The Annals of Probability
1989, Vol. 17, No. 1, 301–316

RATES OF CONVERGENCE FOR THE FUNCTIONAL LIL

BY VICTOR GOODMAN AND JAMES KUELBS[1]

Indiana University and University of Wisconsin

Rates of convergence are obtained for Strassen's functional law of the iterated logarithm for polygonal processes under classical conditions.

1. Introduction. Let $X, X_1, X_2, \dots$ be i.i.d. random variables such that

$$E(X) = 0 \quad \text{and} \quad E(X^2) = 1, \tag{1.1}$$

and define the polygonal process

$$S(t) = \sum_{j=1}^{[t]} X_j + (t - [t]) X_{[t]+1}, \qquad t \geq 0, \tag{1.2}$$

where $[\cdot]$ is the greatest integer function and $\sum_{j=1}^{0} X_j = 0$. If $C[0,1]$ denotes the continuous functions on $[0,1]$, and

$$\mathscr{K} = \left\{ f\colon f(t) = \int_0^t f'(s)\, ds,\, 0 \leq t \leq 1, \text{ and } \int_0^1 |f'(s)|^2\, ds \leq 1 \right\}, \tag{1.3}$$

then $\mathscr{K}$ is a compact, convex, symmetric subset of $C[0,1]$ such that the random sequence $\{S(n(\cdot))/(2nLLn)^{1/2}\colon n \geq 1\}$ converges to and clusters throughout $\mathscr{K}$ in the uniform norm with probability 1. This is Strassen's law of the iterated logarithm [10], and in [6] we examined the rate at which the convergence and clustering took place in this result.

More precisely, if $A \subseteq C[0,1]$ let $A^\varepsilon = \{g\colon \inf_{f \in A} \|g - f\|_\infty < \varepsilon\}$, where $\|f\|_\infty = \sup_{0 \leq t \leq 1} |f(t)|$, and set

$$E_n = \left\{ S(k(\cdot))/(2kLLn)^{1/2}\colon k = 1, \dots, n \right\}. \tag{1.4}$$

Then, if $\varepsilon_n = \varepsilon/(LLn)^{1/2}$, we proved under (1.1) and the additional assumption $E(X^2 LL|X|) < \infty$, that for each $\varepsilon > 0$,

$$P(E_n \subseteq \mathscr{K}^{\varepsilon_n} \text{ eventually}) = 1 \tag{1.5}$$

and

$$P(\mathscr{K} \subseteq E_n^{\varepsilon_n} \text{ eventually}) = 1 \tag{1.6}$$

(see the remark following Corollary 3 in [6]). Further, under the more restrictive moment condition $E(X^2(LL|X|)^{1+\beta}) < \infty$, $0 < \beta \leq 1$, Corollary 3 in [6] contains better rates of convergence in the sense that ε_n can be taken to be of the form $\varepsilon_n = \gamma/(LLn)^{(1+\beta)/2}$. The method of proof was to prove the analogous results for Brownian motion, and then, by a suitable strong approximation of the

Received August 1987; revised February 1988.
[1]Supported in part by NSF Grant DMS-85-21586.
AMS 1980 *subject classifications*. 60B12, 60F15, 60G15, 60G17.
Key words and phrases. Brownian motion, functional law of the iterated logarithm, Gaussian measures.

polygonal process in (1.2) by Brownian motion, to obtain the results for the polygonal process.

The purpose of this paper is to establish these results under conditions closer to the classical moment assumptions $E(X) = 0$ and $E(X^2) = 1$. The main result is the following, and it holds under these assumptions.

THEOREM 1. *Let $S(t)$: $t \geq 0\}$ be defined by* (1.2) *with* $E(X) = 0$ *and* $E(X^2) = 1$. *Let* $\mathscr{K}$ *be as in* (1.3) *and define* E_n *by* (1.4). *Let* $\varepsilon_n = \varepsilon/(LLn)^{1/2}$. *Then, for each* $\varepsilon > 0$,

$$P(E_n \subseteq \mathscr{K}^{\varepsilon_n} \text{ eventually}) = 1 \tag{1.7}$$

and, for each $\delta \in (0,1)$,

$$P((1-\delta)\mathscr{K} \subseteq E_n^{\varepsilon_n} \text{ eventually}) = 1. \tag{1.8}$$

Using the ideas in the proof of Theorem 1 and those in Theorem 4 of [6], it is possible to obtain results analogous to those in Corollary 3 of [6] under the additional moment assumption $E(X^2(LL|X|)^\beta) < \infty$, where $0 < \beta \leq 1$. This improves the corresponding moment assumption used in Corollary 3 and is our next theorem. We write $a_n \sim b_n$ if $\lim_n a_n/b_n = 1$ in the following.

THEOREM 2. *Let* $\{S(t): t \geq 0\}$ *be defined by* (1.2) *with* $E(X) = 0$, $E(X^2) = 1$ *and* $E(X^2(LL|X|)^\beta) < \infty$ *for some* $\beta \in (0,1]$. *Let* $\mathscr{K}$ *be as in* (1.3), *set*

$$E_{n,\eta(n)} = \left\{S(k(\cdot))/(2kLLn)^{1/2}: \eta(n) \leq k \leq n\right\},$$

where $\eta(n) \leq n$ *is such that as* $n \to \infty$,

$$LL\eta(n) \sim LLn,$$

and define

$$F_n = \left\{S(k(\cdot))/(2nLLn)^{1/2}: 1 \leq k \leq n\right\}.$$

If $\varepsilon_n = \gamma/(LLn)^{(1+\beta)/2}$, *then for each* $\delta > 0$, *there is a* $\gamma > 0$ *sufficiently large such that*

$$P\Big(E_{n,\eta(n)} \subseteq ((1+\delta)\mathscr{K})^{\varepsilon_n} \text{ eventually}\Big) = 1 \tag{1.9}$$

and

$$P\Big(F_n \subseteq ((1+\delta)\mathscr{K})^{\varepsilon_n} \text{ eventually}\Big) = 1. \tag{1.10}$$

Further, if $\delta \in (0,1)$ *and* $\eta(n)$ *is nondecreasing with* $\eta(n) \leq n^{1/4}$, *then there is a* γ *sufficiently large such that*

$$P\Big((1-\delta)\mathscr{K} \subseteq \left(E_{n,\eta(n)}\right)^{\varepsilon_n} \text{ eventually}\Big) = 1. \tag{1.11}$$

