,0}$ and ${}_hU_t^2$ is the second component of ${}_hU_{0,t}$. We will do so under the condition that

$$(7.7) \qquad {}_{h_n}P^{x,y}\big(U_s^1 \neq \Delta, U_t^2 \neq \Delta, \text{but } (s,t) \notin \zeta\big) \to 0 \quad \text{as } n \to \infty, \text{ for each } s, t > 0.$$

7.8 PROPOSITION. *Let (h_n), h be strictly positive biexcessive functions on $D \times D$. Let ${}_{h_n}U^{x,y}$ and ${}_hU^{x,y}$ be bitransforms of bi-Brownian motion by h_n and h, respectively. Assume* (7.7) *and that $\zeta({}_hU^{x,y})$ is rectangular. If $h_n \to h$ a.e. and $h_n(x, y) \to h(x, y)$, then the law of $({}_{h_n}U^1, {}_{h_n}U^2)$ converges weakly to that of $({}_hU^1, {}_hU^2)$.*

PROOF. For tightness, it suffices to show separately that the laws of ${}_{h_n}U^1$ and ${}_{h_n}U^2$ are tight. Since they are (one-parameter) transforms by $h_n(\cdot, y)$ and $h_n(x, \cdot)$ we need only check (7.2) for these. But biexcessive functions are excessive [Avanissian (1961)], so as in Proposition 7.6, in fact, (7.2) holds for $h_n(\cdot,\cdot)$ (which is even stronger).

Turning to the finite dimensional joint distributions, let $\sigma \in \mathscr{F}_s$, $\tau \in \mathscr{F}_t$. Then

$$\begin{aligned} {}_{h_n}E^{x,y}&\left[\sigma(U^1)\tau(U^2), U_s^1 \neq \Delta, U_t^2 \neq \Delta\right] \\ &\geq {}_{h_n}E^{x,y}\left[\sigma(U^1)\tau(U^2), (s,t) \in \zeta\right] \\ &\geq {}_{h_n}E^{x,y}\left[\sigma(U^1)\tau(U^2), U_s^1 \neq \Delta, U_t^2 \neq \Delta\right] \\ &\quad - {}_{h_n}P^{x,y}\left[U_s^1 \neq \Delta, U_t^2 \neq \Delta \text{ but } (s,t) \notin \zeta\right]. \end{aligned}$$

Moreover,

$${}_{h_n}E^{x,y}\left[\sigma(U^1)\tau(U^2), (s,t) \in \zeta\right] = E^{x,y}\left[\sigma(X)\tau(Y)h_n(X_s, Y_t)/h_n(x,y)\right],$$

by definition. The integrand converges a.e. by assumption, and the appropriate tightness is shown as before (using Avanissian's theorem again). Combined with (7.7), this establishes the result. □

Note that some path condition such as (7.7) is required. In fact, though this need not concern us here, simple examples show that h need not uniquely determine the law of an h-bitransform. If the h_n-bitransforms in question were known to have rectangular ζ's, then it turns out their laws would be determined, but in this case (7.7) would be trivially satisfied as well.

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