DIMENSIONAL PROPERTIES OF ONE-DIMENSIONAL BROWNIAN MOTION

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For each closed set \( F \subseteq [0, 1] \), \( \dim X(F + t) = \min(1, 2 \dim F) \) for almost all \( t > 0 \). (\( X \) is one-dimensional Brownian motion). For each closed set \( F \subseteq [0, 1] \) of dimension greater than \( 1/2 \), \( m(X(F + t)) > 0 \) for almost all \( t > 0 \). These statements are true outside a single null-set in the sample space.

Introduction. \( X(t) \) is the standard one-dimensional Wiener process on \( 0 \leq t < +\infty \). We are interested in the Hausdorff dimension \( \dim X(F) \) for closed sets \( F \) in \( R^+ \). Since \( X \) is almost surely in every class \( \text{Lip}^{1/2-\epsilon} \) on every bounded set, we obtain easily \( \dim X(F) \leq \min(1, 2 \dim F) \) for all sets \( F \), outside a single null set. For fixed closed sets \( F \) the inequality is an equality [Kahane (1968, 1986)], the exceptional closed set depends on \( F \). Since \( X^{-1}(0) \) has almost surely dimension 1/2, it is clear that results valid for all closed sets \( F \) must have a different form. Theorems 1 and 2 name properties of \( X \) valid outside a single null set for all closed sets \( F \). After presenting their proofs, we make some comments of a more speculative nature.

**Theorem 1.** For each closed set \( F \subseteq [0, 1] \), \( \dim X(F + t) = \min(1, 2 \dim F) \) for almost all \( t > 0 \).

**Theorem 2.** For each closed set \( F \subseteq [0, 1] \) of dimension greater than \( 1/2 \), \( m(X(F + t)) > 0 \) for almost all \( t > 0 \).

**Proof of Theorem 1.** It is convenient to define \( H(u) = 1 \) if \( |u| < 1 \), \( H(u) = 0 \) otherwise and

\[
I(x, y, R) = \int_0^1 H(RX(x + t) - RX(y + t)) \, dt
\]

provided \( R > 0 \), \( 0 \leq x < y \leq 1 \).

**Lemma 1.** \( E(I(x, y, R)^p) \leq 2p!3^pR^{-p}(y-x)^{-p/2} \) for \( p = 1, 2, \ldots, 0 \leq x < y \leq 1, R > 1 \).

**Proof.** The \( p \)th moment is a multiple integral,

\[
p! \int \cdots \int P(|X(x + t_i) - X(y + t_i)| < R^{-1}, 1 \leq i \leq p) \, dt_1 \cdots dt_p,
\]

where the integral is extended over the set defined by \( 0 \leq t_1 \leq t_2 \leq \cdots \leq t_p \leq 1 \).

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We estimate the conditional probability

\[ P(|X(x + t_j) - X(y + t_j)| < R^{-1}|X(s), 0 \leq s < y + t_{j-1}^1), \]

for \(2 \leq j \leq p\). Now \(P \leq 1\) always,

\[ \begin{align*}
P & \leq R^{-1}(t_j - t_{j-1})^{-1/2}, & \text{when } R^{-2} \leq t_j - t_{j-1} < y - x, \\
& \leq R^{-1}(y - x)^{-1/2}, & \text{when } y - x < t_j - t_{j-1}. 
\end{align*} \]

(We assume that \(R^{-2} < y - x\), since the inequality is trivial otherwise.) Integration on \(t_j\) yields an upper bound \(3R^{-1}(y - x)^{-1/2}\), and iteration of this yields the inequality of Lemma 1. \(\square\)

For use in Theorem 2, we observe that the integral of the square of the probability has magnitude \(O(R^{-2}\log R) + O(R^{-2})(y - x)^{-1}\) and that this too is trivial if \(y - x < R^{-2}\).

To prove Theorem 1 we use Lemma 1 with \(R_n = 2^{-n} (n = 1, 2, 3, \ldots)\) and \(x, y\) all possible choices from the set \(T_n\) of rationals \(k8^{-n}\) in \([0, 1]\). The number of pairs \(x < y\) in question is at most \(8^{2n+1}\). Hence for \(A > 1\),

\[ P(I(x, y, 2^n) > nA2^{-n}(y - x)^{-1/2} \text{ for some } x \in T_n, y \in T_n) < 8^{2n+1}p!(An)^{-p}. \]

The optimal estimation of \(P(\cdot)\) is easily estimated by Stirling’s formula and is summable for large \(A\) (e.g., \(A > 24\log 2\)).