2. Some lemmas. The first lemma adapts a result of [1] in a manner similar to that in [3].

LEMMA 2.1. *If* μ *and* ν *are centered Gaussian measures on the Borel subsets of* $C[0,1]$ *with covariance functions*

$$R_\mu(s,t) = \int_{C[0,1]} x(s)x(t)\,d\mu(x)$$

and

$$R_\nu(s,t) = \int_{C[0,1]} x(s)x(t)\,d\nu(x)$$

such that

(2.1) $$R_\mu(s,t) - R_\nu(s,t) \text{ is nonnegative definite},$$

then for each symmetric, convex, open subset U *of* $C[0,1]$,

(2.2) $$\mu(U) \le \nu(U).$$

PROOF. Let $\{t_j\colon j \ge 1\}$ be a countable dense subset of $[0,1]$ and let $\pi_n\colon C[0,1] \to \mathbb{R}^n$ be defined by

$$\pi_n(x) = (x(t_1), \dots, x(t_n)).$$

If K is any compact subset of $C[0,1]$ and $A_k = \pi_k^{-1}(\pi_k(K))$, then

(2.3) $$A_{k+1} \subseteq A_k, \qquad k \ge 1,$$

and

(2.4) $$\bigcap_{k=1}^{\infty} A_k = K.$$

To verify (2.3), simply observe that

$$A_k = K + \text{null}(\pi_k), \qquad k \ge 1.$$

For (2.4) take $x \in \bigcap_{k=1}^{\infty} A_k$. Then $x = y_k + z_k$, where $y_k \in K$ and $z_k \in \text{null}(\pi_k)$. Since K is compact, there is a subsequence $\{y_{k'}\}$ such that $\lim_{k'} y_{k'} = y \in K$. Let $z = x - y$. Then

$$\pi_k(z) = \lim_{k'} \pi_k(x - y_{k'}) = \lim_{k'} \pi_k(z_{k'}) = 0,$$

for all $k \ge 1$. Since $z \in C[0,1]$, this implies $z = 0$ and hence $x = y \in K$.

If K is any compact, convex, symmetric subset of $C[0,1]$, then the sets $A_k = \pi_k^{-1}(\pi_k(K))$ are convex and symmetric cylinder sets which satisfy (2.3) and (2.4). Further, by Corollary 3 of [1]

(2.5) $$\mu(A_k) \le \nu(A_k),$$

and hence

(2.6) $$\mu(K) = \lim_k \mu(A_k) \le \lim_k \nu(A_k) = \nu(K).$$

Thus if $\mathscr{F}$ denotes the class of convex, symmetric, compact subsets of $C[0,1]$ and U is open, convex and symmetric, then

(2.7) $$\mu(U) = \sup_{\substack{K \subseteq U \\ K \in \mathscr{F}}} \mu(K) \le \sup_{\substack{K \subseteq U \\ K \in \mathscr{F}}} \nu(K) = \nu(U).$$

Hence Lemma 2.1 is proved. □

LEMMA 2.2. *Let $Y_1, Y_2, \dots$ be independent centered Gaussian random variables with*

$$E(Y_k^2) = \sigma_k^2 \le 1, \qquad k \ge 1, \tag{2.8}$$

and define the polygonal process

$$T_Y(t) = \sum_{j=1}^{[t]} Y_j + (t - [t]) Y_{[t]+1}, \qquad t \ge 0. \tag{2.9}$$

If $\{W(t)\colon t \ge 0\}$ is standard Brownian motion and U is any convex, open, symmetric subset of $C[0,1]$, then for all $n \ge 1$,

$$P(T_Y(n(\cdot)) \in U) \ge P(W(n(\cdot)) \in U). \tag{2.10}$$

PROOF. Let $\mu = \mathscr{L}(W(n(\cdot)))$ and $\nu = \mathscr{L}(T_Y(n(\cdot)))$ denote the probability measures induced on $C[0,1]$ by the given sample path continuous processes, fix $n \ge 1$, and let U be a convex, open, symmetric subset of $C[0,1]$.

For the purpose of the proof it is useful to consider three centered Gaussian polygonal processes defined for $t \ge 0$ by

$$\begin{aligned} T_g(t) &= \sum_{j=1}^{[t]} g_j + (t - [t]) g_{[t]+1}, \\ T_Y(t) &= \sum_{j=1}^{[t]} Y_j + (t - [t]) Y_{[t]+1}, \\ T_Z(t) &= \sum_{j=1}^{[t]} Z_j + (t - [t]) Z_{[t]+1}, \end{aligned} \tag{2.11}$$

where $\{g_j\colon j \ge 1\}$, $\{Y_j\colon j \ge 1\}$ and $\{Z_j\colon j \ge 1\}$ are independent sequences of centered, independent, Gaussian random variables such that for each $j \ge 1$,

$$E(g_j^2) = 1, \qquad E(Y_j^2) = \sigma_j^2 \le 1, \qquad E(Z_j^2) = 1 - \sigma_j^2. \tag{2.12}$$

Then for each real number $n \ge 1$ an easy calculation gives that the covariance of the process $\{T_Y(nt) + T_Z(nt)\colon t \ge 0\}$ is the same as that of the process $\{T_g(nt)\colon t \ge 0\}$. Further, letting $\Sigma_g(s,t)$, $\Sigma_Y(s,t)$ and $\Sigma_Z(s,t)$ denote the corresponding covariances, the independence gives for all s, t that

$$\Sigma_g(s,t) = \Sigma_Y(s,t) + \Sigma_Z(s,t). \tag{2.13}$$

Hence it follows that

$$\Sigma_g(s,t) - \Sigma_Y(s,t) = \Sigma_Z(s,t) \tag{2.14}$$

is nonnegative definite.

If $\{W(t)\colon t \ge 0\}$ is standard Brownian motion, then $\{W(nt)\colon t \ge 0\}$ has covariance

$$\Sigma_W(s,t) = \min(ns, nt) \tag{2.15}$$

and

(2.16) $$\Sigma_W(s,t) - \Sigma_g(s,t)$$

can be shown to be nonnegative definite. Once (2.16) is verified and using the elementary fact that sums of nonnegative definite functions are nonnegative definite, (2.14) added to (2.16) gives

(2.17) $$\Sigma_W(s,t) - \Sigma_Y(s,t)$$

is nonnegative definite. Since the covariances of μ and ν, call them R_μ and R_ν, satisfy $R_\mu = \Sigma_W$ and $R_\nu = \Sigma_Y$, (2.17) implies

$$R_\mu(s,t) - R_\nu(s,t)$$

is nonnegative definite. Hence by Lemma 2.1 the equation (2.10) holds, and the proof of Lemma 2.2 is complete once we verify (2.16).

