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STABILITY RESULTS AND STRONG INVARIANCE PRINCIPLES FOR PARTIAL SUMS OF BANACH SPACE VALUED RANDOM VARIABLES

By Uwe Einmahl

Universität zu Köln

A general stability theorem for B-valued random variables is obtained which refines a result of Kuelbs and Zinn. Our proof is based on two exponential inequalities for sums of independent B-valued r.v.'s essentially due to Yurinskii and appears particularly simple. We then use our theorem to prove strong invariance principles, LIL results and other related stability results for sums of i.i.d. B-valued r.v.'s in the domain of attraction of a Gaussian law. Most of these results seem to be still unknown for real-valued r.v.'s.

1. Introduction. Let B be a real separable Banach space with norm $\|\cdot\|$. Write Lt for $\log(t \vee e)$ and set $L_2 t := L(Lt)$ and $L_3 t := L(L_2 t)$, $t \geq 0$. Let X: $\Omega \to B$ be a random variable defined on a p-space $(\Omega, \mathscr{A}, P)$. Let further $\{X_n\}$ be a sequence of independent copies of X. Suppose that X satisfies the central limit theorem, i.e.,

$$P \circ \frac{1}{\sqrt{n}} \sum_1^n X_k \text{ converges weakly to } \mu = P \circ Y, \tag{1.1}$$

where Y: $\Omega \to B$ is a nondegenerate Gaussian mean zero random variable.

Let H_μ be the reproducing kernel Hilbert space of μ and denote by K_μ the unit ball of H_μ. Then K_μ is a compact subset of B, which plays a crucial role in the subsequent compact law of the iterated logarithm (LIL) for partial sums of i.i.d. B-valued r.v.'s satisfying the central limit theorem (CLT).

Theorem A [Goodman, Kuelbs and Zinn (1981) and Heinkel (1979)]. *Suppose that X satisfies* (1.1). *Then we have*: *With probability* 1, $\{\sum_1^n X_k / \sqrt{2nL_2 n}: n \in \mathbb{N}\}$ *is relatively compact in B and its limit set equals K_μ iff*

$$\sum_1^\infty P\left\{\|X\| > \sqrt{nL_2 n}\right\} < \infty. \tag{1.2}$$

It is now of great interest to find out whether related results can hold true when X does not satisfy CLT, but it is still in the domain of attraction of a Gaussian law, i.e.,

$$P \circ \frac{1}{a_n} \sum_1^n X_k \text{ converges weakly to } \mu = P \circ Y \text{ for some sequence } a_n \uparrow \infty. \tag{1.3}$$

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It is always possible to choose the above sequence $\{a_n\}$ in a way such that

$$b_n := a_n/\sqrt{n} \text{ is nondecreasing,} \tag{1.4}$$

which can therefore be assumed in the sequel.

Let b: $[0,\infty) \to [0,\infty)$ be the nondecreasing continuous function with $b(0) = 0$ and $b(n) = b_n$, $n \in \mathbb{N}$, obtained by linear interpolation between the integers.

Noticing that for real-valued random variables the CLT and the finiteness of the second moment are equivalent, we see that Theorem A in this case is just a refined version of the classical Hartman–Wintner LIL. The more general question in this setting, whether an appropriate LIL holds for random variables with infinite variances, was studied by Feller (1968) and Kesten (1972). From their work follows

THEOREM B. *Let X: $\Omega \to \mathbb{R}$ be a random variable satisfying* (1.3) *with $\mu = N(0,1)$ (standard normal distribution). Then we have: With probability* 1, *$\{\sum_1^n X_k/\sqrt{2nL_2n}\, b(nL_2n): n \in \mathbb{N}\}$ is relatively compact in $\mathbb{R}$ (bounded in $\mathbb{R}$) and the set of its limit points is a nontrivial subset of $K_\mu = [-1,1]$, iff*

$$\sum_1^\infty P\left\{|X| > \sqrt{nL_2n}\, b(nL_2n)\right\} < \infty. \tag{1.5}$$

In the two above-mentioned papers it was also discussed whether it is possible to obtain LIL results with respect to norming sequences other than that one considered in Theorem B. A basic result in this direction is the subsequent Theorem C which can be obtained from Theorem 7 of Kesten (1972).

THEOREM C. *Let X: $\Omega \to \mathbb{R}$ be a random variable satisfying* (1.3) *with $\mu = N(0,1)$. Let $\{c_n\}$ be a sequence of positive real numbers such that $c_n/\sqrt{n}$ is nondecreasing and*

$$c_n/\sqrt{nL_2n}\, b(nL_2n) \to \infty. \tag{1.6}$$

Then we have: With probability 1,

$$\limsup_{n\to\infty} \left|\sum_1^n X_k/c_n\right| = 0 \quad or \ = \infty$$

according as $\sum_1^\infty P\{|X| > c_n\}$ is finite or infinite.

Theorem C shows that there are no LIL results with respect to sequences $\{c_n\}$ satisfying (1.6), but that it is still possible to obtain stability results for these sequences.

Theorem C was later refined by Klass [(1976), (1977)]. From his work it follows that this result remains valid for sequences $\{c_n\}$ satisfying instead of (1.6) only

$$c_n/\sqrt{nL_2n}\, b(n/L_2n) \to \infty. \tag{1.7}$$

Since b is nondecreasing [see (1.4)], this clearly improves Theorem C.

It is now natural to ask whether an LIL can hold with respect to the norming sequence $\{\sqrt{2nL_2n}\, b(n/L_2n)\}$. This problem was partially solved by Klass who was able to establish an analogue of Theorem B. However, similarly as in Theorem B, the question remained open whether the limit set in this LIL equals the unit interval. Kuelbs (1985) finally extended Klass's LIL to the B-valued case. He was also able to determine the limit set, thereby answering the above question in the affirmative.

THEOREM D [Kuelbs (1985)]. *Let X: $\Omega \to B$ be a r.v. satisfying* (1.3). *Then we have*: *With probability* 1, $\{\sum_1^n X_k/\sqrt{2nL_2n}\, b(n/L_2n)\colon n \in \mathbb{N}\}$ *is relatively compact in B and its limit set equals K_μ iff*

$$(1.8) \qquad \sum_1^\infty P\Big\{\|X\| > \sqrt{nL_2n}\, b(n/L_2n)\Big\} < \infty.$$

Moreover, Kuelbs obtained a functional LIL under assumption (1.8).

In our paper [Einmahl (1988)] we established a strong invariance principle for partial sums of i.i.d. B-valued r.v.'s in the domain of attraction of a Gaussian law implying a compact and a functional LIL with canonical limit set under condition (1.5), which is weaker than (1.8).

In the present paper we want to further discuss the problem of which regular norming sequences one can obtain compact (functional) LIL's. The starting point of our investigations is a refined version of Theorem C being valid for sequences $\{c_n\}$ satisfying, for some $q \in \mathbb{R}$,

$$(1.9) \qquad c_n/\sqrt{nL_2n}\, b\big(n(L_2n)^q\big) \to \infty.$$

Our method of proof works for Banach-valued r.v.'s. Thus, we are able to establish this refined version of Theorem C even in the infinite-dimensional setting. Having obtained this result, it appears now reasonable to ask whether LIL results for B-valued r.v.'s can hold with respect to the norming sequences $\{\sqrt{2nL_2n}\, b(n(L_2n)^q)\}$. Our answer to this question is affirmative. In fact, we prove somewhat more: strong invariance principles implying compact (functional) LIL's with respect to the norming sequences $\{\sqrt{2nL_2n}\, b(n(L_2n)^q)\}$, $q \in \mathbb{R}$.

