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INVARIANCE PRINCIPLE AND EMPIRICAL MEAN LARGE DEVIATIONS OF THE CRITICAL ORNSTEIN–UHLENBECK PROCESS[1]

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We consider a lattice system of linear interacting diffusion processes with infinitely many invariant distributions. We first prove a nonstandard central limit theorem and identify the equation of the fluctuation field. We then derive dimension dependent large deviation results for the empirical mean.

1. Introduction. Let $I = Z^d$ be the d-dimensional lattice. Consider a diffusion process $X_t = (X_t(i), i \in I)$ on $\mathbb{R}^I$ of the form

$$X_t(i) = x(i) + \tfrac{1}{2}\int_0^t \Big(-X_s(i) + \sum_{k \neq i} \alpha(i-k) X_s(k) \Big)\, ds + W_t(i), \qquad i \in I, \tag{1.1}$$

where α is nonnegative and symmetric and $(W(i), i \in I)$ is a collection of independent Wiener processes.

Consider first the subcritical case where

$$\sum_{k \neq 0} \alpha(k) = \eta < 1.$$

Then the process has a unique Gaussian invariant distribution and a standard rescaling yields a classical central limit theorem. More precisely let $\mathscr{S}(\mathbb{R}^d)$ be the Schwarz space of rapidly decreasing functions and $\mathscr{S}'(\mathbb{R}^d)$ be the tempered distributions. Define the $\mathscr{S}'(\mathbb{R}^d)$-valued process

$$\tilde{Y}_t^{(\lambda)}(\psi) \equiv \lambda^{-d/2} \sum_i \psi(i/\lambda)\big(X_t(i) - E[X_t(i)]\big), \qquad \psi \in \mathscr{S}'(\mathbb{R}^d).$$

Then one can show that $\tilde{Y}^{(\lambda)}$ converges in law as $\lambda \to \infty$ to a Gaussian process $\tilde{Y}$ which satisfies the partial stochastic differential equation

$$\tilde{Y}_t(\psi) = \tilde{Y}_0(\psi) - \frac{1-\eta}{2}\int_0^t \tilde{Y}_s(\psi)\, ds + B_t(\psi), \qquad \psi \in \mathscr{S}(\mathbb{R}^d),$$

where $(B_t(\psi), \psi \in \mathscr{S}(\mathbb{R}^d))$ is the $\mathscr{S}'(\mathbb{R}^d)$-valued Wiener process with variance $\langle B(\psi)\rangle_t = t\int_{\mathbb{R}^d} |\psi(x)|^2\, dx$; cf. [3, 8]. Moreover in the subcritical case the process is hypercontractive; thus one can apply the large deviation theory of Donsker

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and Varadhan [4] to the normalized occupation time functional of X,

$$L_T \equiv \frac{1}{T}\int_0^T \delta_{X_s}\, ds;$$

cf. [15, 9].

In the critical case

$$\sum_{k\neq 0} \alpha(k) = 1,$$

for $d \geq 3$, the process has infinitely many invariant distributions and one is likely to expect nonstandard invariance principle and large deviation results. The aim of this paper is to illustrate these phenomena in the critical situation.

First set

$$Y_t^{(\lambda)}(\psi) \equiv \lambda^{-(d+2)/2}\sum_i \psi(i/\lambda)\big(X_{\lambda^2 t}(i) - E[X_{\lambda^2 t}(i)]\big), \qquad \psi \in \mathscr{S}(\mathbb{R}^d).$$

We are going to show in Section 3 that $Y^{(\lambda)}$ converges in law as $\lambda \to \infty$ to a Gaussian process Y, which is the solution of the partial stochastic differential equation

$$Y_t(\psi) = 0 + \int_0^t Y_s\big(\tfrac{1}{4}\Delta_\alpha \psi\big)\, ds + B_t(\psi), \qquad \psi \in \mathscr{S}'(\mathbb{R}^d),$$

where $\Delta_\alpha \psi(x) \equiv \sum_{k,i=1}^d a(i,k)\partial_i \partial_k \psi(x)$ and $(a(i,k), 1 \leq i,k \leq d)$ is the positive definite matrix associated with the bilinear form

$$Q(\theta) \equiv \sum_{k\neq 0} (k\theta)^2 \alpha(k) = \sum_{i,k=1}^{d} a(i,k)\theta_i\theta_j. \tag{1.2}$$

This result is very similar to the ones derived by Holley and Stroock for critical branching Brownian motions [7] or the voter model [8]. This coincidence is due to the fact that the covariance matrix of the invariant measures of the process X is precisely given by the Green function of the d-dimensional random walk associated to the coefficients $(\alpha(k), k \in I)$; cf. [13]. Actually both ideas and methods of this section are greatly inspired by Holley and Stroock [8]. Moreover it should be mentioned that Hsiao [10] has obtained the same type of results for an infinite system with Gaussian interaction of components.

Next we will describe in Section 4 the asymptotic distribution of the empirical mean

$$Z_T \equiv \frac{1}{T}\int_0^T X_s(0)\, ds.$$

We shall see that there exists $s_\alpha(d) \in (0,\infty)$, such that

$$\lim_{T\to\infty} \frac{1}{r_T(d)} \log P(Z_T > \beta) = -\frac{1}{2s_\alpha(d)}\beta^2,$$

with a rate $r_T(d)$ depending on the dimension,

$$\begin{aligned} r_T(d) &= T^{1/2}, && d = 3,\\ &= T/\log T, && d = 4,\\ &= T, && d \geq 5. \end{aligned} \tag{1.3}$$

For $d \geq 5$, one can actually compute the coefficient $s_\alpha(d)$ in terms of the covariance matrix of the invariant distributions of X; for $d = 3$, $s_\alpha(3)$ depends on the initial measure μ. Here again the similarity to related models is very appealing; cf. Bramson, Cox and Griffeath [1] or Cox and Griffeath [2].

Although the demonstrations are based on the linearity of the model, our method could probably be applied to different dynamics such as the stochastic Heisenberg model. Furthermore it would be interesting to derive a variational formula for the rate of convergence associated with the more general functional L_T; cf. [4].

In Section 2 we describe the covariances and the reversible distributions of the critical process. In Section 3 we show the invariance principle. Finally in Section 4 we derive the large deviations for the empirical mean.