To prove (2.16), let $g_j = W(j) - W(j-1)$ for $j \geq 1$, and define $\{T_g(t): t \geq 0\}$ from (2.11). Then a simple computation of covariances shows that $\{T_g(nt): t \geq 0\}$ and $\{(W(nt) - T_g(nt)): t \geq 0\}$ are independent centered Gaussian processes. Letting $\Sigma_{W-T_g}(s,t)$ denote the covariance of the process $\{(W(nt) - T_g(nt)): t \geq 0\}$, we have

$$\Sigma_{W-T_g} + \Sigma_g = \Sigma_W$$

and hence

$$\Sigma_W - \Sigma_g = \Sigma_{W-T_g}$$

is nonnegative definite as claimed. □

LEMMA 2.3. *Suppose* $\varepsilon_n = \varepsilon/(LLn)^{1/2}$ *and for each* $\varepsilon > 0$,

(2.18) $$P(S(n(\cdot))/d(n) \varepsilon \mathscr{K}^{\varepsilon_n} \text{ eventually}) = 1,$$

where $d(n) = (2nLLn)^{1/2}$. *Then* (1.7) *holds.*

PROOF. Once (2.18) holds we have $n_0(\omega)$ such that for $k \geq n_0(\omega)$,

(2.19) $$S(k(\cdot))/d(k) \in \mathscr{K}^{\varepsilon_k},$$

for a set of ω's of probability 1. Further, $f/(2LLk)^{1/2} \in \mathscr{K}^{\varepsilon_k}$ implies $f/(2LLn)^{1/2} \in \mathscr{K}^{\varepsilon_n}$ for $n \geq k$, and hence (2.19) implies

$$S(k(\cdot))/(2kLLn)^{1/2} \in \mathscr{K}^{\varepsilon_n},$$

for $n_0(\omega) \leq k \leq n$. Now

$$P\big(S(k(\cdot))/k^{1/2} \in (2LLn)^{1/2}\mathscr{K}^{\varepsilon_n} \text{ for } 0 \geq k \geq n_0(\omega) \text{ eventually in } n\big) = 1$$

since $(2LLn)^{1/2}\mathscr{K}^{\varepsilon_n} \nearrow C[0,1]$. Thus (1.7) holds. □

LEMMA 2.4. *Suppose* $\varepsilon_n = \gamma/(LLn)^{(1+\beta)/2}$ *for some* $\beta \in (0,1]$ *and*

(2.20) $$P\big(S(n(\cdot))/d(n) \in ((1+\delta)\mathscr{K})^{\varepsilon_n/2} \text{ eventually}\big) = 1,$$

where $d(n) = (2nLLn)^{1/2}$. *Then* (1.9) *and* (1.10) *hold under the conditions on* $\eta(n)$ *in Theorem* 2.

PROOF. To obtain (1.9), use (2.20) to select $h_k \in (1+\delta)\mathscr{K}$ such that for $k \geq n_0(\omega)$,

$$\|S(k(\cdot))/d(k) - h_k\|_\infty < \varepsilon_k/2.$$

Thus for $n_0(\omega) \leq \eta(n) \leq k \leq n$, as $n \to \infty$,

$$\begin{aligned}\|S(k(\cdot))/(2kLLn)^{1/2} - (LLk/LLn)^{1/2} h_k\|_\infty &\leq \gamma/\big(2(LLk)^{\beta/2}(LLn)^{1/2}\big)\\ &\sim \varepsilon_n/2\end{aligned}$$

because $LL\eta(n) \sim LLn$. Since $h_k \in (1+\delta)\mathscr{K}$ and $k \leq n$ throughout, it follows that (1.9) holds.

To establish (1.10), observe that (1.10) follows from (1.9) if

$$\sup_{k \leq \eta(n)} \|S(k(\cdot))/d(n)\|_\infty = O(n^{-1/4}).$$

Choosing $\eta(n) = n^{1/4}$ and setting

$$M(\omega) = \sup_{k \geq 1} \|S(k(\cdot))\|_\infty / d(k),$$

we have $M(\omega) < \infty$ with probability 1 by the law of the iterated logarithm for the polygonal process. Thus

$$\begin{aligned}\sup_{k \leq \eta(n)} \frac{\|S(k(\cdot))\|_\infty}{d(n)} &\leq M(\omega) \sup_{k \leq \eta(n)} \frac{d(k)}{d(n)}\\ &\leq n^{-1/4} M(\omega),\end{aligned}$$

and hence (1.10) holds. □

To verify the inner results, namely (1.8) and (1.11), we need a Cameron–Martin translation theorem for the centered Gaussian measures

$$\mu_j = \mathscr{L}\Big(\big(T_Y(\lambda(n_j,\cdot)) - T_Y(n_{j-1})\big)/n_j^{1/2}\Big) \tag{2.21}$$

on $C[0,1]$, where $n_r = \exp\{rLr\}$ for $r \geq 1$, $\lambda(n_j, t) = (n_j - n_{j-1})t + n_{j-1}$, $0 \leq t \leq 1$, and $T_Y(t)$ is as in (2.9). This is given in the following lemma.

LEMMA 2.5. *Let* μ_j *be the probability on* $C[0,1]$ *given by* (2.21) *for* $j \geq 2$ *and define* Q_n: $C[0,1] \to C[0,1]$ *by*

$$Q_n(g)(t) = \begin{cases} g(k/n), & t = k/n,\ k = 0,1,\dots,n,\\ g((k-1)/n) + n(t-(k-1)/n)\Delta g(k,n), & \\ \qquad (k-1)/n \leq t \leq k/n,\ k = 0,1,\dots,n, & \end{cases}$$

where

$$\Delta g(k,n) = g(k/n) - g((k-1)/n),$$

for $g \in C[0,1]$. *Let* $l_j = n_j - n_{j-1}$ *for* $j \geq 2$ *and*

$$\sigma_{j,k}^2 = \sigma_{k+n_{j-1}}^2,$$

where σ_k^2 *is as in* (2.8). *If* A *is a Borel subset of* $C[0,1]$, *then*

$$\mu_j(A) = \mu_j\big(A \cap Q_{l_j}(C[0,1])\big). \tag{2.22}$$

Further, if $A \subseteq Q_{l_j}(C[0,1])$ *and* $f \in Q_{l_j}(C[0,1])$, *then*

$$\mu_j(A + f) = \exp\left\{-n_j \sum_{k=1}^{l_j} \big(\Delta f(k, l_j)\big)^2 / \big(2\sigma_{j,k}^2\big)\right\} I_j(A), \tag{2.23}$$

where

$$I_j(A) = \int_A \exp\left\{-n_j \sum_{k=1}^{l_j} \big(\Delta g(k, l_j)\, \Delta f(k, l_j)\big)/\sigma_{j,k}^2\right\} d\mu_j(g). \tag{2.24}$$