2. The results.

Let us first state our refined version of Theorem C.

THEOREM 1. *Let X: $\Omega \to B$ be a r.v. satisfying* (1.3). *Let $\{c_n\}$ be a sequence of positive real numbers such that $c_n/\sqrt{n}$ is nondecreasing and for some $q \in \mathbb{R}$,*

$$(2.1) \qquad c_n/\sqrt{nL_2n}\, b\big(n(L_2n)^q\big) \to \infty.$$

Then we have: *With probability* 1,

$$\limsup_{n\to\infty} \left\| \sum_1^n X_k / c_n \right\| = 0 \quad or \ = \infty$$

according as $\sum_1^\infty P\{\|X\| > c_n\}$ *is finite or infinite.*

Our next result contains the related strong invariance principles.

THEOREM 2. *Let* $X\colon \Omega \to B$ *be a r.v. satisfying* (1.3). *Let* $q \in \mathbb{R}$. *If the underlying p-space* $(\Omega, \mathscr{A}, P)$ *is rich enough, one can find a sequence* $\{Y_n\}$ *of independent copies of* Y *such that*

$$\max_{1\le m\le n} \left\| \sum_1^m X_k - b\big(n(L_2 n)^q\big) \sum_1^m Y_k \right\| = o\Big(\sqrt{nL_2 n}\, b\big(n(L_2 n)^q\big)\Big) \quad a.s. \tag{2.2}$$

iff

$$\sum_1^\infty P\Big\{\|X\| > \sqrt{nL_2 n}\, b\big(n(L_2 n)^q\big)\Big\} < \infty. \tag{2.3}$$

Combining (2.2) with the compact (functional) LIL for the sequence $\{Y_n\}$ of i.i.d. Gaussian mean zero random variables, we immediately obtain

COROLLARY 1. *Let* $X\colon \Omega \to B$ *be a r.v. satisfying* (1.3). *Let* $q \in \mathbb{R}$. *Then we have*: *With probability* 1, $\{\sum_1^n X_k / \sqrt{2nL_2 n}\, b(n(L_2 n)^q)\colon n \in \mathbb{N}\}$ *is relatively compact in* B *and its limit set equals* K_μ, *iff* (2.3) *holds.*

Let $C_B[0,1]$ be the space of all continuous B-valued functions on $[0,1]$, endowed with the sup-norm. Define $\eta_n\colon \Omega \to C_B[0,1]$ by

$$\eta_n(t) := \begin{cases} \sum_1^m X_k, & t = m/n,\ 0 \le m \le n, \\ \text{linearly interpolated elsewhere for } t \in [0,1]. \end{cases} \tag{2.4}$$

Denote by $\mathscr{K}_\mu$ the canonical limit set in the functional LIL for the Gaussian r.v. Y [cf. Kuelbs (1985), (2.6)].

COROLLARY 2. *Let* $X\colon \Omega \to B$ *be a r.v. satisfying* (1.3). *Let* $q \in \mathbb{R}$. *Then we have*: *With probability* 1, $\{\eta_n / \sqrt{2nL_2 n}\, b(n(L_2 n)^q)\colon n \in \mathbb{N}\}$ *is relatively compact in* $C_B[0,1]$ *and its limit set is* $\mathscr{K}_\mu$ *iff* (2.3) *holds.*

We now consider some special cases of Corollary 1. If $q = 1$, we obtain the compact LIL with respect to the norming sequence $\sqrt{2nL_2 n}\, b(nL_2 n) \sim a_{[2nL_2 n]}$, as proved by Einmahl (1988). If $q = -1$, we obtain Theorem D. Finally, applying Corollary 1 with $q = 0$, we see that the compact LIL holds with norming sequence $\{\sqrt{2nL_2n}\,b(nL_2n)\}$ as well as $\{\sqrt{2nL_2n}\,b(n)\}$, iff

$$\sum_1^\infty P\Big\{\|X\| > \sqrt{nL_2n}\,b(n)\Big\} < \infty. \tag{2.5}$$

This is now an infinite-dimensional version of Feller's LIL. Unfortunately, the original result of Feller is in error. From Theorem 1 of Feller (1968), specialized to r.v.'s in the domain of attraction of $N(0,1)$, it would follow that one always has an LIL with norming sequence $\{\sqrt{2nL_2n}\,b(n)\}$ when (1.5) holds. It was however shown in Section 5 of Einmahl (1988) that this is not true . [See also Kesten (1972), Remark 9.] As it now turns out, Feller's result can be proved—at least for r.v.'s satisfying (1.3)—under the more restrictive (and necessary) condition (2.5).

The proof of both Theorems 1 and 2 is based on the following general stability result.

THEOREM 3. *Let X: $\Omega \to B$ be a r.v. with $E[\|X\|^\eta] < \infty$ for some $\eta > 0$. Suppose that we have for some $t_0 > 0$,*

$$\limsup_{n\to\infty} P\left\{\left\|\sum_1^n X_k\right\| > t_0 a(n)\right\} \le 1/17e^2,$$

where a: $[1,\infty) \to (0,\infty)$ is a continuous function satisfying

$$b(t) := a(t)/\sqrt{t} \text{ is nondecreasing.} \tag{2.6}$$

Then, if $\{c_n\}$ is a sequence of positive real numbers such that

$$c_n/n^{1/3+\delta} \text{ is nondecreasing for some } \delta > 0 \tag{2.7}$$

and for some $p > 1$ and all $n \ge N_0$,

$$c_n \ge \max\Big\{a\big(n(L_3n)^2\big), \sqrt{nL_2n}\,b\big(n/(L_2n)^p\big)\Big\}, \tag{2.8}$$

we have

$$\limsup_{n\to\infty}\left\|c_n^{-1}\sum_1^n\big(X_k - E[X1\{\|X\| \le c_k\}]\big)\right\| \le t_0(960+160p) \quad a.s. \tag{2.9}$$

iff

$$\sum_1^\infty P\{\|X\| > c_k\} < \infty. \tag{2.10}$$

A closely related result is Theorem 5 of Kuelbs and Zinn (1983). This result, however, is only applicable to sequences $\{c_n\}$ such that c_n/n is bounded and $c_n \ge \sqrt{nL_2n}\,b(n/L_2n)$ for $n \ge N_0$. Therefore, it is neither possible to infer from it a result like Theorem C nor is it possible to prove LIL results with respect to the norming sequences $\{\sqrt{2nL_2n}\,b(n(L_2n)^q)\}$ if $q < -1$.

The remaining part of this paper is now organized as follows: In Section 3 we prove Theorem 3 and from it we then infer Theorem 1. Our proof is based on two exponential inequalities for partial sums of independent B-valued r.v.'s essentially due to Yurinskii (1976). The first one is an infinite-dimensional version of the Fuk–Nagaev inequality. Let us emphasize that the use of this inequality enables us to give a much easier proof of Theorem 3 than the one given by Kuelbs and Zinn (1983) for their weaker result. Our proof even leads to simplifications in the real-valued case [cf. Feller (1968) and Kesten (1972)]. The second exponential inequality—an infinite-dimensional Bernstein type inequality—is decisively needed to handle sequences $\{c_n\}$ satisfying (2.8), but not (1.7). Some technical lemmas used in the proof of Theorems 1 and 3 can be found in the Appendix.