2. Reversible distributions and covariances of the critical Ornstein–Uhlenbeck process. Let us first introduce some notation. Let

$$E \equiv \left\{x = (x(i), i \in I) \in \mathbb{R}^I \colon \sum_i (1 + |i|)^{2p} |x(i)|^2 < \infty \text{ for all } p \geq 1\right\}$$

be the space of all rapidly decreasing sequences and

$$E' \equiv \left\{x = (x(i), i \in I) \in \mathbb{R}^I \colon \sum_i (1 + |i|)^{-2p} |x(i)|^2 < \infty \text{ for some } p \geq 1\right\}$$

be the tempered sequences. To a sequence $\beta \in E'$ such that

$$|\beta|_1 \equiv \sum_k |\beta(k)| < \infty,$$

we associate its Fourier transformation

$$\hat{\beta}(\theta) \equiv \sum_k \beta(k) e^{ik\theta},$$

where $k\theta = k_1\theta_1 + \cdots + k_d\theta_d$. Note that $\hat{\beta}$ is bounded and that

$$\beta(k) = (2\pi)^{-d} \int_{\mathscr{C}} \hat{\beta}(\theta) e^{-ik\theta}\, d\theta, \tag{2.1}$$

where $\mathscr{C}$ is the d-dimensional cube $\mathscr{C} \equiv [-\pi, \pi]^d$. For $1 \leq p < \infty$

$$\|\varphi\|_p \equiv \left(\int_{\mathbb{R}^d} |\varphi(x)|^p\, dx\right)^{1/p}$$

will denote the usual L^p-norm. Next let $M_1(E')$ be the set of probability measures μ on E' with finite second moments $E^\mu[|x(k)|^2] < \infty$ and $M_{1,e}(E')$ be the set of $\mu \in M_1(E')$ which are invariant under the shift operation on the lattice I.

Let $\Omega \equiv C([0, \infty); E')$ and denote by $X = (X_t(i), t \in [0, \infty), i \in I)$ the coordinates on Ω and by $\mathscr{F} = (\mathscr{F}_t, t \in [0, \infty))$ the canonical filtration. For a given

$x \in E'$, let P_x be the law on $(\Omega, \mathscr{F})$ of the diffusion process satisfying the linear stochastic differential equation (1.1); cf. [7]. For $\mu \in M_1(E')$, we write $P_\mu = \int_{E'}\mu(dx)P_x$. Clearly if $\mu \in M_{1,e}(E')$, then P_μ is also shift invariant on the lattice. Let $\mathscr{D}$ denote the set of smooth functions with compact support on E' which depends on only a finite number of coordinates. For $f \in \mathscr{D}$, define

$$\mathscr{L}f(x) = \tfrac{1}{2}\sum_i\left(\left[-x(i) + \sum_{k\neq i}\alpha(i-k)x(k)\right]\partial_i f(x) + \partial_i^2 f(x)\right),$$

where

$$\partial_i f(x) = \frac{\partial}{\partial x(i)} f(x).$$

Then X is the diffusion on E' associated with the generator $\mathscr{L}$.

For $d \geq 3$, we shall call X the *critical Ornstein–Uhlenbeck process* if α satisfies the conditions:

(C)(i) $\alpha \geq 0$ and $\alpha(k) > 0$ for only finitely many $k \in I$.

(C)(ii) $$\alpha(k) = \alpha(-k).$$

(C)(iii) $$\sum_{k\neq 0}\alpha(k) = 1.$$

(C)(iv) $\hat{\alpha}(\theta) = 1$ if and only if all the coordinates of θ are integer multiples of 2π.

The symmetry (C)(ii) implies that the operator $\mathscr{L}$ can be written in divergence form. Condition (C)(iv) is equivalent to the aperiodicity of the random walk on I associated with the transition probabilities $(\alpha(k), k \in I)$; cf. Theorem 7.1 of [14]. A standard example will be the *isotropic* case where

$$\alpha(k) = 1/2d, \qquad |k| = 1,$$
$$= 0, \qquad |k| \neq 1$$

and $\hat{\alpha}(\theta) = 1/d\sum_1^d\cos(\theta_j)$. From conditions (C) it follows that $\varphi(\theta) \equiv 1 - \hat{\alpha}(\theta)$ satisfies

(2.2) $$\varphi(\theta) = \tfrac{1}{2}Q(\theta) + O(|\theta|^4),$$

where $Q: \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ is the positive definite bilinear form defined by (1.2). Moreover there exists $\varepsilon \in (0, \infty)$ such that

(2.3) $$\varphi(\theta) \geq \varepsilon|\theta|^2, \qquad \theta \in \mathscr{C};$$

cf. [14].

We shall now describe the reversible measures of the process X. We first introduce the potential $\mathscr{P} = \{J_F, F \subseteq I \text{ and } |F| < \infty\}$:

$$J_F(x) \equiv \begin{cases} J_{\{i\}}(x) = \frac{1}{2}(x(i))^2, & \\ J_{\{i,k\}}(x) = -\alpha(i-k)x(i)x(k), & i \neq k, \\ J_F(x) = 0, & |F| > 2, \end{cases}$$

and define

$$H_i(x(i)|\tilde{x}^i) = H_i(x) = \sum_{F \ni i} J_F(x),$$

where $\tilde{x}^i = (x(k), k \in I - \{i\})$. With $\mathscr{P}$ we associate the class $\mathscr{G}(\mathscr{P}) \subseteq M_1(E')$ of *Gibbs states with potential* $\mathscr{P}$: $\mu \in \mathscr{G}(\mathscr{P})$ if and only if for all $i \in I$, $\mu^i(\cdot|\tilde{x}^i)$, the regular conditional probability distribution of μ given $\sigma(x(k), k \in I - \{i\})$ has a density of the form

$$\frac{\mu^i(dy|\tilde{x}^i)}{dy} = \exp(-H_i(y|\tilde{x}^i)) \Big/ \int_{\mathbb{R}} \exp(-H_i(z|\tilde{x}^i))\, dz.$$

It is easy to see that

$$\mu^i(\cdot|\tilde{x}^i) = \mathscr{N}\Big(\sum_{k \neq i} \alpha(i-k)x(k); 1\Big), \quad \mu \text{ a.e.,} \tag{2.4}$$

where $\mathscr{N}(m; 1)$ is the Gaussian law on $\mathbb{R}$ with expectation m and variance 1; cf. [11]. In the critical case we have a phase transition, i.e., the set $\mathscr{G}(\mathscr{P})$ is infinite. More precisely, let $\mathscr{G}_e(\mathscr{P}) \equiv \mathscr{G}(\mathscr{P}) \cap M_{1,e}(E')$. Then:

PROPOSITION 2.5 (Rozanov). *Let $d \geq 3$ and assume Conditions* (C). *Then the set of extremal points of $\mathscr{G}_e(\mathscr{P})$ coincides with the set of Gaussian measures $\mu \in M_{1,e}(E')$ with*

$$E^\mu[x(k)] = m \quad \textit{and} \quad \operatorname{cov}^\mu(x(j), x(k+j)) = R(k),$$

where $m \in \mathbb{R}$ is arbitrary and $(R(k), k \in I)$ is determined by

$$R(k) = (2\pi)^{-d} \int_{\mathscr{C}} \frac{1}{\varphi(\theta)} e^{-i\theta k}\, d\theta. \tag{2.6}$$