PROOF. Since $\mu_j(Q_{l_j}(C[0,1])) = 1$, (2.22) is immediate. Thus assume $A \subseteq Q_{l_j}(C[0,1])$ and $f \in Q_{l_j}(C[0,1])$. Then

$$A + f \subseteq Q_{l_j}(C[0,1]),$$

and for $u_0 = 0$, $\tilde{Q}_{l_j}(A) = \{(g(1/l_j), \dots, g(l_j/l_j)): \ g \in A\}$, $\Lambda(j) = \prod_{k=1}^{l_j}(2\pi\sigma_{j,k}^2/n_j)^{1/2}$ and $v_k = u_k - f(k/l_j)$ for $k = 0, \dots, l_j$,

$$\begin{aligned}
&\mu_j(A + f) \\
&= \int \cdots \int_{\tilde{Q}_{l_j}(A+f)} \exp\left\{-n_j \sum_{k=1}^{l_j} \frac{(u_k - u_{k-1})^2}{\big(2\sigma_{j,k}^2\big)}\right\} du_1 \cdots du_{l_j} \Big/ \Lambda(j) \\
&= \int \cdots \int_{\tilde{Q}_{l_j}(A)} \exp\left\{-n_j \sum_{k=1}^{l_j} \left(\frac{(v_k - v_{k-1}) + \Delta f(k, l_j)^2}{\big(2\sigma_{j,k}^2\big)}\right)\right\} dv_1 \cdots dv_{l_j} \Big/ \Lambda(j) \\
&= \exp\left\{-n_j \sum_{k=1}^{l_j} \frac{\big(\Delta f(k, l_j)\big)^2}{\big(2\sigma_{j,k}^2\big)}\right\} I_j(A),
\end{aligned}$$

where $I_j(A)$ is as in (2.24). Thus (2.23) is proved. □

3. Proof of Theorem 1. Let $Y_1, Y_2, \dots$ be independent centered Gaussian random variables with

$$E\big(Y_k^2\big) = \sigma_k^2, \qquad 2^n \leq k < 2^{n+1},$$

where

$$\sigma_k^2 = E\big(X^2 I(X^2 \leq 2^n)\big) - \big(E\big(XI(X^2 \leq 2^n)\big)\big)^2,$$

and define the polygonal process $\{T_Y(t): t \geq 0\}$ as in (2.9). Then by [8] there is a probability space on which we can define copies of $\{X_j\colon j \geq 1\}$ and $\{Y_j\colon j \geq 1\}$ such that with probability 1,

$$\lim_n \|S(n(\cdot)) - T_Y(n(\cdot))\|_\infty / n^{1/2} = 0. \tag{3.1}$$

Since (3.1) holds we will have

$$P(S(n(\cdot))/d(n) \in \mathscr{K}^{\varepsilon_n} \text{ eventually}) = 1, \tag{3.2}$$

for each $\varepsilon > 0$, provided

$$P(T_Y(n(\cdot))/d(n) \in \mathscr{K}^{\varepsilon_n} \text{ eventually}) = 1, \tag{3.3}$$

for each $\varepsilon > 0$. Hence we turn to the proof of (3.3).

To prove (3.3), we first show

$$\sum_r P\left(T_Y(n_r(\cdot))/d(n_r) \notin \mathscr{K}^{\delta_r}\right) < \infty, \tag{3.4}$$

provided $n_r = \exp\{r/(Lr)^2\}$ and $\delta_r = \varepsilon/(4(LLn_r)^{1/2})$. This is immediate since $\mathscr{K}^{\delta_r}$ is an open, convex, symmetric subset of $C[0,1]$ and we can apply (2.10) along with the fact that

$$\sum_r P\left(W(n_r(\cdot))/d(n_r) \notin \mathscr{K}^{\delta_r}\right) < \infty. \tag{3.5}$$

That is, (3.5) holds by the argument in (2.7), (2.8) and (2.9) of [6] with $A(n_r) = LLn_r$, and hence (3.4) holds. Further, the Borel–Cantelli lemma and (3.4) imply

$$P\left(T_Y(n_r(\cdot))/d(n_r) \in \mathscr{K}^{\delta_r} \text{ eventually}\right) = 1. \tag{3.6}$$

If $h_r \in \mathscr{K}$ is such that

$$\|T_Y(n_{r+1}(\cdot))/d(n_{r+1}) - h_r\|_\infty < \delta_{r+1},$$

then for $n \in I(r) = [n_r, n_{r+1}]$ and $g(t) = h_r(nt/n_{r+1})$,

$$\begin{aligned} &\|T_Y(n(\cdot))/d(n) - h_r(\cdot)\|_\infty \\ &\quad \leq \|h_r - g\|_\infty + \|g - T_Y(n(\cdot))/d(n_{r+1})\|_\infty \\ &\qquad + \|T_Y(n(\cdot))/d(n)\|_\infty |1 - d(n)/d(n_{r+1})|. \end{aligned} \tag{3.7}$$

Elementary calculations, the definition of $\mathscr{K}$, the law of the iterated logarithm and (3.7) imply that for large r,

$$\sup_{n \in I(r)} \|T_Y(n(\cdot))/d(n) - h_r(\cdot)\|_\infty \leq 3\delta_{r+1}. \tag{3.8}$$

Combining (3.6) and (3.8) implies (3.3) for each $\varepsilon > 0$: Hence (3.2) holds for each $\varepsilon > 0$ and applying Lemma 2.3 yields (1.7).