In Section 4 we prove Theorem 2. Using Theorem 3, we are able to reduce the proof to the finite-dimensional case. Combining the multidimensional strong approximation technique developed in our papers [Einmahl (1987a), (1987b)] with a double truncation argument obtained from the proof of Theorem 3, we obtain Theorem 2 for $q \leq 1$. We finally prove this result for $q \geq 1$ by an application of Theorem 2 of Einmahl (1988).

3. Proof of Theorems 1 and 3. We first state the two exponential inequalities needed in the proof.

Let $Z_1, \ldots, Z_n \colon \Omega \to B$ be independent random variables. Suppose that $E[\|Z_1 + \cdots + Z_n\|] \leq \beta_n$ and that $E[\|Z_j\|^2] \leq \sigma_j^2$, $1 \leq j \leq n$. Put $B_n := \sum_1^n \sigma_j^2$.

INEQUALITY 1. *We have for* $t \geq 4\beta_n$,

$$P\left\{\left\|\sum_1^n Z_j\right\| \geq t\right\} \leq 9 \cdot 2^{11} t^{-3} \sum_1^n E\left[\|Z_j\|^3\right] + \exp\left(-t^2/96B_n\right).$$

INEQUALITY 2. *Suppose additionally that*

$$\max_{1 \leq j \leq n} \left(E\left[\|Z_j\|^m\right]/\sigma_j^2\right) \leq (m!/2)H^{m-2}, \qquad m = 3, 4, \ldots.$$

Then we have for $t \geq 4\beta_n$,

$$P\left\{\left\|\sum_1^n Z_j\right\| \geq t\right\} \leq \exp\left(-t^2/16B_n\right) \vee \exp(-t/64H).$$

PROOF. If $t \geq 16(\sum_1^n E[\|Z_j\|^3]/\beta_n)^{1/2} \vee 4\beta_n$, we obtain Inequality 1 immediately from Theorem 5.1 of Yurinskii (1976).

If $t \leq 16(\sum_1^n E[\|Z_j\|^3]/\beta_n)^{1/2}$, we get, from the Markov inequality,

$$P\left\{\left\|\sum_1^n Z_j\right\| \geq t\right\} \leq t^{-1}\beta_n \leq 256 t^{-3} \sum_1^n E\left[\|Z_j\|^3\right],$$

hence, Inequality 1 for $t \geq 4\beta_n$.

Inequality 2 follows from Theorem 2.1 of Yurinskii (1976), after some straightforward calculations. □

We still need the following simple, but nevertheless useful lemma.

LEMMA 1. *Let* $\{X_n\}$ *be a sequence of i.i.d. B-valued r.v.'s. Suppose that* $\{c_n\}$ *is a sequence of positive real numbers satisfying* (2.7) *such that* $\sum_1^\infty P\{\|X_n\| > c_n\} < \infty$. *Then we have*:

$$\text{(i)} \qquad \sum_1^\infty c_n^{-3} E\left[\|X_n\|^3 1\{\|X_n\| \le c_n\}\right] < \infty.$$

$$\text{(ii)} \qquad \sum_1^\infty c_n^{-1} E\left[\|X_n\| 1\{\varepsilon c_n \le \|X_n\| \le c_n\}\right] < \infty, \qquad \varepsilon \in (0,1).$$

$$\text{(iii)} \qquad \sum_1^\infty P\{\|X_n\| > \delta c_n\} < \infty, \qquad \delta > 0.$$

PROOF. (i) can be shown by the same argument as in Lemma 1 of Einmahl (1988).

(ii) follows from (i) using the simple inequality

$$E\left[\|X_n\| 1\{\varepsilon c_n \le \|X_n\| \le c_n\}\right] \le \varepsilon^{-2} c_n^{-2} E\left[\|X_n\|^3 1\{\|X_n\| \le c_n\}\right].$$

(iii) has only to be shown for $\delta \in (0,1)$. Since we have $\sum_1^\infty P\{\|X_n\| > c_n\} < \infty$, it suffices to prove $\sum_1^\infty P\{\delta c_n \le \|X_n\| \le c_n\} < \infty$. This follows immediately from (ii). □

We now proceed to the proof of Theorem 3. W.l.o.g. we assume $t_0 = 1$. We put $\alpha = 1/24$ and we set

$$X_n' := X_n 1\left\{\|X_n\| \le \alpha c_n/(L_2 n)^{(1+p)/2}\right\},$$

$$X_n'' := X_n 1\left\{\alpha c_n/(L_2 n)^{(1+p)/2} < \|X_n\| \le \alpha c_n\right\},$$

$$X_n''' := X_n 1\{\|X_n\| > \alpha c_n\}, \qquad n \in \mathbb{N}.$$

We further define the subsequences $\{m_k\}, \{n_k\}$ by the recursion

$$m_1 := 1, \qquad m_k := \min\left\{m \ge m_{k-1}\colon c_m \ge 2c_{m_{k-1}}\right\},$$

$$k \ge 2, \qquad n_k := m_{k+1} - 1, \qquad k \ge 1.$$

It suffices to show that

$$(3.1) \qquad \sum_{k=1}^\infty P\left\{\max_{m_k \le n \le n_k} \left\|\sum_{m_k}^n \left(X_j'' - E\left[X_j''\right]\right)\right\| \ge 40(1+p)c_{n_k}\right\} < \infty$$

and

$$\sum_{k=1}^{\infty} P\left\{ \max_{m_k \le n \le n_k} \left\| \sum_{m_k}^{n} \left(X_j' - E\left[X_j' \right] \right) \right\| \ge 200 c_{n_k} \right\} < \infty \tag{3.2}$$

when (2.10) holds.

This can be seen as follows: Using the Borel–Cantelli lemma and the definition of $\{m_k\}$, we obtain from (3.1) and (3.2)

$$\begin{aligned} &\limsup_{n\to\infty} \left\| c_n^{-1} \sum_{1}^{n} \left(X_k' + X_k'' - E\left[X1\{\|X\| \le \alpha c_k\} \right] \right) \right\| \\ &\qquad \le 960 + 160p \quad \text{a.s.} \end{aligned} \tag{3.3}$$

Since $\Sigma_1^n X_k''' = O(1)$ a.s. [use Lemma 1(iii)], (2.9) follows from (3.3) provided that we can show

$$\left\| \sum_{1}^{n} E\left[X1\{\|X\| \le \alpha c_k\} \right] - \sum_{1}^{n} E\left[X1\{\|X\| \le c_k\} \right] \right\| = o(c_n). \tag{3.4}$$

Employing Lemma 1(ii) and the Kronecker lemma, we immediately obtain (3.4) from (2.10).

Moreover, it is trivially seen that (2.10) is necessary for (2.9).

We use Inequality 1 for the proof of (3.1), whereas the proof of (3.2) is based on Inequality 2. In order to apply these inequalities we still need estimates for $E[\|\Sigma_m^n(X_j' - E[X_j'])\|]$ and for $E[\|\Sigma_m^n(X_j'' - E[X_j''])\|]$, $m_k \le m \le n \le n_k$.