PROOF. First note that by (2.3), $1/\varphi \in L^1(\mathscr{C})$,

$$(2\pi)^{-d} \int_{\mathscr{C}} \frac{1}{\varphi(\theta)}\, d\theta \leq (2\pi)^{-d} \int_{\mathscr{C}} \varepsilon^{-1}|\theta|^{-2}\, d\theta = \frac{c_d}{\varepsilon} \int_0^\pi \theta^{d-3}\, d\theta < \infty.$$

Thus the integral in (2.6) is well defined and one can verify that $(R(k), k \in I)$ is the unique solution in E' to the equation

$$R(k) = \sum_{j \neq k} \alpha(k-j)R(j) + \delta(0,k), \qquad k \in I. \tag{2.7}$$

One the other hand, it follows from (2.4) that $\mu \in \mathscr{G}_e(\mathscr{P})$ if and only if the covariances of μ satisfy (2.7) and m fulfills

$$m = E^{\mu}[x(i)] = \sum_{k \neq i} \alpha(i-k) E^{\mu}[x(k)] = \sum_{k \neq i} \alpha(i-k) m. \tag{2.8}$$

Clearly by (C)(iii) any $m \in \mathbb{R}$ satisfies the preceding equation. Finally for $\mu \in \mathscr{G}_e(\mathscr{P})$, the conditional law of any finite set is Gaussian. This implies that extremal points of $\mathscr{G}_e(\mathscr{P})$ are Gaussian. □

COROLLARY 2.9. *The process X has infinitely many reversible measures: Any $\mu \in \mathscr{G}(\mathscr{P})$ has the property that, for each $T > 0$, $(X_t, t \in [0, T])$ and $(X_{T-t}, t \in [0, T])$ have the same distribution under P^{μ}.*

PROOF. Note that the operator $\mathscr{L}$ can be written in divergence form

$$\mathscr{L} f(x) = \tfrac{1}{2} \sum_i e^{H_i(x)} \partial_i \big(e^{-H_i(x)} \, \partial_i f(x) \big), \qquad f \in \mathscr{D}. \tag{2.10}$$

The result follows from (2.10) by a classical argument; cf. [7]. □

REMARK 2.11. (i) Define $\mathscr{A}(m) \equiv \{\mu \in M_{1,e}(E') \colon E^{\mu}[x(k)] = m\}$. Then it follows immediately from (1.1) that for each $\mu \in \mathscr{A}(m)$,

$$E^{P_\mu}[X_t(k)] = m \tag{2.12}$$

for all $t \in [0, \infty)$ and $k \in I$.

(ii) One can see from (2.7) that $(R(k), k \in I)$ is identical to the Green function of the d-dimensional random walk associated with the transition probabilities $(\alpha(k), k \in I)$; cf. [13]. Set

$$g_{a,d}(k) \equiv \kappa(d) (\det a)^{-1/2} (k, a^{-1} k)^{-(d-2)/2},$$

where a is the matrix associated with Q [cf. (1.2)] and $\kappa(d) \in (0, \infty)$ is a constant depending only on the dimension. Then the covariances of $\mu \in \mathscr{G}_e(\mathscr{P})$ have the asymptotic behavior

$$R(k) = g_{a,d}(k) \tag{2.13}$$

as $|k| \to \infty$; cf. Proposition 26.1 of [14].

Next we will give a spectral representation for the covariances of the process X. For $\mu \in M_{1,e}(E')$, set

$$R_t(k) \equiv \operatorname{cov}^{P_\mu}(X_t(0), X_t(k)),$$

$$\Gamma_{t,s}(k) \equiv \operatorname{cov}^{P_\mu}(X_t(0), X_s(k)).$$

PROPOSITION 2.14. *Let $\mu \in M_{1,e}(E')$ be such that $|R_0|_1 < \infty$. Then*

$$|R_t|_1 \leq |R_0|_1 + t. \tag{2.15}$$

Moreover we have the following Fourier transformations for R_t and $\Gamma_{t,s}$:

$$\hat{R}_t(\theta) = e^{-\varphi(\theta)t}\hat{R}_0(\theta) + \frac{1}{\varphi(\theta)}(1 - e^{-\varphi(\theta)t}), \tag{2.16}$$

$$\hat{\Gamma}_{t,s}(\theta) = e^{-1/2\varphi(\theta)(t-s)}\hat{R}_s(\theta), \qquad 0 \le s < t < \infty. \tag{2.17}$$

PROOF. We first derive the equation

$$E^{P_\mu}[X_t(k)|X_s] = \sum_j A_{t-s}(k-j)X_s(j), \qquad s \le t, \tag{2.18}$$

where $\hat{A}_{t-s}(\theta) = e^{(-1/2)\varphi(\theta)(t-s)}$. Clearly by the Markov property it is enough to take $s = 0$. Set $M_t^x(k) = E^{P_x}[X_t(k)]$. Then from (1.1), $M_t^x(k) = \sum_j B_t(k, j)x(j)$, where the matrix B_t is the solution to the linear equation

$$B_t(k, j) = \delta(k, j) + \tfrac{1}{2}\int_0^t \Big(-B_s(k, j) + \sum_{i \ne k} \alpha(k-i)B_s(i, j)\Big)\, ds;$$

cf. [3]. B_t is shift invariant and symmetric: $B_t(k, j) = A_t(k-j) = A_t(j-k)$, with

$$A_t(k) = \delta(k,0) + \tfrac{1}{2}\int_0^t \Big(-A_s(k) + \sum_{i \ne k} \alpha(k-i)A_s(i)\Big)\, ds.$$

Hence $|A_t|_1 = 1$ and its Fourier transformation satisfies

$$\hat{A}_t(\theta) = 1 - \tfrac{1}{2}\int_0^t \varphi(\theta)\hat{A}_s(\theta)\, ds.$$

This implies (2.18). We now turn to the computation of the covariances. First note that for all $j \in I$,

$$\operatorname{cov}^{P_\mu}(X_t(j), X_t(j+k)) = \operatorname{cov}^{P_\mu}(X_t(0), X_t(0+k)) = R_t(k). \tag{2.19}$$

Next by (2.12) we may assume that $\mu \in \mathscr{A}(0)$. Itô's formula yields

$$\begin{aligned} dX_t(0)X_t(k) = \Big(-X_t(0)X_t(k) + \tfrac{1}{2}\sum_{i \ne 0} \alpha(i)\{X_t(-i)X_t(k) \\ + X_t(0)X_t(k-i)\} + \delta(0, k)\Big)\, dt \\ + X_t(0)\, dW_t(k) + X_t(k)\, dW_t(0). \end{aligned}$$

Using the symmetry of α and (2.19), we get

$$R_t(k) = R_0(k) + \int_0^t \Big(-R_s(k) + \sum_{j \ne k} \alpha(k-j)R_s(j) + \delta(0, k)\Big)\, ds, \qquad k \in I. \tag{2.20}$$