To prove (1.8), fix $\delta \in (0,1)$ and in the remainder of the section set $n_r = \exp\{rLr\}$ for $r \geq 1$. Since (3.1) holds it suffices to show that for each $\varepsilon > 0$,

$$P\left((1-\delta)\mathscr{K} \subseteq \left(\hat{E}_n\right)^{3\varepsilon_n} \text{ eventually}\right) = 1, \tag{3.9}$$

where

$$\hat{E}_n = \{T_Y(k(\cdot))/(2kLLn)^{1/2}: Ln \le k \le n\}. \tag{3.10}$$

To verify (3.9), we first prove that

$$P\Big(\big\{\big(T_Y(\lambda(n_j,\cdot)) - T_Y(n_{j-1})\big)/(2n_jLLn_r)^{1/2}: r/2 \le j \le r\big\} \subseteq \big(\hat{F}_{n_r}\big)^{\xi\varepsilon_{n_r}} \text{ eventually}\Big) = 1, \tag{3.11}$$

for each $\xi > 0$, $\lambda(n_j, t)$ as in (2.21) and

$$\hat{F}_n = \{T_Y(k(\cdot))/(2kLLn)^{1/2}: (Ln)^2 \le k \le n\}.$$

To verify (3.11), we prove that with probability 1,

$$\lim_r \varepsilon_{n_r}^{-1} \sup_{r/2 \le j \le r} \|T_Y(n_{j-1}(\cdot))/(2n_jLLn_r)^{1/2}\|_\infty = 0 \tag{3.12}$$

and

$$z_r = \sup_{r/2 \le j \le r} \|T_Y(\lambda(n_j,\cdot)) - T_Y(n_j(\cdot))\|_\infty/(2n_jLLn_r)^{1/2} \le \xi\varepsilon/\big(2(LLn_r)^{1/2}\big) \tag{3.13}$$

eventually as $r \to \infty$. Then, since $\{T_Y(n_j(\cdot))/(2n_jLLn_r)^{1/2}: r/2 \le j \le\}$ is a subset of $\hat{F}_{n_r}$ for r large, (3.11) will hold.

Now (3.12) follows from (3.3) and the inequality

$$\sup_{r/2 \le j \le r} (n_{j-1}/n_j)^{1/2} \le 2/(Lr)^{-3/2}.$$

To verify (3.13), use (3.3). That is, (3.3) implies that for $\xi > 0$, with probability 1,

$$T_Y(n_j(\cdot))/(2n_jLLn_r)^{1/2} \in \mathscr{K}^{\xi\varepsilon_{n_r}/4},$$

for all sufficiently large r and $j \in [(Lr)^3, r]$. Hence choose $g_j \in \mathscr{K}$ such that

$$\|T_Y(n_j(\cdot))/(2n_jLLn_r)^{1/2} - g_j(\cdot)\|_\infty < \varepsilon_{n_r}/4 \tag{3.14}$$

and define

$$h_j(t) = g_j(\lambda(n_j, t)/n_j), \qquad 0 \le t \le 1.$$

Then

$$z_r \le \sup_{Lr^3 \le j \le r} \Big\{\|T_Y(n_j(\cdot))/(2n_jLLn_r)^{1/2} - g_j\|_\infty + \|g_j - h_j\|_\infty + \|h_j - T_Y(\lambda(n_j,\cdot))/(2n_jLLn_r)^{1/2}\|_\infty\Big\}, \tag{3.15}$$

and since $g_j \in \mathscr{K}$ we have for large r,

$$\begin{aligned}
\sup_{(Lr)^3 \le j \le r} \|g_j - h_j\|_\infty &= \sup_{(Lr)^3 \le j \le r} \sup_{0 \le t \le 1} |g_j(t) - g_j(\lambda(n_j, t)/n_j)| \\
&\le \sup_{(Lr)^3 \le j \le r} \sup_{0 \le t \le 1} |t - \lambda(n_j, t)/n_j|^{1/2} \\
&\le \sup_{(Lr)^3 \le j \le r} |n_{j-1}/n_j|^{1/2} \\
&\le 2(Lr)^{-3/2}.
\end{aligned} \tag{3.16}$$

Further,

$$\begin{aligned}
&\sup_{(Lr)^3 \le j \le r} \sup_{0 \le t \le 1} \left| h_j(t) - T_Y(\lambda(n_j, t))/(2n_j LLn_r)^{1/2} \right| \\
&\quad = \sup_{(Lr)^3 \le j \le r} \sup_{n_{j-1}/n_j \le u \le 1} \left| g_j(u) - T_Y(n_j u)/(2n_j LLn_r)^{1/2} \right| \\
&\quad < \xi \varepsilon_{n_r}/4
\end{aligned} \tag{3.17}$$

by (3.14), and hence by (3.14)–(3.17) we get (3.13). Thus (3.11) is verified.

Now let

$$\Lambda_{n_r} = \left\{ \left(T_Y(\lambda(n_j, \cdot)) - T_Y(n_{j-1}) \right)/(2n_j LLn_r)^{1/2} \colon r/2 \le j \le r \right\}. \tag{3.18}$$

Next we show that for $0 < \xi < 1$ and $\delta \in (0,1)$,

$$P\left((1-\delta)\mathscr{K} \subseteq \Lambda_{n_r}^{(1-\xi)\varepsilon_{n_r}} \text{ eventually} \right) = 1. \tag{3.19}$$

To prove (3.19), fix $\varepsilon > 0$, $\delta \in (0,1)$ and let $\mathscr{K}_{n_r}$ be a finite subset of $(1-\delta)\mathscr{K}$ such that

(i) balls centered at points of $\mathscr{K}_{n_r}$ of sup-norm radius $(1-\xi)\varepsilon_{n_r}/8$ are disjoint, and

(ii) $\mathscr{K}_{n_r}$ is maximal, i.e., if we add a point of $(1-\delta)\mathscr{K}$ to $\mathscr{K}_{n_r}$ we get overlap among balls of sup-norm radius $(1-\xi)\varepsilon_{n_r}/8$ centered at this larger set.

Then

$$\begin{aligned}
&\mathscr{P}\left(\mathscr{K}_{n_r} \not\subseteq (\Lambda_{n_r})^{(1-\xi)\varepsilon_{n_r}/2} \right) \\
&\quad \le \sum_{f \in \mathscr{K}_{n_r}} P\left(\bigcap_{j=r/2}^{r} \left\{ \| \left(T_Y(\lambda(n_j, \cdot)) - T_Y(n_{j-1}) \right)/n_j^{1/2} - (2LLn_r)^{1/2} f \|_\infty \ge (1-\xi)\varepsilon/2^{1/2} \right\} \right) \\
&\quad = \sum_{f \in \mathscr{K}_{n_r}} \prod_{j=r/2}^{r} \left[1 - \mu_j\left(g \colon \|g - (2LLn_r)^{1/2} f\|_\infty \le (1-\xi)\varepsilon/2^{1/2} \right) \right],
\end{aligned} \tag{3.20}$$

where μ_j is as in (2.21).