LEMMA 2. *Under the assumptions of Theorem* 3, *we have for* $m_k \le m \le n \le n_k$, *if k is large enough,*

$$\max\left(E\left[\left\| \sum_{m}^{n} \left(X_j' - E\left[X_j' \right] \right) \right\| \right], E\left[\left\| \sum_{m}^{n} \left(X_j'' - E\left[X_j'' \right] \right) \right\| \right] \right) \le 2 c_{n_k}. \tag{3.5}$$

PROOF. We show for $n \ge n_0$ (say),

$$\begin{aligned} E\left[\left\| \sum_{1}^{n} \left(X_j' + X_j'' - E\left[X_j' + X_j'' \right] \right) \right\| \right] &\le c_n, \\ E\left[\left\| \sum_{1}^{n} \left(X_j' - E\left[X_j' \right] \right) \right\| \right] &\le c_n. \end{aligned} \tag{3.6}$$

Taking into account Lemma 2.7 from Chapter 3 of Araujo and Giné (1980), we see that (3.6) implies (3.5).

Since $c_n/a(n) \to \infty$, (3.6) easily follows from Lemma A.1 because we have, for sufficiently large n,

$$\max_{1\le k\le n} P\left\{\left\|\sum_1^k X_j\right\| \ge \alpha c_n\right\} \le 1/16e^2. \qquad \square$$

PROOF OF (3.1). Using Lemma 2, we easily obtain from the Markov inequality if k is large enough,

$$(3.7)\qquad \max_{m_k\le m\le n_k} P\left\{\left\|\sum_m^{n_k}\left(X_j'' - E\left[X_j''\right]\right)\right\| \ge 20(1+p)c_{n_k}\right\} \le 1/10.$$

Employing Ottaviani's inequality, we get from (3.7),

$$(3.8)\qquad \begin{aligned} &P\left\{\max_{m_k\le n\le n_k}\left\|\sum_{m_k}^{n}\left(X_j'' - E\left[X_j''\right]\right)\right\| \ge 40(1+p)c_{n_k}\right\}\\ &\quad\le (10/9)P\left\{\left\|\sum_{m_k}^{n_k}\left(X_j'' - E\left[X_j''\right]\right)\right\| \ge 20(1+p)c_{n_k}\right\}. \end{aligned}$$

Applying Inequality 1 with $\beta_n = 2c_{n_k}$ and $\sigma_j^2 := 4E[\|X''_{n_{k-1}+j}\|^2]$, $1 \le j \le n_k - n_{k-1}$, we obtain

$$\begin{aligned} &P\left\{\left\|\sum_{m_k}^{n_k}\left(X_j'' - E\left[X_j''\right]\right)\right\| \ge 20(1+p)c_{n_k}\right\}\\ &\quad\le 20^{-3}9\cdot 2^{11}\cdot 8c_{n_k}^{-3}\sum_{m_k}^{n_k} E\left[\|X_j''\|^3\right] + \exp\left(-(1+p)^2c_{n_k}^2\Big/\sum_{m_k}^{n_k}E\left[\|X_j''\|^2\right]\right). \end{aligned}$$

Since $\sum_{k=1}^{\infty} c_{n_k}^{-3}\sum_{m_k}^{n_k}E[\|X_j''\|^3] < \infty$ [use Lemma 1(i)], it suffices to show

$$(3.9)\qquad \sum_{k=1}^{\infty}\exp\left(-(1+p)^2c_{n_k}^2\Big/\sum_{m_k}^{n_k}E\left[\|X_j''\|^2\right]\right) < \infty.$$

To simplify our notations we set $d_j := c_j/(L_2 j)^{(1+p)/2}$, $j \in \mathbb{N}$. Using integration by parts, Lemma A.4 and noticing that we have by virtue of (2.6), for the inverse function a^{-1} of a, $a^{-1}(u) = u^2/S(u)$, where S is nondecreasing, we easily obtain, for sufficiently large j,

$$\begin{aligned} E\left[\|X_j''\|^2\right] &\le \alpha^2 d_j^2 P\{\|X_j\| > \alpha d_j\} + 2\int_{\alpha d_j}^{\alpha c_j} uP\{\|X_j\| > u\}\,du\\ &\le \tfrac{1}{2}S(\alpha d_j) + \int_{\alpha d_j}^{\alpha c_j} u^{-1}S(u)\,du \le (1+p)(L_3 j)S(c_j)\\ &= (1+p)c_j^2 L_3 j/a^{-1}(c_j). \end{aligned}$$

Using the trivial inequality $\exp(-t) \le 2t^{-1}\exp(-t/2)$, $t > 0$, we infer, for $k \ge k_1$ (say),

$$\exp\left(-(1+p)^2 c_{n_k}^2 \Big/ \sum_{m_k}^{n_k} E\left[\|X_j''\|^2\right] \right)$$

$$\le c_{n_k}^{-2} \sum_{m_k}^{n_k} E\left[\|X_j''\|^2\right] \exp\left(-\tfrac{1}{2}(1+p)^2 c_{n_k}^2 \Big/ \sum_{m_k}^{n_k} E\left[\|X_j''\|^2\right] \right)$$

$$\le 48 c_{n_k}^{-3} \sum_{m_k}^{n_k} E\left[\|X_j''\|^3\right] (L_2 n_k)^{(1+p)/2} \exp\left(-\tfrac{1}{2}(1+p) a^{-1}(c_{n_k})/n_k L_3 n_k \right)$$

$$\le 48 c_{n_k}^{-3} \sum_{m_k}^{n_k} E\left[\|X_j''\|^3\right].$$

[Notice that $c_n \ge a(n(L_3 n)^2)$, $n \ge N_0$.]

Recalling Lemma 1(i), we see that (3.9) holds and our proof of (3.1) is complete. □

Proof of (3.2). Using the same argument as in the proof of (3.1), we obtain, for sufficiently large k,

$$P\left\{ \max_{m_k \le n \le n_k} \left\| \sum_{m_k}^{n} \left(X_j' - E\left[X_j'\right] \right) \right\| \ge 200 c_{n_k} \right\} \le (50/49) P\left\{ \left\| \sum_{m_k}^{n_k} \left(X_j' - E\left[X_j'\right] \right) \right\| \ge 100 c_{n_k} \right\}. \tag{3.10}$$

Set

$$r_k := \left[n_k/(L_2 n_k)^p \right], \qquad q_k := \left[(n_k - n_{k-1})/r_k \right] + 1,$$

$$U_j := U_j(k) = \sum_{(j-1)r_k}^{j r_k - 1} \left(X_{m_k+l}' - E\left[X_{m_k+l}'\right] \right), \qquad 1 \le j < q_k,$$

$$U_{q_k} := U_{q_k}(k) = \sum_{m_k}^{n_k} (X_m' - E[X_m']) - \sum_{1}^{q_k - 1} U_j.$$

We now want to apply Inequality 2 with $Z_j := U_j$, $1 \le j \le q_k$, $n = q_k$. Since $c_n/(L_2 n)^{(1+p)/2} \ge a(n/(L_2 n)^p)$ for sufficiently large n, we obtain from Lemma A.1,

$$E\left[\|U_j\|^2\right] \le 4 \cdot 12^2 c_{n_k}^2/(L_2 n_k)^{1+p} =: \sigma_j^2, \qquad 1 \le j \le q_k, \tag{3.11}$$

and

$$\max_{1 \le j \le q_k} \left(E\left[\|U_j\|^m\right]/\sigma_j^2 \right) \le (m!/2) H_k^{m-2}, \qquad m = 3, 4, \dots, \tag{3.12}$$

where $H_k := 12 c_{n_k}/(L_2 n_k)^{(1+p)/2}$.