We claim now that \(I(x, y, 2^n) < A'n2^{-n}(y - x)^{-1/2}\) for all \(x, y\) and \(n > n_0(\omega),\) almost surely. This is trivial unless \(A'n2^{-n} < y - x\), which we therefore assume to be true. Let \(\bar{x}\) and \(\bar{y}\) be the closest points in \(T_{n-1}\) to \(x\) and \(y\), respectively. For \(n > n_0(\omega),\) we get from Lévy’s modulus of continuity \(I(x, y, 2^n) \leq I(\bar{x}, \bar{y}, 2^{n-1})\). But

\[ I(\bar{x}, \bar{y}, 2^{n-1}) < An2^{-n}(\bar{y} - \bar{x})^{-1/2} < 4An2^{-n}(y - x)^{-1/2} \text{ for } n > n_0(\omega). \]

Let now \(e < \dim F\) and \(0 < \eta < 1, 0 < \eta < 2e\). By a theorem of Frostman [see Carleson (1967), page 28 or Kahane and Salem (1962), page 62], \(F\) carries a probability measure \(\mu\) such that \(\mu(S) \leq c(diam S)^{\alpha}\) for every measurable set \(S\). Let \(\lambda_t\) be the transform of \(\mu\) by the mapping \(x \to X(x + t) (0 < x < 1, 0 < t < 1)\). A further theorem of Frostman [Carleson (1967), page 28 or Kahane and Salem (1962), page 34] shows that \(X(F + t)\), the support of \(\lambda_t\), has dimension at least \(\eta\) if

\[ \int \int [s_1 - s_2]^{-\eta} \lambda_t(ds_1)\lambda_t(ds_2) = \int \int |X(x + t) - X(y + t)|^{-\eta} \mu(dx)\mu(dy) \]

is finite. The second formula for “energy of \(\lambda_t\) in dimension \(\eta\)” can be transformed (using the function \(H\) introduced above) into

\[ \eta \int \int H(RX(x + t) - RX(y + t))R^{\eta-1}\mu(dx)\mu(dy) dR \]

\[ \leq 1 + \int_1^\infty \int H(RX(x + t) - RX(y + t))R^{\eta-1}\mu(dx)\mu(dy) dR. \]
To prove that the integral on the right is finite for almost all $t \in (0, 1)$, we integrate on $(0, 1)$ obtaining

$$2 \int \int_{x<y} \int_1^\infty I(x, y, R) R^{n-1} \mu(dx)\mu(dy).$$

The product measure of the set defined by $0 < y - x < R^{-2}$ is $O(R^{-2e})$, and the consequent estimation converges because $-2e + \eta - 1 < -1$. On the complementary domain we have $(y - x)^{-1/2} < R$ and then we have $I(x, y, R) < B(\omega) \log(e + R)R^{-1}(y - x)^{-1/2}$ (with $B$ depending only on the path). Integrating with respect to $R$ first, we obtain $O(y - x)^{-\eta/2} \log(e + |y - x|^{-1})$, and the integral converges because $\eta < 2e$. This completes the proof of Theorem 1.

**Proof of Theorem 2.** The argument applies to sets $E$ of positive $h$-measure, where $h(u) = u^{1/2} \log^{-3}(e + u^{-1})$, $0 < u < 1$. Obviously the method used for Theorem 1 must fail, since the energy in dimension 1 is always infinite. The standard technique involves the Plancherel formula; we employ the notations $e(t) = \exp 2\pi it$, $\hat{\mu}(u) = \int e(us)\mu(ds)$. In proving that $X(F + t)$ has positive measure for almost all $t \in (0, 1)$, it is natural to consider

$$\int_{-\infty}^{\infty} \int \int \left[ \int e(-uX(x + t) + uX(y + t)) dt \right]$$

$$\times \mu(dx)\mu(dy) du,$$

for an appropriate measure $\mu$ on $F$, determined by Frostman's theorem. The inner integral, however, cannot be brought down to $o(u^{-1})$ even for $x = 0$, $y = 1$, and so this method, too, seems to fail. To overcome this difficulty, we choose and fix a smooth, even function $\psi \geq 0$, such that $\psi(u) = 1$ when $1 \leq |u| \leq 2$ and $\psi(u) = 0$ outside $1/2 < |u| < 5/2$. Then for any function $g(u)$,

$$\int_{|u| > 1} \left| g(u) \right|^2 du < \sum_{0}^{\infty} \int_{0}^{\psi(2^{-n}u)} |g(u)|^2 du.$$

Writing $g(u) = \hat{\lambda}_t(u)$, we find a formula for the $n$th integral on the right ($n = 0, 1, 2, 3, \ldots$),

$$2^n \int \int \psi(2^n X(x + t) - 2^n X(y + t)) \mu(dx)\mu(dy).$$

Bearing in mind that this integral is positive, we see that Theorem 2 can be proved by verifying the convergence of

$$\sum_{1}^{\infty} 2^n \int \int \psi(2^n X(x + t) - 2^n X(y + t)) \mu(dx)\mu(dy)$$

for all measures $\mu$ on $(0, 1)$ with the appropriate Lipschitz-type property. From the $n$th integral in (1) we remove the set defined by $|x - y| < 4^{-n}(n + 1)^{-2}$,
allowing thereby an error $O(n^{-2})$. For the remaining points $(x, y)$, we define

$$J(x, y, n) = \int_0^1 \hat{\psi}(2^n X(x + t) - 2^n X(y + t)) \, dt$$

and state

**Lemma 2.** For $n \geq n(\omega)$ and $|y - x| \geq 4^{-n}n^2$, $|J(x, y, n)| \leq (2 + c)^{-n}(y - x)^{-1/2}$, for some $c > 1/2$.