Put $\bar{R}_t(k) \equiv e^t R_t(k)$. Then (2.20) yields

$$d\bar{R}_t(k) = \Big(\sum_{j \ne k} \alpha(k-j)\bar{R}_t(j) + e^t\delta(0, k)\Big)\, dt.$$

Thus by (C)(i) and (C)(ii) we get $|\overline{R}_t|_1 \le e^t|R_0|_1 + e^t t$, which implies (2.15). From (2.9) $\hat{R}_t(\theta)$ is the solution to

$$\hat{R}_t(\theta) = \hat{R}_0(\theta) - \int_0^t \varphi(\theta)\hat{R}_s(\theta)\,ds + t.$$

Thus $\hat{R}_t(\theta)$ is given by (2.16). Finally note that by (2.18) and (2.19)

$$\begin{aligned}\Gamma_{t,s}(k) &= E^{P_\mu}\big[X_s(k)E^{P_\mu}[X_t(0)|X_s]\big]\\ &= \sum_j A_{t-s}(j)E^{P_\mu}[X_s(k)X_s(j)] = \sum_j A_{t-s}(j)R_s(k-j).\end{aligned}$$

This yields (2.17) by (2.16) and (2.18). □

Remark 2.21. In Proposition 2.14 we have restricted ourselves to initial measures μ with bounded spectral density $\hat{R}_0$. Clearly we can extend (2.16) and (2.17) to $\hat{R}_0 \in L^1(\mathscr{C})$. In this case the covariances $(R_t(k), k \in I)$ are not necessarily summable. In particular if $\mu \in \mathscr{G}_e(\mathscr{P})$, one has $\hat{R}_t = 1/\varphi$ and $\hat{\Gamma}_{t,s} = e^{-1/2(t-s)\varphi}/\varphi$.

Our computations give us the following information about the convergence of R_t as $t \to \infty$:

Corollary 2.22. *Let $d \ge 3$ and $\mu \in M_{1,e}(E')$ satisfy $|R_0|_1 < \infty$. Then there exist $A, B \in (0,\infty)$ such that*

$$\sup_k |R_t(k) - R(k)| \le (A + B|R_0|_1/t)t^{1-d/2}. \tag{2.23}$$

Proof. By (2.1), (2.6) and (2.16) we have

$$R_t(k) - R(k) = (2\pi)^{-d}\int_{\mathscr{C}} \hat{f}_t(\theta)e^{-ik\theta}\,d\theta,$$

with $\hat{f}_t(\theta) = e^{-\varphi(\theta)t}\hat{R}_0(\theta) - 1/(\varphi(\theta))e^{-\varphi(\theta)t}$. Thus we will obtain the result once we verify

$$(2\pi)^{-d}\int_{\mathscr{C}} |\hat{f}_t(\theta)|\,d\theta \le (A + B|R_0|_1/t)t^{1-d/2}.$$

But this is just a consequence of (2.3) as

$$\begin{aligned}(2\pi)^{-d}\int_{\mathscr{C}} |\hat{f}_t(\theta)|\,d\theta &\le c_d\int_0^\pi e^{-\theta^2\varepsilon t}\left(|R_0|_1 + \frac{1}{\varepsilon\theta^2}\right)\theta^{d-1}\,d\theta\\ &= t^{1-d/2}c_d\int_0^\pi e^{-\theta^2\varepsilon}\left(\frac{|R_0|_1}{t} + \frac{1}{\varepsilon\theta^2}\right)\theta^{d-1}\,d\theta.\end{aligned}$$ □

Remark 2.24. In the subcritical case, one sees from (2.8) that $|\mathscr{G}(\mathscr{P})| = 1$. The unique element of $\mathscr{G}(\mathscr{P})$ is the Gaussian measure μ with $E^\mu[x(k)] \equiv 0$ and covariances $(R(k), k \in I)$ given by (2.6). $R(k)$ decays exponentially as $|k| \to \infty$; cf. [11]. Moreover, since

$$\varphi(\theta) \ge 1 - \sum_{k\ne 0}\alpha(k) = 1 - \eta \in (0,1],$$

there exists $A \in (0, \infty)$ such that

$$\sup_k |R_t(k) - R(k)| \le Ae^{-(1-\eta)t}.$$

3. The invariance principle. Let X be the critical Ornstein–Uhlenbeck starting at $\mu \in \mathscr{A}(m)$. For $\lambda \in (0, \infty)$ we introduce the $\mathscr{S}'(\mathbb{R}^d)$-valued process

$$Y_t^{(\lambda)}(\psi) \equiv \lambda^{-(d+2)/2} \sum_i \psi(i/\lambda)(X_{\lambda^2 t}(i) - m), \qquad \psi \in \mathscr{S}(\mathbb{R}^d). \tag{3.1}$$

The aim of this section is to prove

THEOREM 3.2. *Let $d \ge 3$ and assume conditions* (C). *Then for all $\mu \in \mathscr{A}(m)$ with $|R_0|_1 < \infty$, $Y^{(\lambda)}$ converges weakly as $\lambda \to \infty$ to the Gaussian $\mathscr{S}'(\mathbb{R}^d)$-valued process Y determined by the partial stochastic differential equation*

$$Y_t(\psi) = 0 + \int_0^t Y_s\left(\tfrac{1}{4}\Delta_a\psi\right) ds + B_t(\psi), \qquad \psi \in \mathscr{S}(\mathbb{R}^d), \tag{3.3}$$

where

$$\Delta_a\psi(x) \equiv \sum_{i,j=1}^{d} a(i,j)\partial_i\partial_j\psi(x)$$

[*cf.* (1.2)] *and* $(B_t(\psi), \psi \in \mathscr{S}(\mathbb{R}^d))$ *is the $\mathscr{S}'(\mathbb{R}^d)$-valued Wiener process with quadratic variation $\langle B(\psi)\rangle_t = t\|\psi\|_2^2$.*

PROOF. First note that by (2.12) we may assume $m = 0$. Next we will derive an estimate for the variance of $Y_t^{(\lambda)}(\psi)$. Put

$$\sigma_t^{2(\lambda)}(\psi) \equiv E^{P_\mu}\left[|Y_t^{(\lambda)}(\psi)|^2\right].$$

Then (2.15) yields

$$\begin{aligned} \sigma_t^{2(\lambda)}(\psi) &= \lambda^{-(d+2)} \sum_{i,j} \psi(i/\lambda)\psi(j/\lambda)R_{\lambda^2 t}(i-j) \\ &\le \lambda^{-d} \sum_i |\psi(i/\lambda)| \, \|\psi\|_\infty \lambda^{-2} |R_{\lambda^2 t}|_1 \qquad (3.4) \\ &\le \lambda^{-d} |\psi(\cdot/\lambda)|_1 \|\psi\|_\infty \left(|R_0|_1 \lambda^{-2} + t\right). \end{aligned}$$

Thus $\lim_{\lambda\to\infty}\sigma_0^{2(\lambda)}(\psi) = 0$ and $Y_0^{(\lambda)}(\psi)$ converges in law to 0.