We now get from Inequality 2 (applied with $\beta_n = 2c_{n_k}$) for sufficiently large k,

$$P\left\{ \left\| \sum_1^{q_k} U_j \right\| \ge 100 c_{n_k} \right\} \le \exp\left(-100^2 c_{n_k}^2 / 16 \sum_1^{q_k} \sigma_j^2 \right) \vee \exp\left(-100 c_{n_k}/64 H_k \right)$$

$$\le (L n_k)^{-1.05}.$$

When $\sum_1^\infty (L n_k)^{-1.05} < \infty$, we immediately have (3.2). But it may happen that this series is not convergent. Therefore, we give a further bound for the above probability.

Using Theorem 2.1, de Acosta (1981) and Lemma 2 above, we have, if k is large enough,

$$P\left\{ \left\| \sum_{m_k}^{n_k} \left(X_j' - E[X_j'] \right) \right\| \ge 100 c_{n_k} \right\}$$

$$\le P\left\{ \left| \left\| \sum_{m_k}^{n_k} \left(X_j' - E[X_j'] \right) \right\| - E\left[\left\| \sum_{m_k}^{n_k} \left(X_j' - E[X_j'] \right) \right\| \right] \right| \ge 98 c_{n_k} \right\}$$

$$\le 4 \cdot 98^{-2} c_{n_k}^{-2} \sum_{m_k}^{n_k} E\left[\|X_j' - E[X_j']\|^2 \right] \le 16 \cdot 98^{-2} c_{n_k}^{-2} \sum_{m_k}^{n_k} E\left[\|X_j'\|^2 \right]$$

$$\le 16 \cdot 98^{-2} n_k E\left[\|X\|^\gamma \right] c_{n_k}^{-\gamma} \le C n_k 2^{-k\gamma},$$

where C is a positive constant, $\gamma := \eta \wedge 2$. (Notice that $c_{n_k} \ge 2^{k-1} c_1$.) Set $\mathbb{N}_1 := \{k \in \mathbb{N} \colon n_k < 2^{k\gamma/2}\}$, $\mathbb{N}_2 := \mathbb{N} - \mathbb{N}_1$. Then we have

$$\sum_{k=k_2}^{\infty} P\left\{ \left\| \sum_{m_k}^{n_k} \left(X_j' - E[X_j'] \right) \right\| \ge 100 c_{n_k} \right\}$$

$$\le \sum_{k \in \mathbb{N}_1} C \cdot 2^{-k\gamma/2} + \sum_{k \in \mathbb{N}_2} (L n_k)^{-1.05} < \infty.$$

Recalling (3.10), we see that (3.2) holds. □

In order to prove Theorem 1, we need the following corollary to Theorem 3.

COROLLARY 3. *Let X be a B-valued random variable satisfying* (1.3). *Let $t_0 > 0$ be such that $P\{\|Y\| \ge t_0\} \le 1/17e^2$. Then, for any sequence $\{c_n\}$ satisfying* (2.7) *as well as*

$$\liminf_{n \to \infty} \left(c_n / \sqrt{n L_2 n}\, b\left(n/(L_2 n)^p \right) \right) \ge 1 \quad \textit{for some } p > 1, \tag{3.13}$$

we have

$$\limsup_{n\to\infty}\left\|\frac{1}{c_n}\sum_1^n X_k\right\| \le t_0(960+160p) \quad a.s.$$

iff $\sum_1^\infty P\{\|X\| > c_n\} < \infty$.

Observing that condition (2.1) implies for any $\varepsilon > 0$,

$$\liminf_{n\to\infty}\left(\varepsilon c_n/\sqrt{nL_2n}\, b\big(n/(L_2n)^p\big)\right) \ge 1,$$

where $p := (-q) \vee 2$, we see that Theorem 1 is contained in Corollary 3. [Notice Lemma 1(iii).] Thus, it remains to show Corollary 3.

PROOF OF COROLLARY 3. We set $a(t) := \sqrt{t}\, b(t)$, $t \ge 1$. Recalling (1.4), we see that (2.6) holds. Moreover, (1.3) implies

$$\limsup_{n\to\infty} P\left\{\left\|\sum_1^n X_k\right\| \ge t_0 a(n)\right\} \le P\{\|Y\| \ge t_0\} \le 1/17e^2.$$

From (1.3) it also follows that $E[\|X\|^\eta] < \infty$, $\eta < 2$. Thus, all assumptions of Theorem 3 are fulfilled. To prove Corollary 3, it suffices to show

$$\sqrt{nL_2n}\, b\big(n/(L_2n)^p\big)/a\big(n(L_3n)^2\big) \to \infty \tag{3.14}$$

and

$$\left\|\sum_1^n E\left[X1\{\|X\| \le c_k\}\right]\right\| = o(c_n) \quad \text{as } n\to\infty. \tag{3.15}$$

(3.14) easily follows from the well-known fact that the function b is slowly varying at infinity when (1.3) holds. (3.15) is an immediate consequence of Proposition 2 of Einmahl (1988). □

4. Proof of Theorem 2. We prove

THEOREM 2′. *Let* X: $\Omega \to B$ *be a r.v. satisfying* (1.3). *Let* $q \in \mathbb{R}$. *Suppose that* $\sum_1^\infty P\{\|X\| > \sqrt{nL_2n}\, b(n(L_2n)^q)\} < \infty$. *Then one can construct a p-space* $(\Omega_0, \mathscr{A}_0, P_0)$ *and two sequences of i.i.d. r.v.'s* $\{\hat{X}_n\}, \{\hat{Y}_n\}$ *with* $P_0 \circ \hat{X}_1 = P \circ X$ *and* $P_0 \circ \hat{Y}_1 = P \circ Y$ *such that*

$$\left\|\sum_1^n \hat{X}_k - \sum_1^n b\big(k(L_2k)^q\big)\hat{Y}_k\right\| = o\left(\sqrt{nL_2n}\, b\big(n(L_2n)^q\big)\right) \quad a.s. \tag{4.1}$$

Using analogous arguments as in Section 4 of Einmahl (1988), we infer, from (4.1),

$$\max_{1\le m\le n}\left\|\sum_1^m \hat{X}_k - b\big(n(L_2n)^q\big)\sum_1^m \hat{Y}_k\right\| = o\left(\sqrt{nL_2n}\, b\big(n(L_2n)^q\big)\right) \quad \text{a.s.} \tag{4.2}$$

Thus, we obtain from Theorem 2′ a particular p-space $(\Omega_0, \mathscr{A}_0, P_0)$ such that (2.2) holds for an appropriate construction. This seems to be still weaker than Theorem 2, but the assertion follows from (4.2) by an application of Theorem 1 of Skorokhod (1976).

It remains to show Theorem 2′. To simplify our notations, we set for $q \in \mathbb{R}$, $\sigma_n(q) := b(n(L_2n)^q)$ and $\gamma_n(q) := \sqrt{nL_2n}\,\sigma_n(q)$.