Taking into account the Hölder-continuity of $X$ and the smoothness of $\hat{\psi}$, we see that it will be sufficient to prove an inequality

$$E(J(x, y, n)^{2p}) \leq A_p(2 + c_1)^{-2p n}(y - x)^{-p}$$

with a constant $c_1 > 1/2$. ($J$ is real because $\psi$ is even.) The moment is the expected value of a multiple integral,

$$\int \cdots \int (\prod_{k=1}^{2p} \hat{\psi}(2^n X(x + t_K) - 2^n X(y + t_K)) \, dt_1 \cdots dt_{2p}.$$  

We can assume that $0 < x < y$ and claim that the expected value is exceedingly small if, for a certain $K$, $|t_K - t_j| \geq 4^{-n}(n + 1)^2$ for $j \neq K$ and $|y + t_K - x - t_j| \geq 4^{-n}(n + 1)^2$ for $j \neq K$. To verify this we let $r_n = 4^{-n}(n + 1)^2$ so that the interval $(t_K + y - r_n, t_K + y + r_n)$ is entirely contained in $(0, + \infty)$ and contains none of the $4p$ values appearing in the product $\Pi$ except $y + t_K$. Thus $X(y + t_K - r_n) - 2X(y + t_K) + X(y + t_K - r_n)$ is orthogonal to all values $X(\cdot)$ appearing there, except $X(y + t_K)$, with which it has inner product $-r_n$, its variance being $2r_n$. Hence $X(y + t_K) = h + Z$, where $h$ is measurable over the $\sigma$-field of the remaining values $X(\cdot)$, and $Z$ is Gaussian and independent of those values, $\sigma^2(Z) = r_n/2$, $\sigma^2(2^n Z) \geq 4t_r/2 = (n + 1)^2/2$. Here we invoke a formula from Fourier analysis: When $\psi \in L^1(R)$ and $Y$ is a random variable, $E(\hat{\psi}(Y)) = \int_{-\infty}^{\infty} \psi(s)E(e(sY)) \, ds$. We use the requirement that $\psi(u) = 0$ when $|u| < 1/2$, and first take the expected value with respect to the variable $Z$. The expectation is indeed minuscule, being bounded by $c_1\exp(-c_2 n^2)$ ($c_1 > 0, c_2 > 0$). This argument is valid for $K = 1, 2, \ldots, 2p$; a bit of combinatorics shows that it applies to all values $t_1, \ldots, t_{2p}$ except a set of product measure $A_p r_n^p = A_p 2^{-n p}(n + 1)^{2p}$, which we call $T_n(x, y)$.

$$\int \cdots \int_{T_n} E(\prod \hat{\psi}(2^n X(x + t_K) - 2^n X(y + t_K)) \, dt_1 \cdots dt_{2p}$$

by means of the Cauchy–Schwarz inequality and a remark made in the proof of Lemma 1. Let $B$ be any (large) positive number; since $\hat{\psi}$ is a rapidly decreasing function, the product $\Pi$ is bounded by $C(B)2^{-nB}$ outside the set defined by the inequalities $|X(x + t_K) - X(y + t_K)| \leq 2^{-n/8}$. The Cauchy–Schwarz inequality, the estimate for the measure of $T_n$ and the remark cited above
therefore yield (with $R = 2^{-\frac{9n}{10}}$) an estimate
\[ A_{p}'''((y-x)^{-2np}2^{-7np/2}2^{-np})^{1/2}n^{n}\leq A_{p}''''((y-x)^{-np})(2\cdot1)^{-2np}. \]

The $n$th integral in the sum (1) has magnitude
\[ O(n^{-2}) + (2 + c)^{-n}2^{n}\int\int|y-x|^{-1/2}\mu(dx)\mu(dy), \]
where the integral $\int f^*$ extends over the subset $|x - y| \geq 4^{-n}(n + 1)^2$. Since $\int_0^1 h(t)t^{-3/2}dt < +\infty$, the sum (1) converges. □

Remarks and problems. For Brownian motion $(X_1, X_2)$ with range in $R^2$, Theorem 1 has no interest in view of Kaufman (1969b) and Hawkes (1970). The following problem analogous to Theorem 1 seems very difficult.

For each closed set $F$, a number $\theta$ in $[0, \pi]$ is exceptional if $X_1 \cos \theta + X_2 \sin \theta$ maps $F$ onto a linear set of dimension less than $\min(1, 2 \dim F)$. Is there a random closed set $F$ whose exceptional set of angles has positive dimension?

Returning to one-dimensional Brownian motion $X$, $t$ is exceptional if $\dim X(F + t) < \min(1, 2 \dim F)$. What about the exceptional sets? On these topics compare Kaufman (1968, 1969a) and Kaufman and Mattila (1975).

When $F$ is a fixed set of dimension greater than $1/2$, then $X(F)$ has almost surely an interior point [Kahane (1986)]. Is it true that for every closed set $F$ of dimension greater than $1/2$, $X(F + t)$ has an interior point for some $t$?

REFERENCES