Using (1.1) we have

$$\begin{aligned} \sum_i X_{\lambda^2 t}(i)\psi(i/\lambda) &= \sum_i X_0(i)\psi(i/\lambda) \\ &\quad + \tfrac{1}{2}\int_0^{\lambda^2 t} \sum_i X_s(i)\left(\sum_{k\ne i} a(k-i)\{\psi(k/\lambda) - \psi(i/\lambda)\}\right) ds \qquad (3.5) \\ &\quad + \sum_i \psi(i/\lambda) W_{\lambda^2 t}(i). \end{aligned}$$

On the other hand,

$$\psi(k/\lambda) - \psi(i/\lambda) = (1/\lambda)D\psi(i/\lambda)(k-i) + 1/(2\lambda^2)D^2\psi(i/\lambda)(k-i) + 1/(6\lambda^3)D^3\psi(i/\lambda + \delta(k/\lambda - i/\lambda))(k-i),$$

where D^k denotes the kth derivative of ψ acting on $\mathbb{R}^k$ and $\delta = \delta(i/\lambda, k/\lambda) \in [0,1]$. Set

$$\zeta^{(\lambda)}(i/\lambda) \equiv \frac{1}{6}\sum_{k\neq i} a(k-i)D^3\psi(i/\lambda + \delta(k/\lambda - i/\lambda))(k-i)$$

and remember the definition of the matrix $a(i, j)$. Then by (C)(ii) we have

$$\sum_{k\neq i} a(k-i)\{\psi(k/\lambda) - \psi(i/\lambda)\} = 1/(2\lambda^2)\Delta a\psi(i/\lambda) + (1/\lambda^3)\zeta^{(\lambda)}(i/\lambda).$$

Substituting this into (3.5), we obtain by the definition of $Y_t^{(\lambda)}$,

$$Y_t^{(\lambda)}(\psi) = Y_0^{(\lambda)}(\psi) + \int_0^t Y_s^{(\lambda)}\left(\tfrac{1}{4}\Delta_a\psi\right) ds + M_t^{(\lambda)}(\psi) + 1/(2\lambda)\int_0^t Y_s^{(\lambda)}(\zeta^{(\lambda)})\, ds, \tag{3.6}$$

where $M_t^{(\lambda)}(\psi)$ is a martingale with quadratic variation,

$$\langle M^{(\lambda)}(\psi)\rangle_t = \lambda^{-d}\sum_i |\psi(i/\lambda)|^2 \underset{\lambda\to\infty}{\longrightarrow} \|\psi\|_2^2. \tag{3.7}$$

We shall now see that the distributions of $Y^{(\lambda)}$, $\lambda \in (0,\infty)$, are tight in $C([0,\infty); \mathscr{S}'(\mathbb{R}^d))$. By Mitoma's theorem (cf. [12]) it suffices to show the tightness on $C([0,\infty);\mathbb{R})$ for fixed $\psi \in \mathscr{S}(\mathbb{R}^d)$. This will follow from (3.6), once we verify that for all $0 \le s < t \le T$,

$$E^{P_\mu}\left[\left|\int_s^t dM_u^{(\lambda)}(\psi)\right|^4\right] \le k_1|t-s|^2, \tag{3.8}$$

$$E^{P_\mu}\left[\left|\int_s^t Y_u^{(\lambda)}\left(\tfrac{1}{4}\Delta_a\psi + 1/(2\lambda)\zeta^{(\lambda)}\right) du\right|^2\right] \le k_2|t-s|^2, \tag{3.9}$$

where $k_1, k_2 \in (0,\infty)$ are constants depending on ψ, T and $|R_0|_1$. Clearly (3.7) implies (3.8). On the other hand we get (3.9) from (3.4) as

$$E^{P_\mu}\left[\left|\int_s^t Y_u^{(\lambda)}\left(\tfrac{1}{4}\Delta_a\psi + 1/(2\lambda)\zeta^{(\lambda)}\right) du\right|^2\right] \le |t-s|\int_s^t \sigma_u^{2(\lambda)}\left(\tfrac{1}{4}\Delta_a\psi + 1/(2\lambda)\zeta^{(\lambda)}\right) du.$$

From (3.4), (3.6) and (3.7) we see that any limiting process must satisfy (3.3). This concludes the proof, since (3.3) has a unique solution; cf. [7]. □

We conclude this section with a description of the rate of convergence of $\sigma_t^2(\psi) \equiv E^{P_\mu}[|Y_t(\psi)|^2]$ as $t \to \infty$. The reader is referred to Section 5 of [7] for a discussion of the ergodic properties of the process determined by (3.3).

PROPOSITION 3.10. *Let $d \geq 3$. Then under conditions* (C)

$$\lim_{t\to\infty} \sigma_t^2(\psi) = \sigma^2(\psi) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \psi(x)\psi(y) g_{a,d}(x-y)\,dx\,dy, \qquad \psi \in \mathscr{S}(\mathbb{R}^d);$$

cf. (2.13). *Moreover there exists $A \in (0,\infty)$ such that*

$$|\sigma_t^2(\psi) - \sigma^2(\psi)| \leq A\|\psi\|_1^2 t^{1-d/2}.$$

PROOF. From the preceding theorem we know that

$$\sigma_t^2(\psi) = \lim_{\lambda\to\infty} \sigma_t^{2(\lambda)}(\psi).$$

Replacing (2.23) into (3.4), we get

$$|\sigma_t^{2(\lambda)}(\psi) - \sigma^{2(\lambda)}(\psi)| \leq \left(\lambda^{-d}|\psi(\cdot/\lambda)|_1\right)^2\left(A + B|R_0|_1/(t\lambda^2)\right)t^{1-d/2},$$

where $\sigma^{2(\lambda)}(\psi) \equiv \lambda^{-d-2}\sum_{i,j}\psi(i/\lambda)\psi(j/\lambda)R(i-j)$. Applying (2.13), we see that

$$\begin{aligned}\lim_{\lambda\to\infty} \sigma^{2(\lambda)}(\psi) &= \lim_{\lambda\to\infty} \lambda^{-2d} \sum_{i,j,i\neq j} \psi(i/\lambda)\psi(j/\lambda)\lambda^{d-2}R(i-j)\\ &= \lim_{\lambda\to\infty} \lambda^{-2d} \sum_{i,j,i\neq j} \psi(i/\lambda)\psi(j/\lambda)g_{a,d}(i/\lambda - j/\lambda)\\ &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} \psi(x)\psi(y)g_{a,d}(x-y)\,dx\,dy.\end{aligned}$$

□

REMARK 3.11. In the subcritical case one can easily verify that

$$\lim_{\lambda\to\infty} \sigma_t^{2(\lambda)}(\psi) \equiv 0,$$

and $Y^{(\lambda)}$ converges in law to 0 as $\lambda \to \infty$.