4.1. *The finite-dimensional case when* $q \leq 1$. Let $(\mathbb{R}^d, |\cdot|)$ be the d-dimensional Euclidean space and let $X\colon \Omega \to \mathbb{R}^d$ be a random vector in the domain of attraction of $N(0, I)$, where I denotes the d-dimensional unit matrix. Let $\{X_n\}$ be independent copies of X so that we have

$$P \circ \frac{1}{a_n} \sum_1^n X_k \text{ converges weakly to } N(0, I). \tag{4.3}$$

Let $\xi\colon \Omega \to \mathbb{R}$ be the first component of X and set $G(t) := E[\xi^2 1\{|\xi| \leq t\}]$, $t \geq 0$. Then it follows from Proposition 1 of Einmahl (1988) that

$$\frac{1}{G(t)} \operatorname{cov}(X1\{|X| \leq t\}) \to I \quad \text{as} \quad t \to \infty. \tag{4.4}$$

Let the function $a\colon [1, \infty) \to (0, \infty)$ be defined as in the proof of Corollary 3, i.e., $a(t) = \sqrt{t}\, b(t)$, $t \geq 1$.

Setting $X_n' := X_n 1\{|X_n| \leq a(n(L_2n)^q)\}$, $n \in \mathbb{N}$, we infer from relation (3.14) and Lemma 3 below that

$$\left| \sum_1^n X_k - \sum_1^n (X_k' - E[X_k']) \right| = o(\gamma_n(q)) \quad \text{a.s.} \tag{4.5}$$

[Notice that $a(n(L_2n)^q) = \gamma_n(q)/(L_2n)^{(1-q)/2}$.]

Using the same arguments as in the proof of Theorem 2 of Einmahl (1988), we obtain from Einmahl [(1987b), Theorem 2] a p-space $(\Omega_0, \mathscr{A}_0, P_0)$ and two sequences of independent random vectors $\{\hat{X}_k\}, \{\hat{Y}_k\}$ with $P_0 \circ \hat{X}_k = P \circ X$ and $P_0 \circ \hat{Y}_k = N(0, I)$ such that

$$\left| \sum_1^n \hat{X}_k - \sum_1^n \operatorname{cov}(X_k')^{1/2} \hat{Y}_k \right| = o(\gamma_n(q)) \quad \text{a.s.} \tag{4.6}$$

Since $G(a(n(L_2n)^q)) \sim b(n(L_2n)^q)^2 = \sigma_n^2(q)$ [use (2.6) of Einmahl (1988)], it easily follows from (4.4) and (4.6) that

$$\left| \sum_1^n \hat{X}_k - \sum_1^n \sigma_k(q) \hat{Y}_k \right| = o(\gamma_n(q)) \quad \text{a.s.,} \tag{4.7}$$

hence Theorem 2′ holds for finite-dimensional random vectors when $q \leq 1$.

LEMMA 3. *Let* $X\colon \Omega \to B$ *be a r.v. satisfying* (1.3) *and* $\sum_1^\infty P\{\|X_n\| > c_n\} < \infty$, *where* $\{c_n\}$ *is a sequence of positive real numbers such that* $c_n/\sqrt{n}$ *is* *nondecreasing and* $c_n/a_{[n(L_3 n)^2]} \to \infty$. *Let* $p \geq 0$ *be fixed. Set* $X_n' := X_n 1\{\|X_n\| \leq c_n/(L_2 n)^p\}$, $n \in \mathbb{N}$. *Then we have*

$$\left\| \sum_1^n (X_k' - E[X_k']) - \sum_1^n X_k \right\| = o(c_n) \quad a.s.$$

PROOF. Fix $\varepsilon > 0$ and put

$$X_n'' := X_n''(\varepsilon) = X_n 1\{c_n/(L_2 n)^p < \|X_n\| \leq \varepsilon c_n\},$$

$$X_n''' := X_n'''(\varepsilon) = X_n 1\{\|X_n\| > \varepsilon c_n\}, \qquad n \in \mathbb{N}.$$

A straightforward modification of the proof of (3.1) yields

$$\limsup_{n\to\infty} c_n^{-1} \left\| \sum_1^n (X_k'' - E[X_k'']) \right\| \leq 1536\varepsilon \quad \text{a.s.} \tag{4.8}$$

Combining Lemma 1(iii) and the Borel–Cantelli lemma, we obtain

$$\left\| \sum_1^n X_k''' \right\| = O(1) \quad \text{a.s.} \tag{4.9}$$

Moreover, we have according to Proposition 2 of Einmahl (1988),

$$\left\| \sum_1^n E[X_k'''] \right\| = o(c_n). \tag{4.10}$$

Since $E[X] = E[X_k'] + E[X_k''] + E[X_k'''] = 0$, we finally conclude

$$\limsup_{n\to\infty} c_n^{-1} \left\| \sum_1^n X_k - \sum_1^n (X_k' - E[X_k']) \right\| \leq 1536\varepsilon \quad \text{a.s.} \qquad \square$$

4.2. *The finite-dimensional case when* $q \geq 1$. Since $\sum_1^\infty P\{|X| > \gamma_n(q)\} < \infty$, we obtain from Theorem 2 of Einmahl (1988) a p-space $(\Omega_0, \mathscr{A}_0, P_0)$ and two sequences of i.i.d. random vectors $\{\hat{X}_n\}, \{\hat{Y}_n\}$ with $P_0 \circ \hat{X}_1 = P \circ X$ and $P_0 \circ \hat{Y}_1 = N(0, I)$ such that

$$\left| \sum_1^n \hat{X}_k - \sum_1^n \Gamma_k^{1/2} \hat{Y}_k \right| = o(\gamma_n(q)) \quad \text{a.s.,} \tag{4.11}$$

where $\Gamma_n := \operatorname{cov}(X1\{|X| \leq \gamma_n(q)\})$, $n \in \mathbb{N}$.

Using (4.4), we infer from (4.11),

$$\left| \sum_1^n \hat{X}_k - \sum_1^n \sqrt{G(\gamma_k(q))}\, \hat{Y}_k \right| = o(\gamma_n(q)) \quad \text{a.s.} \tag{4.12}$$

[Notice that according to (2.6) of Einmahl (1988),

$$nL_2nG(\gamma_n(q)) \sim nL_2n(\gamma_n(q))^2/a^{-1}(\gamma_n(q)) = O(\gamma_n(q)^2)$$

when $q \geq 1$.]

Since $\sigma_k^2(q) \sim G(\gamma_k(q)(L_2k)^{(q-1)/2}) =: G(\bar{\gamma}_k(q))$, it suffices to show

$$\left|\sum_1^n \left(\sqrt{G(\gamma_k(q))} - \sqrt{G(\bar{\gamma}_k(q))}\right)\hat{Y}_k\right| = o(\gamma_n(q)) \quad \text{a.s.} \tag{4.13}$$

We set $m_k := 2^{k-1}$, $n_k := m_{k+1} - 1$, $k \in \mathbb{N}$. We show that we have for any $\varepsilon > 0$,

$$\sum_{k=1}^{\infty} \exp\left(-\varepsilon^2\gamma_{n_k}^2(q)/2d\sum_{m_k}^{n_k}(G(\bar{\gamma}_m(q)) - G(\gamma_m(q)))\right) < \infty. \tag{4.14}$$

(4.13) easily follows from (4.14), when using a well-known exponential inequality for normally distributed r.v.'s and the Borel–Cantelli lemma.