Now for each $f \in \mathscr{K}_{n_r}$ and each integer j such that $r/2 \le j \le r$,

$$\|f - Q_{l_j} f\|_\infty \le \left(l_j^{-1/2}\right) \le 2\exp\{-rLr/8\},$$

where $l_j = n_j - n_{j-1}$, and hence for large r,

$$(2LLn_r)^{1/2}\|f - Q_{l_j} f\|_\infty \le 4Lr\exp\{-rLr/8\}.$$

Thus for large r, uniformly in j satisfying $r/2 \le j \le r$,

$$\begin{aligned} &\mu_j\left(g: \|g - (2LLn_r)^{1/2} f\|_\infty < (1-\xi)\varepsilon/2^{1/2}\right) \\ (3.21)\quad &\ge \mu_j\left(g: \|g - (2LLn_r)^{1/2} Q_{l_j} f\|_\infty < (1-\xi)\varepsilon/4\right) \\ &= \exp\left\{-n_j LLn_r \sum_{k=1}^{l_j} \left(\Delta f(k, l_j)\right)^2/\sigma_{j,k}^2\right\} I_j(\{g: \|g\|_\infty < (1-\xi)\varepsilon/4\}) \end{aligned}$$

by Lemma 2.5 and the notation therein. Applying Jensen's inequality, it is easy to see that

$$(3.22)\qquad I_j(\{g: \|g\|_\infty < (1-\xi)\varepsilon/4\}) \ge \mu_j(g: \|g\|_\infty < (1-\xi)\varepsilon/4).$$

Since $\lim_k \sigma_k^2 = 1$, Donsker's invariance principle implies, for r large and uniformly in j satisfying $r/2 \le j \le r$, that

$$(3.23)\qquad I_j(\{g: \|g\|_\infty < (1-\xi)\varepsilon/4\}) \ge \rho > 0.$$

Combining (3.21), (3.22) and (3.23), it follows that for r large, uniformly in j satisfying $r/2 \le j \le r$,

$$\begin{aligned} &\mu_j\left(g: \|g - (2LLn_r)^{1/2} f\|_\infty < (1-\xi)\varepsilon/2^{1/2}\right) \\ (3.24)\quad &\ge \rho\exp\left\{-n_j LLn_r \sum_{k=1}^{l_j} \left(\Delta f(k, l_j)\right)^2/\sigma_{j,k}^2\right\}. \end{aligned}$$

Further, since $f \in (1-\delta)\mathscr{K}$, $\lim_k \sigma_k^2 = 1$, and uniformly in j satisfying $r/2 \le j \le r$, $\lim_r (n_j/l_j) = 1$, the lemma in [9], page 75, implies

$$\begin{aligned} (3.25)\quad &\mu_j\left(g: \|g - (2LLn_r)^{1/2} f\|_\infty < (1-\xi)\varepsilon/2^{1/2}\right) \\ &\ge \rho\exp\left\{-(LLn_r)(1-\delta/2)^2\right\} \end{aligned}$$

uniformly in $f \in (1-\delta)\mathscr{K}$ and j satisfying $r/2 \le j \le r$ provided r is sufficiently large. Combining (3.20) and (3.25) for r sufficiently large,

$$\begin{aligned} (3.26)\quad &P\left(\mathscr{K}_{n_r} \not\subseteq \Lambda_{n_r}^{(1-\xi)\varepsilon_{n_r}/2}\right) \\ &\le \operatorname{card}\left(\mathscr{K}_{n_r}\right)\left(1 - \rho\exp\left\{-(LLn_r)(1-\delta/2)^2\right\}\right)^{r/3}. \end{aligned}$$

Since $\operatorname{card}(\mathscr{K}_{n_r}) \le \exp\{LLn_r\}$ by Lemma 3.3 of [5] and $1 - x \le e^{-x}$, (3.26) implies that for large r,

$$(3.27)\quad P\left(\mathscr{K}_{n_r} \not\subseteq \left(\Lambda_{n_r}\right)^{(1-\xi)\varepsilon_{n_r}/2}\right) \le \exp\left\{LLn_r - \rho(r/3)e^{-(LLn_r)(1-\delta/2)^2}\right\}.$$

Since $n_r = \exp\{rLr\}$ and $0 < \delta < 1$, the right-hand terms of (3.27) form a convergent series as r varies and hence by the Borel–Cantelli lemma

$$P\Big(\mathscr{K}_{n_r} \subseteq \big(\Lambda_{n_r}\big)^{(1-\xi)\varepsilon_{n_r}/2} \text{ eventually}\Big) = 1.$$

Hence by the construction of $\mathscr{K}_{n_r}$,

$$P\Big((1-\delta)\mathscr{K} \subseteq \big(\Lambda_{n_r}\big)^{(1-\xi)\varepsilon_{n_r}} \text{ eventually}\Big) = 1 \tag{3.28}$$

and (3.19) holds. By combining (3.11) and (3.19) it follows that

$$P\Big((1-\delta)\mathscr{K} \subseteq \big(\hat{F}_{n_r}\big)^{\varepsilon_{n_r}} \text{ eventually}\Big) = 1. \tag{3.29}$$

To prove (3.9), and hence (1.8), assume (3.29). If $I(r) = [n_r, n_{r+1})$, then by (1.7)

$$\begin{aligned} &\sup_{n \in I(r)} \sup_{1 \le k \le n_r} \|T_Y(k(\cdot))/(2kLLn)^{1/2} - T_Y(k(\cdot))/(2kLLn_r)^{1/2}\|_\infty \\ &\quad = O\Big(\sup_{n \in I(r)} \big|1 - (LLn_r/LLn)^{1/2}\big|\Big) \qquad (3.30) \\ &\quad = O\big((rLr)^{-1}\big). \end{aligned}$$

Further, since $\varepsilon_n \downarrow$ with $n_r = \exp\{rLr\} \uparrow$, it follows that for $n \in I(r)$ and r large

$$\big(\hat{E}_n\big)^{2\varepsilon_n} \supseteq \big\{T_Y(k(\cdot))/(2kLLn)^{1/2}\colon (Ln_r)^2 \le k \le n_r\big\}^{\varepsilon_{n_r}}. \tag{3.31}$$

Combining (3.30), (3.31) and (3.29),

$$P\Big((1-\delta)\mathscr{K} \subseteq \big(\hat{E}_n\big)^{2\varepsilon_n} \text{ eventually}\Big) = 1, \tag{3.32}$$

and since $\varepsilon > 0$ is arbitrary (1.8) holds. □

4. Proof of Theorem 2. Using Theorem 1 of [4], rather than the result of [8], there is a probability space on which we can define $\{X_j\colon j \ge 1\}$ and a sequence of independent centered Gaussian random variables $\{Y_j\colon j \ge 1\}$ such that

$$E\big(Y_k^2\big) = \sigma_k^2 \le 1, \qquad \lim_k \sigma_k^2 = 1,$$

and with probability 1,

$$\lim_n (LLn)^{\beta/2} \|S(n(\cdot)) - T_Y(n(\cdot))\|_\infty / n^{1/2} = 0, \tag{4.1}$$

where $\{T_Y(t)\colon t \ge 0\}$ is defined from the sequence $\{Y_j\colon j \ge 1\}$ as in (2.9).