4. The large deviation of the empirical mean. Consider the critical Ornstein–Uhlenbeck process with initial distribution $\mu \in \mathscr{A}(m)$ and set

$$Z_T \equiv \frac{1}{T}\int_0^T X_t(0)\,dt$$

for the empirical mean of $X(0)$. The aim of this section is to find the rate at which $P_\mu(Z_T > \beta + m)$ converges to 0. We start with an asymptotic description of the variance of Z_T,

$$S_T \equiv E^{P_\mu}\left[(Z_T - m)^2\right].$$

PROPOSITION 4.1. *Let $d \geq 3$ and define $r_T(d)$ as in* (1.3). *Assume conditions* (C). *Then for each $\mu \in \mathscr{A}(m)$ with $|R_0|_1 < \infty$,*

$$\lim_{T\to\infty} r_T(d)S_T = s_\alpha(d), \tag{4.2}$$

where $s_\alpha(d) \in (0,\infty)$ is independent of μ. Moreover if $d \geq 5$,

$$s_\alpha(d) = 4\sum_k |R(k)|^2 = (2\pi)^{-d}\int_{\mathscr{C}} \frac{4}{|\varphi(\theta)|^2}\,d\theta. \tag{4.3}$$

Proof. We shall proceed in several steps, assuming as usual $m = 0$.

(i) We first give a spectral representation for S_T. We have

$$S_T = \frac{2}{T^2}\int_0^T\int_0^t E^{P_\mu}[X_t(0)X_s(0)]\,ds\,dt$$
$$= \frac{2}{T^2}\int_0^T\int_0^t \Gamma_{t,s}(0)\,ds\,dt.$$

Using (2.1) and (2.17), we get by Fubini's theorem

$$S_T = (2\pi)^{-d}\int_{\mathscr{C}} \hat{S}_T(\theta)\,d\theta, \tag{4.4}$$

where

$$\hat{S}_T(\theta) = \frac{2}{T^2}\int_0^T\int_0^t \hat{\Gamma}_{t,s}(\theta)\,ds\,dt$$
$$= \frac{2}{T^2}\int_0^T e^{-1/2\varphi(\theta)t}\int_0^t\left(e^{1/2\varphi(\theta)s}\frac{1}{\varphi(\theta)} + e^{-1/2\varphi(\theta)s}\left(\hat{R}_0(\theta) - \frac{1}{\varphi(\theta)}\right)\right)ds\,dt$$
$$= \frac{\hat{R}_0(\theta)}{\varphi(\theta)T/2}\left(\frac{1}{\varphi(\theta)T/2}\left(1 - e^{-\varphi(\theta)T/2}\right)^2\right)$$
$$+ \frac{4}{\varphi(\theta)^2T}\left(1 - \frac{2}{\varphi(\theta)T/2}\left(1 - e^{-\varphi(\theta)T/2}\right) + \frac{1}{\varphi(\theta)T}\left(1 - e^{-\varphi(\theta)T}\right)\right).$$

(ii) Let $d \geq 5$. Then one can easily verify from (2.3) that $1/\varphi^2 \in L^1(\mathscr{C})$. Noting that the function $t \to 1/t(1 - e^{-t})$ is bounded on $[0, \infty)$. We get by the Lebesgue convergence theorem

$$\lim_{T\to\infty} T\hat{S}_T = \frac{4}{\varphi^2} \quad \text{in } L^1(\mathscr{C}).$$

This together with (4.4) imply (4.2) and (4.3).

(iii) Let $d = 3, 4$. Since $\hat{R}_0$ is bounded by assumption, using the preceding argument, we conclude that

$$\lim_{T\to\infty} r_T(d)\frac{\hat{R}_0}{\varphi T/2}\left(\frac{1}{\varphi T/2}\left(1 - e^{-\varphi T/2}\right)^2\right) = 0 \quad \text{in } L^1(\mathscr{C}).$$

Thus we may assume that $\hat{R}_0 \equiv 0$.

(iv) We can rewrite $\hat{S}_T$ as

$$\hat{S}_T(\theta) = Tf(\varphi(\theta)T/2), \tag{4.5}$$

where $f\colon [0, \infty) \to (0, \infty)$ is the function given by

$$f(t) = \frac{1}{t^2}\left(1 - \frac{1}{2t}\left\{\left(2 - e^{-t}\right)^2 - 1\right\}\right).$$

Using a Taylor expansion, we have

$$f(t) = \sum_0^\infty 2(-1)^k\left(2^{k+1} - 1\right)\frac{t^k}{(k+3)!}$$

and we see that f is continuous on $[0, \infty)$. The following equation holds for all $t \in (0, \infty)$:

$$f(t) \leq \frac{1}{t^2}. \tag{4.6}$$

Moreover for all $\delta \in (0,1)$, if $t > 3/2\delta$, then

$$f(t) \geq \frac{1-\delta}{t^2}. \tag{4.7}$$

(v) Replacing θ by $\theta T^{-1/2}$ into (4.4), we obtain

$$S_T = T^{1-d/2}(2\pi)^{-d} \int_{\mathscr{C}(T)} f(\varphi_T(\theta)/2)\, d\theta, \tag{4.8}$$

where $\mathscr{C}(T) = [-\pi T^{1/2}, \pi T^{1/2}]^d$ and $\varphi_T(\theta) = \varphi(\theta T^{-1/2})T$.

(vi) Take $d = 3$ and let $B(\rho) \equiv \{\theta: |\theta| \leq \rho\}$. Then (4.8) yields

$$T^{1/2} S_T = (2\pi)^{-3} \int_{B(1)} f(\varphi_T(\theta)/2)\, d\theta + (2\pi)^{-3} \int_{\mathscr{C}(T)-B(1)} f(\varphi_T(\theta)/2)\, d\theta.$$

Note that f is bounded on $B(1)$ and by (2.3) and (4.6),

$$f\left(\frac{\varphi_T(\theta)}{2}\right) \leq \frac{4}{|\varphi_T(\theta)|^2} \leq \frac{4}{\varepsilon^2 |\theta|^4},$$

where $4/|\theta|^4 \in L^1(\mathbb{R}^3 - B(1))$. Since f is continuous and by (2.2),

$$\varphi_T(\theta) = \tfrac{1}{2} Q(\theta) + T^{-1} O(|\theta|^4), \tag{4.9}$$

we get by the Lebesgue convergence theorem

$$\lim_{T\to\infty} (2\pi)^{-3} \int_{\mathscr{C}(T)} f(\varphi_T(\theta)/2)\, d\theta = (2\pi)^{-3} \int_{\mathbb{R}^3} f(Q(\theta)/4)\, d\theta \equiv s_\alpha(3).$$