In order to prove (4.14), we use a similar argument as in the proof of (3.9). We first note that we have for $k \geq k_1$ (say),

$$\sum_{m_k}^{n_k}(G(\bar{\gamma}_m(q)) - G(\gamma_m(q))) \leq n_k G(\bar{\gamma}_{n_k}(q)) \leq n_k(L_2n_k)^{1/4}G(\gamma_{n_k}(q)).$$

Notice that, as a consequence of (1.3), G is slowly varying at infinity.

Since $a^{-1}(t) \sim t^2/G(t)$ as $t \to \infty$, we further have, for $k \geq k_2$ (say),

$$\begin{aligned}
&\exp\left(-\varepsilon^2\gamma_{n_k}(q)^2/2d\sum_{m_k}^{n_k}\{G(\bar{\gamma}_m(q)) - G(\gamma_m(q))\}\right)\\
&\leq 4d\varepsilon^{-2}\gamma_{n_k}(q)^{-2}\sum_{m_k}^{n_k}\{G(\bar{\gamma}_m(q)) - G(\gamma_m(q))\}\\
&\quad \times \exp\left(-(\varepsilon^2/4d)\gamma_{n_k}(q)^2/n_k(L_2n_k)^{1/4}G(\gamma_{n_k}(q))\right)\\
&\leq 8d\varepsilon^{-2}\gamma_{n_k}(q)^{-3}\sum_{m_k}^{n_k}E\left[|\xi|^3 1\{|\xi| \leq \bar{\gamma}_m(q)\}\right]\\
&\quad \times \exp\left(-(\varepsilon^2/5d)a^{-1}(\gamma_{n_k}(q))/n_k(L_2n_k)^{1/4}\right)\\
&\leq \bar{\gamma}_{n_k}(q)^{-3}\sum_{m_k}^{n_k}E\left[|\xi|^3 1\{|\xi| \leq \bar{\gamma}_m(q)\}\right].
\end{aligned}$$

[Notice that $\gamma_{n_k}(q) \geq a(n_kL_2n_k)$, $k \in \mathbb{N}$.]

Since we have, by virtue of Lemma 1,

$$\sum_1^\infty \bar{\gamma}_m(q)^{-3} E\left[|\xi|^3 1\{|\xi| \le \bar{\gamma}_m(q)\}\right] < \infty,$$

we obtain (4.14). This completes the proof of Theorem 2′ for finite-dimensional random vectors when $q \ge 1$.

4.3. *The general case.* Let Π_N: $B \to H_\mu$, $N \in \mathbb{N}$, be the maps obtained from $\mu = P \circ Y$ according to Lemma 2.1 of Kuelbs (1976). Let Q_N: $B \to B$ be defined by $Q_N(x) = x - \Pi_N(x)$, $x \in B$.

Let $\varepsilon > 0$ be fixed. Then it follows for sufficiently large N_0,

$$P\left\{\|Q_{N_0}(Y)\| \ge \varepsilon/(1920 + 320p)\right\} \le 1/17e^2, \tag{4.15}$$

where $p := (-q) \vee 2$.

Since Q_{N_0} is continuous and linear, we obtain from (1.3)

$$P \circ \frac{1}{a_n} \sum_1^n Q_{N_0}(X_k) \text{ converges weakly to } P \circ Q_{N_0}(Y). \tag{4.16}$$

Combining (4.16) and Corollary 3 [applied with $Q_{N_0}(X)$], we get

$$\limsup_{n\to\infty} \gamma_n(q)^{-1} \left\| \sum_1^n Q_{N_0}(X_k) \right\| \le \varepsilon/2 \quad \text{a.s.} \tag{4.17}$$

Applying Theorem 2′ to the finite-dimensional random vector $\Pi_{N_0}(X)$, we obtain a p-space $(\Omega_1, \mathscr{A}_1, P_1)$ and two sequences of independent r.v.'s $\{x_n\}, \{y_n\}$ with $P_1 \circ x_n = P \circ \Pi_{N_0}(X)$ and $P_1 \circ y_n = P \circ \Pi_{N_0}(Y)$ such that

$$\left\| \sum_1^n x_k - \sum_1^n y_k \right\| = o(\gamma_n(q)) \quad \text{a.s.} \tag{4.18}$$

Using the same argument as in the proof of (3.26) of Einmahl (1988), we also get, for N_0 large enough,

$$\limsup_{n\to\infty} \gamma_n(q)^{-1} \left\| \sum_1^n \sigma_k(q) Q_{N_0}(Y_k) \right\| \le \varepsilon/2 \quad \text{a.s.} \tag{4.19}$$

Combining (4.17), (4.18) and (4.19) with Lemma A.1 of Berkes and Philipp (1979), we can find a p-space $(\Omega_2, \mathscr{A}_2, P_2)$ and two sequences of independent random variables $\{\overline{X}_n\}, \{\overline{Y}_n\}$ (possibly depending on ε) with $P_2 \circ \overline{X}_n = P \circ X$ and $P_2 \circ \overline{Y}_n = P \circ Y$ such that

$$\limsup_{n\to\infty} \gamma_n(q)^{-1} \left\| \sum_1^n \overline{X}_k - \sum_1^n \sigma_k(q) \overline{Y}_k \right\| \le \varepsilon \quad \text{a.s.} \tag{4.20}$$

Observing that (4.20) holds for arbitrary $\varepsilon > 0$, we finally obtain the p-space $(\Omega_0, \mathscr{A}_0, P_0)$ with the desired r.v.'s $\{\hat{X}_n\}, \{\hat{Y}_n\}$ by a known argument of Major (1976).

APPENDIX

LEMMA A.1. *Let $Z_1, \dots, Z_n$: $\Omega \to B$ be i.i.d. random variables satisfying* $\max_{1 \le k \le n} P\{\|\sum_1^k Z_j\| \ge K_n\} \le 1/16e^2$. *Put* $Z_j' := Z_j 1\{\|Z_j\| \le \tau_j\}$, $1 \le j \le n$, *where* τ_j, $1 \le j \le n$, *are positive real numbers such that* $\max_{1 \le j \le n} \tau_j \le K_n$. *Then we have*:

(i) $$E\left[\exp\left(\left\|\sum_1^n \left(Z_j' - E\left[Z_j'\right]\right)\right\| / 12K_n\right)\right] \le e.$$

(ii) $$E\left[\left\|\sum_1^n \left(Z_j' - E\left[Z_j'\right]\right)\right\|^m\right] \le m!(e-1)(12K_n)^m, \qquad m \in \mathbb{N}.$$

PROOF. Let $\bar{Z}_1, \dots, \bar{Z}_n$ be independent r.v.'s with $P \circ \bar{Z}_j = P \circ Z_j$, $1 \le j \le n$, and let further $\varepsilon_1, \dots, \varepsilon_n$ be a Rademacher sequence. Suppose that the three sequences $\{Z_1, \dots, Z_n\}$, $\{\bar{Z}_1, \dots, \bar{Z}_n\}$ and $\{\varepsilon_1, \dots, \varepsilon_n\}$ are independent of each other. Put finally $\bar{Z}_j' := \bar{Z}_j 1\{\|\bar{Z}_j\| \le \tau_j\}$, $1 \le j \le n$.