Since (4.1) holds and $\varepsilon_n = \gamma/(LLn)^{(1+\beta)/2}$ with $\beta \in (0,1]$ we will have (1.9) and (1.10) from Lemma 2.4 provided we show

$$P\Big(T_Y(n(\cdot))/d(n) \in ((1+\delta)\mathscr{K})^{\gamma/LLn} \text{ eventually}\Big) = 1, \tag{4.2}$$

for all γ sufficiently large.

Let $n_r = \exp\{r/(Lr)^2\}$ for $r \geq 1$. To prove (4.2), we argue as in (3.4) and (3.5) using Lemma 2.2 and (2.10) since

$$\sum_r P\Big(W(n_r(\cdot))/d(n_r) \notin ((1+\delta)\mathscr{K})^{\gamma/(4LLn_{r+1})}\Big) < \infty, \tag{4.3}$$

for all γ sufficiently large (see (5.8) of [6]). Thus (2.10) and (4.3) imply

$$P\Big(T_Y(n_r(\cdot))/d(n_r) \in ((1+\delta)\mathscr{K})^{\gamma/(4LLn_{r+1})} \text{ eventually}\Big) = 1 \tag{4.4}$$

and arguing as in (3.7) we have

$$P\Big(T_Y(n(\cdot))/d(n) \in ((1+\delta)\mathscr{K})^{\gamma/LLn} \text{ eventually}\Big) = 1, \tag{4.5}$$

for all γ sufficiently large.

Thus (1.9) and (1.10) hold, and it suffices to establish (1.11). Hence fix $\delta \in (0,1)$ and set $n_r = \exp\{rLr\}$ for $r \geq 1$ in the remainder of the section.

Let

$$G_r = \Big\{T_Y(k(\cdot))/(2kLLn_r)^{1/2}: (n_{r+1})^{1/3} \leq k \leq n_r\Big\},$$

for $r \geq 1$, and set

$$\Lambda_{n_r} = \Big\{\big(T_Y(\lambda(n_j,\cdot)) - T_Y(n_{j-1})\big)/(2n_jLLn_r)^{1/2}: r/2 \leq j \leq r\Big\},$$

where $\lambda(n_j, t)$ is as in (2.21). Then

$$P\Big(\Lambda_{n_r} \subseteq (G_r)^{4\gamma/(LLn_r)} \text{ eventually}\Big) = 1 \tag{4.6}$$

provided γ is sufficiently large. To establish (4.6), argue as in the verification of (3.12) and (3.13) with $\gamma/(LLn_r)$ equal to the right-hand side of (3.13) and $\varepsilon_{n_r} = \gamma/(LLn_r)$ in (3.12) [see (3.14)–(3.17)].

Next we prove that

$$P\Big((1-\delta)\mathscr{K} \subseteq \big(\Lambda_{n_r}\big)^{2\gamma/(LLn_r)} \text{ eventually}\Big) = 1, \tag{4.7}$$

for all γ sufficiently large. To verify (4.7), we let $\mathscr{K}_{n_r}$ be a finite subset of $(1-\delta)\mathscr{K}$ such that

(i) balls centered at points in $\mathscr{K}_{n_r}$ of sup-norm radius $\gamma/(8LLn_r)$ are disjoint, and

(ii) $\mathscr{K}_{n_r}$ is a maximal subset of $(1-\delta)\mathscr{K}$ with this property.

Then as in (3.20)

$$\begin{aligned} &P\Big(\mathscr{K}_{n_r} \not\subseteq \big(\Lambda_{n_r}\big)^{\gamma/(LLn_r)} \text{ eventually}\Big) \\ &\quad\leq \sum_{f \in \mathscr{K}_{n_r}} \prod_{j=r/2}^{r} \Big\{1 - \mu_j\Big(g: \|g - (2LLn_r)^{1/2} f\|_\infty < 2^{1/2}\gamma/(LLn_r)^{1/2}\Big)\Big\}, \end{aligned} \tag{4.8}$$

where μ_j is as in (2.21). Further, arguing as in (3.21) and (3.22) and by applying Lemma 2.2 (rather than Donsker's theorem), we have uniformly in j satisfying

$r/2 \le j \le r$, for r sufficiently large,

$$\mu_j\big(g\colon \|g - (2LLn_r)^{1/2} f\|_\infty < 2^{1/2}\gamma/(LLn_r)^{1/2}\big)$$

$$\ge \exp\left\{ -n_j LLn_r \sum_{k=1}^{l_j} \big(\Delta f(k, l_j)\big)^2/\sigma_{j,k}^2 \right\}$$

$$\times I_j\big(g\colon \|g\|_\infty < 2^{1/2}\gamma/(LLn_r)^{1/2}\big)$$

$$\ge \exp\left\{ -n_j LLn_r \sum_{k=1}^{l_j} \big(\Delta f(k, l_j)\big)^2/\sigma_{j,k}^2 \right\} \tag{4.9}$$

$$\times P\big(\|\big(W(\lambda(n_j, \cdot)) - W(n_{j-1})\big)/n_j^{1/2}\|_\infty < 2^{1/2}\gamma/(LLn_r)^{1/2}\big)$$

$$\ge \exp\left\{ -n_j LLn_r \sum_{k=1}^{l_j} \big(\Delta f(k, l_j)\big)^2/\sigma_{j,k}^2 \right\}$$

$$\times \mu_W\big(g\colon \|g\|_\infty < \gamma/(LLn_r)^{1/2}\big),$$

where μ_W is Wiener measure. Using the lemma in [9], page 75, uniformly in $f \in (1-\delta)\mathscr{K}$ and j satisfying $r/2 \le j \le r$, for r sufficiently large and all γ sufficiently large, we can combine (4.8) and (4.9) to give

$$P\big(\mathscr{K}_{n_r} \not\subseteq (\Lambda_{n_r})^{\gamma/(LLn_r)}\big) \le \exp\big\{(LLn_r)^2 - (r/3)e^{-(LLn_r)((1-\delta/2)^2+\gamma^{-1})}\big\} \tag{4.10}$$