(vii) Finally let $d = 4$. For each $\delta \in (0,1)$ we can choose $\rho_\delta \in (0, \infty)$ such that $\varphi_T(\theta)/2 > 3/2\delta$ on $B(\rho_\delta)$. We have by (4.8),

$$\frac{T}{\log T} S_T = \frac{(2\pi)^{-4}}{\log T} \int_{B(\rho_\delta)} f\left(\frac{\varphi_T(\theta)}{2}\right) d\theta + \frac{(2\pi)^{-4}}{\log T} \int_{\mathscr{C}(T)-B(\rho_\delta)} f\left(\frac{\varphi_T(\theta)}{2}\right) d\theta.$$

f is bounded on $B(\rho_\delta)$, thus

$$\lim_{T\to\infty} \frac{(2\pi)^{-4}}{\log T} \int_{B(\rho_\delta)} f\left(\frac{\varphi_T(\theta)}{2}\right) d\theta = 0.$$

On the other hand (4.7) implies

$$\begin{aligned}(1-\delta)\frac{(2\pi)^{-4}}{\log T} \int_{\mathscr{C}(T)-B(\rho_\delta)} \frac{4}{|\varphi_T(\theta)|^2}\, d\theta &\leq \frac{(2\pi)^{-4}}{\log T} \int_{\mathscr{C}(T)-B(\rho_\delta)} f\left(\frac{\varphi_T(\theta)}{2}\right) d\theta \\ &\leq \frac{(2\pi)^{-4}}{\log T} \int_{\mathscr{C}(T)-B(\rho_\delta)} \frac{4}{|\varphi_T(\theta)|^2}\, d\theta.\end{aligned}$$

Since δ is arbitrary, the proof will be complete once we show

$$\lim_{T\to\infty}\frac{(2\pi)^{-4}}{\log T}\int_{\mathscr{C}(T)-B(\rho_\delta)}\frac{4}{|\varphi_T(\theta)|^2}\,d\theta = s_\alpha(4)\in(0,\infty). \tag{4.10}$$

For simplicity we will suppose that $Q(\theta) = 2\lambda|\theta|^2$, where $\lambda\in(0,\infty)$. This actually corresponds to the isotropic case, but our argument can be generalized to conditions (C). From (2.3) and (4.9) we have

$$\begin{aligned}\left|\frac{1}{|\varphi_T(\theta)|^2}-\frac{1}{\lambda^2|\theta|^4}\right| &= |\varphi_T(\theta)-\lambda|\theta|^2|\left(\frac{1}{|\varphi_T(\theta)|^2\lambda|\theta|}+\frac{1}{\varphi_T(\theta)\lambda^2|\theta|^4}\right)\\ &\le\left(\frac{1}{\varepsilon^2\lambda}+\frac{1}{\varepsilon\lambda^2}\right)\frac{|\varphi_T(\theta)-\lambda|\theta|^2|}{|\theta|^6}\le AT^{-1}|\theta|^{-2}\end{aligned}$$

for some $A\in(0,\infty)$. Next note that

$$\begin{aligned}\lim_{T\to\infty}\frac{(2\pi)^{-4}}{\log T}\int_{\mathscr{C}(T)-B(\rho_\delta)}\frac{4}{\lambda^2|\theta|^4}\,d\theta &= \lim_{T\to\infty}\frac{c_4}{\log T}\int_{\rho_\delta}^{\pi T^{1/2}}\frac{4}{\lambda^2\theta}\,d\theta\\ &=\frac{2c_4}{\lambda^2}.\end{aligned}$$

Now (4.10) just follows from before as

$$\begin{aligned}\frac{(2\pi)^{-4}}{\log T}\int_{\mathscr{C}(T)-B(\rho_\delta)}\left|\frac{4}{|\varphi_T(\theta)|^2}-\frac{4}{\lambda^2|\theta|^4}\right|d\theta &\le \frac{(2\pi)^{-4}}{T\log T}\int_{\mathscr{C}(T)-B(\rho_\delta)}\frac{4A}{|\theta|^2}\,d\theta\\ &=\frac{4Ac_4}{T\log T}\int_{\rho_\delta}^{\pi T^{1/2}}\theta\,d\theta\xrightarrow[\lambda\to\infty]{}0.\qquad\square\end{aligned}$$

We introduce the set $\mathscr{H}$ of product measures $\mu=\prod_k\nu$ on $\mathbb{R}^I$ such that the function $\beta\to\Lambda_\nu(\beta)\equiv\log E^\nu[e^{\beta x(k)}]$ is C^2 on some interval $(-\delta,\delta)$. We come to our main result:

Theorem 4.11. *Let $d\ge 3$ and assume conditions* (C). *Then for any $\mu\in\mathscr{A}(m)$ which either belongs to $\mathscr{H}$ or is Gaussian with summable covariances,*

$$\lim_{T\to\infty}\frac{1}{r_T(d)}\log P_\mu(Z_T>m+\beta) = -\frac{1}{2s_\alpha(d)}\beta^2 \tag{4.12}$$

for all $\beta\in(0,\infty)$.

Proof. We will see that for all $\beta\in(0,\infty)$,

$$\lim_{T\to\infty}\frac{1}{r_T(d)}\log E^{P_\mu}[\exp(\beta r_T(d)Z_T)] = \frac{1}{2}s_\alpha(d)\beta^2+m\beta. \tag{4.13}$$

(4.12) follows then by a standard large deviation argument; cf. [5]. For Gaussian μ with $|R_0|_1<\infty$, (4.13) is a trivial consequence of (4.2). When μ is not Gaussian, we can write the solution of (1.1) in the form

$$X_t(k) = \sum_j A_t(k-j)X_0(j) + \overline{X}_t(k), \qquad k \in I,$$

where $(\overline{X}_t(k), k \in I)$ is the solution of (1.1) with $\overline{X}_0(k) \equiv 0$; cf. (2.18). Hence

$$Z_T = \sum_k B_T(k)X_0(k) + \overline{Z}_T,$$

with $B_T(k) = (1/T)\int_0^T A_t(k)\,dt$. This yields

$$\begin{aligned}\log E^{P_\mu}[\exp(\beta r_T(d)Z_T)] &= \log E^{\mu}\left[\exp\left(\beta r_T(d)\sum_k B_T(k)x(k)\right)\right] \\ &\quad + \log E^{P_0}[\exp(\beta r_T(d)\overline{Z}_T)].\end{aligned} \tag{4.14}$$

By the independence of $x(k)$, $k \in I$, under μ we have

$$\log E^{\mu}\left[\exp\left(\beta r_T(d)\sum_k B_T(k)x(k)\right)\right] = \sum_k \Lambda_\nu(\beta r_T(d)B_T(k)). \tag{4.15}$$