Then it easily follows, for $t \ge 0$,

$$E\left[\exp\left(t\left\|\sum_1^n \left(Z_j' - E\left[Z_j'\right]\right)\right\|\right)\right] \le E\left[\exp\left(t\left\|\sum_1^n \left(Z_j' - \bar{Z}_j'\right)\right\|\right)\right]$$
$$= E\left[\exp\left(t\left\|\sum_1^n \varepsilon_j\left(Z_j' - \bar{Z}_j'\right)\right\|\right)\right],$$

where we use Lemma 2.7 from Chapter 3 of Araujo and Giné (1980). (Notice that the r.v.'s $Z_j' - \bar{Z}_j'$, $1 \le j \le n$, are symmetric.)

Applying the Hölder inequality, we further conclude

$$E\left[\exp\left(t\left\|\sum_1^n \varepsilon_j\left(Z_j' - \bar{Z}_j'\right)\right\|\right)\right] \le E\left[\exp\left(t\left\|\sum_1^n \varepsilon_j Z_j'\right\|\right)\exp\left(t\left\|\sum_1^n \varepsilon_j \bar{Z}_j'\right\|\right)\right]$$
$$\le E\left[\exp\left(2t\left\|\sum_1^n \varepsilon_j Z_j'\right\|\right)\right].$$

Thus, it remains to show in order to prove (i),

(A.1) $$E\left[\exp\left(\left\|\sum_1^n \varepsilon_j Z_j'\right\| / 6K_n\right)\right] \le e.$$

Noticing that the r.v.'s $\varepsilon_j Z_j'$, $1 \le j \le n$, are symmetric, we obtain (A.1) from Lemma 6.1(b) of Kuelbs and Zinn (1983), provided that we can show

(A.2) $$P\left\{\left\|\sum_1^n \varepsilon_j Z_j'\right\| \ge 2K_n\right\} \le 1/4e^2.$$

It easily follows from Lemmas A.2 and A.3 below that

$$P\left\{\left\|\sum_1^n \varepsilon_j Z_j'\right\| \geq 2K_n\right\} \leq 2P\left\{\left\|\sum_1^n \varepsilon_j Z_j\right\| \geq 2K_n\right\}$$

$$\leq 4 \max_{1\leq k\leq n} P\left\{\left\|\sum_1^k Z_j\right\| \geq K_n\right\} \leq 1/4e^2.$$

Hence (A.2) holds and our proof of (i) is complete.

(ii) immediately follows from (i) and the fact that

$$E\left[\exp\left(t\left\|\sum_1^n \left(Z_j' - E\left[Z_j'\right]\right)\right\|\right)\right]$$

$$= 1 + \sum_{m=1}^{\infty} E\left[\left\|\sum_1^n \left(Z_j' - E\left[Z_j'\right]\right)\right\|^m\right] t^m/m!, \qquad t \in \mathbb{R}. \qquad \square$$

Lemma A.2. *Let $Z_1, \dots, Z_n\colon \Omega \to B$ be independent symmetric random variables. Put $Z_j' := Z_j 1\{\|Z_j\| \leq \tau_j\}$, $1 \leq j \leq n$, where τ_j, $1 \leq j \leq n$, are positive real numbers. Then we have*

$$P\left\{\left\|\sum_1^n Z_j'\right\| \geq t\right\} \leq 2P\left\{\left\|\sum_1^n Z_j\right\| \geq t\right\}, \qquad t \geq 0.$$

Proof. Setting $\hat{Z}_j := Z_j 1\{\|Z_j\| > \tau_j\}$, we have

$$P \circ \left(Z_j' - \hat{Z}_j\right) = P \circ \left(Z_j' + \hat{Z}_j\right) = P \circ Z_j, \qquad 1 \leq j \leq n.$$

Using the simple inequality $\|x\| \leq \|x + y\| \vee \|x - y\|$, $x, y \in B$, we infer

$$P\left\{\left\|\sum_1^n Z_j'\right\| \geq t\right\} \leq P\left\{\left\|\sum_1^n \left(Z_j' + \hat{Z}_j\right)\right\| \geq t\right\} + P\left\{\left\|\sum_1^n \left(Z_j' - \hat{Z}_j\right)\right\| \geq t\right\}$$

$$= 2P\left\{\left\|\sum_1^n Z_j\right\| \geq t\right\}. \qquad \square$$

Lemma A.3. *Let $Z_1, \dots, Z_n\colon \Omega \to B$ be i.i.d. random variables. Let $\varepsilon_1, \dots, \varepsilon_n\colon \Omega \to \{-1, 1\}$ be a Rademacher sequence independent of the Z_j's. Then we have for $t \geq 0$,*

$$P\left\{\left\|\sum_1^n \varepsilon_j Z_j\right\| \geq t\right\} \leq 2 \max_{1\leq k\leq n} P\left\{\left\|\sum_1^k Z_j\right\| \geq t/2\right\}.$$

PROOF. See Giné and Zinn (1984), Lemma 2.7(a). □

LEMMA A.4. *Let $\{Z_n\}$ be a sequence of i.i.d. B-valued random variables satisfying*

$$\limsup_{n\to\infty} P\left\{\left\|\sum_1^n Z_k\right\| \geq a(n)\right\} \leq c < 1/4,$$

where a: $[1,\infty) \to (0,\infty)$ is a continuous function such that $t^{-1/2}a(t)$ is nondecreasing. Let a^{-1}: $[a(1),\infty) \to [1,\infty)$ be the inverse function of a. Then we have

$$\limsup_{u\to\infty}\left(a^{-1}(u)P\{\|Z_1\| \geq u\}\right) \leq 16c/(1-4c).$$

PROOF. Let $\bar{c} \in (c, 1/4)$ be fixed. Let $\{\varepsilon_n\}$ be a Rademacher sequence independent of the Z_j's. By virtue of Lemma A.3 we have, if n is large enough,

$$P\left\{\left\|\sum_1^n \varepsilon_j Z_j\right\| \geq 2a(n)\right\} \leq 2\bar{c}.$$

Using Theorem 2.6 from Chapter 3 of Araujo and Giné (1980), we infer

$$P\left\{\max_{1\leq j\leq n}\|Z_j\| \geq 2a(n)\right\} = P\left\{\max_{1\leq j\leq n}\|\varepsilon_j Z_j\| \geq 2a(n)\right\} \leq 4\bar{c}.$$

Hence

$$\log(1-4\bar{c}) \leq \log\left(P\left\{\max_{1\leq j\leq n}\|Z_j\| < 2a(n)\right\}\right) = n\log\left(1 - P\{\|Z_1\| \geq 2a(n)\}\right)$$
$$\leq -nP\{\|Z_1\| \geq 2a(n)\}.$$

Using the inequality $\log(1/(1-t)) \leq t/(1-t)$, $t \in [0,1)$, we get

$$nP\{\|Z_1\| \geq 2a(n)\} \leq 4\bar{c}/(1-4\bar{c}).$$

Recalling that $a(t)/\sqrt{t}$ is nondecreasing, we obtain by means of interpolation,

$$\limsup_{u\to\infty}\left(a^{-1}(u)P\{\|Z_1\| \geq u\}\right) \leq 16\bar{c}/(1-4\bar{c}). \qquad \square$$

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DEPARTMENT OF STATISTICS
AND PROBABILITY
MICHIGAN STATE UNIVERSITY
EAST LANSING, MICHIGAN 48824