[see (3.25)–(3.27) and recall

$$\mu_W(g\colon \|g\|_\infty \le \varepsilon) \sim e^{-C\varepsilon^{-2}}$$

as $\varepsilon \downarrow 0$]. Since $n_r = \exp\{rLr\}$ and $0 < \delta < 1$, the right-hand terms in (4.10) form a convergent series as r varies and hence by the Borel–Cantelli lemma

$$P\big(\mathscr{K}_{n_r} \subseteq (\Lambda_{n_r})^{\gamma/(LLn_r)} \text{ eventually}\big) = 1. \tag{4.11}$$

Hence by the construction of $\mathscr{K}_{n_r}$, (4.7) holds and combining (4.6) and (4.7), we have

$$P\big((1-\delta)\mathscr{K} \subseteq (G_r)^{6\gamma/(LLn_r)} \text{ eventually}\big) = 1. \tag{4.12}$$

To prove (1.11), suppose γ is such that (4.12) holds. Since $LLn_r \sim LLn_{r+1}$, (4.12) implies

$$P\big((1-\delta)\mathscr{K} \subseteq (G_r)^{7\gamma/(LLn_{r+1})} \text{ eventually}\big) = 1.$$

Further, for $n \in I(r) = [n_r, n_{r+1})$ the definition of G_r implies

$$G_r \subseteq \bigcap_{n \in I(r)} \big\{T_Y(k(\cdot))/(2kLLn_r)^{1/2}\colon \eta(n) \le k \le n\big\}$$

because $(n_{r+1})^{1/3} \geq \max_{n \in I(r)} \eta(n)$. Hence

$$P\Big((1-\delta)\mathscr{K} \subseteq \big\{T_Y(k(\cdot))/(2kLLn_r)^{1/2}: \eta(n) \leq k \leq n\big\}^{7\gamma/(LLn_{r+1})} \text{ eventually}\Big) = 1,$$

and from an argument similar to that in (3.30) for γ sufficiently large

$$P\Big((1-\delta)\mathscr{K} \subseteq \big\{T_Y(k(\cdot))/(2kLLn)^{1/2}: \eta(n) \leq k \leq n\big\}^{8\gamma/(LLn)} \text{ eventually}\Big) = 1. \tag{4.13}$$

Hence, for γ sufficiently large, (4.1) and (4.13) with $\beta \in (0,1]$ imply (1.11). Thus Theorem 2 is proved. □

5. Some final remarks. The argument in [2] easily shows that the results of this paper are best possible provided we use our method of proof. That is, if we approximate $S(t)$ by a Gaussian process like $T_Y(t)$ such that (3.1) or (4.1) hold with $\lim \sigma_k^2 = 1$, then the argument in [2] easily implies

$$E(X^2) < \infty$$

whenever (3.1) holds, and

$$E\big(X^2(LL|X|)^{\beta}\big) < \infty$$

whenever (4.1) holds.

The paper [6] provides some further references, and a recent paper by Grill [7] presents a related result for Brownian motion. It would be interesting to obtain the uniformity results of this paper with the rates of convergence obtained by Grill under only the classical moment assumptions.

As a final remark, we mention that there is a significant difference in the asymptotic behavior of the sets $E_{n,\eta(n)}$ and F_n which appear in the statement of Theorem 2. The analog of (1.11) does not hold for the set F_n. In fact, for any $\varepsilon > 0$ and $\varepsilon_n \downarrow 0$,

$$P(\varepsilon\mathscr{K} \not\subseteq F_n^{\varepsilon_n} \text{ i.o.}) = 1. \tag{5.1}$$

To see this, note that with probability 1 the sequence

$$\big\{S(n(\cdot))/(2nLLn)^{1/2}\big\}$$

clusters at the zero function [10]. Then

$$P\big(\|S(n(\cdot))\|_\infty/(2nLLn)^{1/2} < \varepsilon/2 \text{ i.o.}\big) = 1.$$

By the definition of F_n, if $f \in F_n$, then

$$\|f\|_\infty \leq \|S(n(\cdot))\|_\infty/(2nLLn)^{1/2}.$$

Then

$$P(F_n^{\varepsilon_n} \subseteq (\varepsilon/2 + \varepsilon_n)U \text{ i.o.}) = 1,$$

where U denotes the unit ball of $C[0,1]$ in the uniform norm. However, $\mathscr{K}$ contains the function $k(t) = t$ and $\|k\|_\infty = 1$ so that

$$\varepsilon\mathscr{K} \not\subseteq (\varepsilon/2 + \varepsilon_n)U,$$

for n sufficiently large. This gives (5.1).

REFERENCES

[1] ANDERSON, T. W. (1955). The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. *Proc. Amer. Math. Soc.* **6** 170–176.

[2] BREIMAN, L. (1967). On the tail behavior of sums of independent random variables. *Z. Wahrsch. verw. Gebiete* **9** 20–25.

[3] CRAWFORD, J. (1977). Elliptically contoured measures on infinite dimensional Banach spaces. *Studia Math.* **60** 15–32.

[4] EINMAHL, U. (1987). Strong invariance principles for partial sums of independent random vectors. *Ann. Probab.* **15** 1419–1440.

[5] GOODMAN, V. (1988). Characteristics of normal samples. *Ann. Probab.* **16** 1281–1290.

[6] GOODMAN, V. and KUELBS, J. (1988). Rates of convergence for increments of Brownian motion. *J. Theoret. Probab.* **1** 27–63.

[7] GRILL, K. (1987). On the rate of convergence in Strassen's law of the iterated logarithm. *Probab. Theory Related Fields* **74** 583–589.

[8] MAJOR, P. (1979). An improvement of Strassen's invariance principle. *Ann. Probab.* **7** 55–61.

[9] REISZ, F. and SZ.-NAGY, B. (1955). *Functional Analysis*. Ungar, New York.

[10] STRASSEN, V. (1964). An invariance principle for the law of the iterated logarithm. *Z. Wahrsch. verw. Gebiete* **3** 211–226.

DEPARTMENT OF MATHEMATICS
INDIANA UNIVERSITY
BLOOMINGTON, INDIANA 47405

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN
MADISON, WISCONSIN 53706