Since $\overline{Z}_T$ is Gaussian, (4.13) holds for P_0 with $m = 0$. Hence by (4.14) and (4.15) the proof is complete once we verify

$$\lim_{T\to\infty} \frac{1}{r_T(d)} \sum_k \Lambda_\nu(\beta r_T(d)B_T(k)) = m\beta. \tag{4.16}$$

Remember that

$$\hat{B}_T(\theta) = \frac{1}{T}\int_0^T \hat{A}_t(\theta)\,dt = \frac{1}{\varphi(\theta)T/2}\left(1 - e^{-\varphi(\theta)T/2}\right)$$

and

$$\sum_k B_T(k) \equiv 1; \tag{4.17}$$

cf. (2.18). As $1/\varphi \in L^1(\mathscr{C})$, we see that uniformly in $k \in I$,

$$\lim_{T\to\infty} r_T(d)B_T(k) = 0. \tag{4.18}$$

On the other hand $\Lambda_\nu \in C^2((-\delta,\delta))$ and $d/(d\beta)\Lambda_\nu(\beta)|_{\beta=0} = m$. Thus by (4.18), there exists $T_0 \in (0,\infty)$ such that, whenever $T > T_0$,

$$\Lambda_\nu(\beta r_T(d)B_T(k)) = m\beta r_T(d)B_T(k) + O\left(|r_T(d)B_T(k)|^2\right)$$

for all $k \in I$. But this together with (4.17) implies (4.16) as

$$\begin{aligned}\left|\frac{1}{r_T(d)}\sum_k \Lambda_\nu(\beta r_T(d)B_T(k)) - m\beta\right| &\le A r_T(d)\sum_k |B_T(k)|^2 \\ &= A r_T(d)(2\pi)^{-d}\int_{\mathscr{C}} |\hat{B}_T(\theta)|^2\,d\theta \underset{\lambda\to\infty}{\longrightarrow} 0,\end{aligned}$$

for some $A \in (0,\infty)$ [cf. (iii) in the proof of Proposition 4.1]. □

REMARK 4.19. By symmetry we have

$$\lim_{T\to\infty} \frac{1}{r_T(d)} \log P_\mu(Z_T < m - \beta) = -\frac{1}{2s_\alpha(d)}\beta^2$$

for all $\beta \in (0, \infty)$.

One may wonder whether coefficient $s_\alpha(d)$ would depend on the initial distribution μ when the covariances are not summable. If, for example, μ is a reversible distribution, the answer is *yes* for $d = 3$ and *no* for $d \geq 4$. Write $s_\alpha^0(d)$ for the number defined by (4.2) for any $\mu \in \mathscr{A}(m)$ with $|R_0|_1 < \infty$.

PROPOSITION 4.20. *Let $\mu \in \operatorname{ext} \mathscr{G}_e(\mathscr{P}) \cap \mathscr{A}(m)$. Then for all $\beta \in (0, \infty)$,*

$$\lim_{T\to\infty} \frac{1}{r_T(d)} \log P_\mu(Z_T > m + \beta) = -\frac{1}{2s_\alpha^\mu(d)}\beta^2,$$

where $s_\alpha^\mu(3) > s_\alpha^0(3)$ and $s_\alpha^\mu(d) = s_\alpha^0(d)$ for $d \geq 4$.

PROOF. Since μ is Gaussian, the result will follow from

$$\lim_{T\to\infty} r_T(d) E^{P_\mu}\left[(Z_T - m)^2\right] = s_\alpha^\mu(d).$$

Looking at the proof of Proposition 4.1, we see by (4.4) and (2.6) that is enough to verify

$$\begin{aligned} &s_\alpha^\mu(d) - s_\alpha^0(d) \\ (4.21)\qquad &= \lim_{T\to\infty} r_T(d)(2\pi)^{-d} \int_{\mathscr{C}} \frac{1}{\varphi(\theta)} \left(\frac{1}{\varphi(\theta)T/2}\left(1 - e^{-\varphi(\theta)T/2}\right)\right)^2 d\theta. \end{aligned}$$

Let $d \geq 5$. Then as $1/\varphi^2 \in L^1(\mathscr{C})$, it is clear that the limit is 0. For $d = 3, 4$, we have

$$s_\alpha^\mu(d) - s_\alpha^0(d) = \lim_{T\to\infty} r_T(d) T^{1-d/2} (2\pi)^{-d} \int_{\mathscr{C}(T)} \tilde{f}(\varphi_T(\theta)/2)\, d\theta,$$

where $\tilde{f}(t) = 1/2t(1/t(1 - e^{-t}))^2$; cf. (4.8). Note that $2\tilde{f}(t) \leq t^{-1} \wedge t^{-3}$ and $|\theta|^{-2} \in L^1(B(\rho))$, respectively, $|\theta|^{-6} \in L^1(\mathbb{R}^d - B(\rho))$. Thus by (2.3), (4.9) and the Lebesgue convergence theorem,

$$\lim_{T\to\infty} (2\pi)^{-d} \int_{\mathscr{C}(T)} \tilde{f}(\varphi_T(\theta)/2)\, d\theta = (2\pi)^{-d} \int_{\mathbb{R}^d} \tilde{f}(Q(\theta)/4)\, d\theta > 0,$$

which implies (4.21). □

REMARK 4.22. (i) The subcritical case behaves much like $d \geq 5$:

$$\lim_{T\to\infty} \frac{1}{T} \log E^{P_\mu}[Z_T > m + \beta] = -\frac{1}{2s_\alpha(d)}\beta^2,$$

where $s_\alpha(d)$ is given by (4.3). Actually the subcritical Ornstein–Uhlenbeck process is hypercontractive; cf. Theorem 2.1 of [9]. Thus the rate functional associated to the large deviations of the occupation time L_T can be identified with the Dirichlet form of the process and the preceding equality follows from the contraction principle; cf. [4, 9].

(ii) Finally one could also look at the large deviations of the Gibbs states: Let $\mu \in \operatorname{ext} \mathscr{G}_e(\mathscr{P}) \cap \mathscr{A}(m)$ and define

$$M_n \equiv |V_n|^{-1} \sum_{i \in V_n} x(i),$$

where V_n is the cube $[-n, n]^d$ and $|V_n|$ is its cardinality. Applying Proposition 3.10 for $d \geq 3$, there exists $\sigma_{a,d}^2 \in (0, \infty)$ such that

$$\lim_{n \to \infty} |V_n|^{(d-2)/d} E^{\mu}\left[|M_n - m|^2\right] = \sigma_{a,d}^2.$$

Since μ is Gaussian, this yields

$$\lim_{n \to \infty} |V_n|^{-(d-2)/d} \log \mu(M_n > m + \beta) = -\frac{1}{2\sigma_{a,d}^2} \beta^2$$

for all $\beta \in (0, \infty)$.

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